

## VI.4 Solutions of the Boltzmann equation

We now turn to a discussion of the solution of the Boltzmann equation.

### VI.4.1 Equilibrium distributions

According to the  $H$ -theorem,  $H(t)$  is a decreasing function of time—at least when the assumption of molecular chaos holds. Besides, the defining formula (VI.23) shows that  $H(t)$  is bounded from below when the spatial volume  $\mathcal{V}$  occupied by the system, and accordingly its total energy, is finite.

This follows from the finiteness of the available phase space—the finite energy of the system translates into an upper bound on momenta—and the fact that the product  $x \ln x$  is always larger than  $-1/e$ .

As a consequence,  $H(t)$  converges in the limit  $t \rightarrow \infty$ , i.e. its derivative vanishes:

$$\frac{dH(t)}{dt} = 0 \quad (\text{VI.32})$$

or equivalently  $dS_B(t)/dt = 0$ . In this section, we define and discuss two types of distributions that fulfill this condition:

- the *global equilibrium distribution*  $\bar{f}_{\text{eq.}}(\vec{r}, \vec{p})$  is defined as a stationary solution of the Boltzmann equation

$$\frac{\partial \bar{f}_{\text{eq.}}}{\partial t} = 0 \quad (\text{VI.33})$$

obeying Eq. (VI.32). In that case, the system has reached global thermodynamic equilibrium, with uniform values of the temperature  $T$  and of the average velocity  $\vec{v}$  of particles, as well as of their number density  $n$  (or equivalently the chemical potential  $\mu$ ) when there is no external force. In particular,  $\bar{f}_{\text{eq.}}$  cancels the collision integral (VI.15c).

- *local equilibrium distributions*, hereafter denoted as  $\bar{f}^{(0)}(t, \vec{r}, \vec{p})$ , cancel the collision term of the Boltzmann equation—and thereby obey condition (VI.32)—, yet are in general not solutions to the whole equation itself.

In practice, an out-of-equilibrium system first tends, under the influence of collisions alone, towards a state of local equilibrium. In a second step, the interplay of collisions and the drift terms drives the system towards global equilibrium.

#### Remarks:

\* If the spatial volume occupied by the system is unbounded, then  $H$  resp.  $S_B$  is not necessarily bounded from below resp. above, and thus a state of global equilibrium may never be reached. This

is for instance the case for a collection of particles expanding initially confined into a small region of space left free to expand into the vacuum.

\* Even if the system is confined in a finite volume, it may still be “stuck” in a persistent non-equilibrium setup. An example is that of a gas performing undamped “monopole” oscillations in a spherically symmetric harmonic trap, as studied theoretically e.g. in Ref. [57] and realized experimentally (to a good approximation) with cold  $^{87}\text{Rb}$  atoms [58].

### VI.4.1 a Global equilibrium

For simplicity, we begin with the case when the system is homogeneous and there is no external force. Consistency implies that the distribution function  $\bar{f}$  is independent of  $\vec{r}$ . This in particular holds for the equilibrium distribution,  $\bar{f}_{\text{eq.}}(\vec{p})$ , and the corresponding particle-number density  $n$ .

Inspecting the derivative (VI.30), one sees that the equilibrium condition  $dH(t)/dt = 0$  can only be fulfilled if the always-negative integrand is actually vanishing itself, which implies the identity

$$\bar{f}_{\text{eq.}}(\vec{p}_1)\bar{f}_{\text{eq.}}(\vec{p}_2) = \bar{f}_{\text{eq.}}(\vec{p}_3)\bar{f}_{\text{eq.}}(\vec{p}_4),$$

for every quadruplet  $(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)$  satisfying the kinetic-energy- and momentum-conservation relationships (VI.3) and allowed by the requirements on the system, in case the latter has a finite energy. From now on we assume that  $\bar{f}_{\text{eq.}}$  is everywhere non-zero.<sup>(91)</sup> Taking the logarithm of the above relation gives

$$\ln \bar{f}_{\text{eq.}}(\vec{p}_1) + \ln \bar{f}_{\text{eq.}}(\vec{p}_2) = \ln \bar{f}_{\text{eq.}}(\vec{p}_3) + \ln \bar{f}_{\text{eq.}}(\vec{p}_4),$$

which expresses that  $\ln \bar{f}_{\text{eq.}}(\vec{p})$  is a quantity conserved in any elastic two-to-two collision, and is thus a linear combination of additive collision invariants  $\chi(\vec{p})$ . Assuming that particle number—amounting to  $\chi(\vec{p}) = 1$ —, momentum [ $\chi(\vec{p}) = \vec{p}$ ] and kinetic energy [ $\chi(\vec{p}) = \vec{p}^2/2m$ ] form a basis of the latter,<sup>(92)</sup> one can write  $\ln \bar{f}_{\text{eq.}}(\vec{p}) = A' + \vec{B}' \cdot \vec{p} + C'\vec{p}^2$ , with constant  $A'$ ,  $\vec{B}'$ ,  $C'$ . Equivalently, one has (with new, related constants  $A$ ,  $C$  and  $\vec{p}_0$ )

$$\bar{f}_{\text{eq.}}(\vec{p}) = C e^{-A(\vec{p}-\vec{p}_0)^2}. \quad (\text{VI.34})$$

Following relation (VI.2), the momentum-space integral of  $\bar{f}_{\text{eq.}}(\vec{p})$  should yield the particle number density  $n$ , which necessitates  $C = (4\pi\hbar^2 A)^{3/2} n$ . The integral of the product  $\bar{f}_{\text{eq.}}(\vec{p}) \vec{p}$  with  $\bar{f}_{\text{eq.}}$  of the form (VI.34) over the whole phase space then gives  $\vec{p}_0$ , which should equal the total momentum of the particles; the latter vanishes if the particles are studied in their global rest frame, yielding  $\vec{p}_0 = \vec{0}$ . Eventually, the momentum-space integral of  $\bar{f}_{\text{eq.}}(\vec{p}) \vec{p}^2/2m$  should give  $n$  times the average kinetic energy per particle, while Eq. (VI.34), together with the relation between  $C$  and  $A$  found above, leads to  $3n/4mA$ . Denoting the average kinetic energy per particle by  $\frac{3}{2}k_B T$ , one thus finds  $A = 1/2mk_B T$ . All in all, one obtains

$$\bar{f}_{\text{eq.}}(\vec{p}) = n \left( \frac{2\pi\hbar^2}{mk_B T} \right)^{3/2} e^{-\vec{p}^2/2mk_B T}, \quad (\text{VI.35})$$

which is the Maxwell–Boltzmann distribution.<sup>(93)</sup>

One easily checks that the latter actually also cancels the left-hand side of the Boltzmann equation in the absence of external force, i.e. it represents the global equilibrium distribution.

<sup>(91)</sup>The identically vanishing distribution  $\bar{f} \equiv 0$  is clearly stationary and does fulfill condition (VI.32), yet it of course leads to a vanishing particle number, i.e. it represents a very boring system!

<sup>(92)</sup>A proof was given by Grad [59], which is partly reproduced by Sommerfeld [60] § 42]. An alternative argument is that if there were yet another kinematic collisional invariant in two-to-two scatterings, it would yield a fifth condition on the outgoing momenta—the requirement of having two particles in the final state leaves the constraints from energy and momentum conservation—, and thus restrict their angles, which is not the case.

<sup>(93)</sup>possibly up to the normalization factor, since in these lecture notes  $\bar{f}$  is dimensionless and normalized to the total number of particles, while the “usual” Maxwell–Boltzmann distribution is rather a probability distribution.

**Remark:** The Maxwell–Boltzmann distribution (VI.35) was probably known to the reader as the single-particle phase-space distribution of a classical ideal monoatomic gas at thermodynamic equilibrium studied in the canonical ensemble. Here it emerges as the (almost) universal long-time limit of the out-of-equilibrium evolution of a Boltzmann gas, in which the average energy per particle was “arbitrarily” denoted by  $\frac{3}{2}k_B T$ , where  $T$  has no other interpretation. This constitutes a totally different vision, yet the fact that both approaches yield the same distribution hints at the consistency of their results.

In case the system is subject to an external force deriving from a time-independent scalar potential,  $\vec{F} = -\vec{\nabla}V(\vec{r})$ , the stationary equilibrium solution of the Boltzmann equation is no longer spatially homogeneous, but becomes

$$\bar{f}_{\text{eq}}(\vec{r}, \vec{p}) = n(\vec{r}) \left( \frac{2\pi\hbar^2}{mk_B T} \right)^{3/2} e^{-\vec{p}^2/2mk_B T} \quad \text{with} \quad n(\vec{r}) \equiv \frac{N e^{-V(\vec{r})/k_B T}}{\int e^{-V(\vec{r})/k_B T} d^3\vec{r}}. \quad (\text{VI.36})$$

Note that the temperature is position-independent.

### Boltzmann entropy of the global equilibrium distribution

Inserting the equilibrium distribution (VI.36) into the Boltzmann entropy (VI.24), one finds with the help of Eq. (A.1c)

$$S_B = Nk_B \ln \left[ \frac{\mathcal{V}}{N} \left( \frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} \right] + \frac{5}{2} Nk_B, \quad (\text{VI.37})$$

where  $\mathcal{V}$  is the volume occupied by the particles. This result coincides with the *Sackur–Tetrode* (cf) formula [61, 62] for the thermodynamic entropy of a classical ideal gas under the same conditions. This shows that at equilibrium the Boltzmann entropy, which is also defined out of equilibrium, coincides with the thermodynamic entropy.

#### VI.4.1 b Local equilibrium

A non-equilibrated macroscopic system of weakly-interacting classical particles will not reach the global equilibrium distribution (VI.35) at once, the approach to equilibrium can be decomposed in “successive” (not in a strict chronological sense, but rather logically) steps involving various time scales [cf. Eqs. (V.16), (V.17)]:

- i. Over the shortest time scale  $\tau_c$ , neighbouring particles scatter on each other. These collisions—which strictly speaking are *not* described by the Boltzmann equation—lead to the emergence of molecular chaos.
- ii. Once molecular chaos holds, scatterings tend to drive the system to a state of local equilibrium, described by local thermodynamic variables as defined in chapter II: particle number density  $n(t, \vec{r})$ , temperature  $T(t, \vec{r})$  and average particle velocity  $\vec{v}(t, \vec{r})$ . That is, the single-particle distribution  $\bar{f}$  relaxes to a *local equilibrium distribution*, of the form

$$\bar{f}^{(0)}(t, \vec{r}, \vec{p}) = n(t, \vec{r}) \left[ \frac{2\pi\hbar^2}{mk_B T(t, \vec{r})} \right]^{3/2} \exp \left\{ -\frac{[\vec{p} - m\vec{v}(t, \vec{r})]^2}{2mk_B T(t, \vec{r})} \right\}, \quad (\text{VI.38})$$

which cancels the collision integral on the right-hand side of the Boltzmann equation. This relaxation takes place over a time scale  $\tau_r$ , which physically should be (much) larger than  $\tau_c$  since it involves “many” local scatterings.

(ce) H. M. SACKUR, 1895–1931 (cf) O. TETRODE, 1880–1914

- iii. Eventually, the state of local equilibrium relaxes to that of global equilibrium over the time scales  $\tau_s, \tau_e$  defined in Eq. (V.16). This step in the evolution of the single-particle distribution  $\bar{f}$ , which is now rather driven by the slow drift terms on the left-hand side of the Boltzmann equation, is more conveniently described in terms of the evolution of local thermodynamic quantities obeying macroscopic equations, in particular those of hydrodynamics, as will be described in Sec. VI.6.

**Remark:** In general, the distribution (VI.38) does not cancel the left-hand side of the Boltzmann equation and is thus *not* a solution of the equation. This is however not a problem, since the single-particle distribution  $\bar{f}$  never really equals a given  $\bar{f}^{(0)}$ , it only tends momentarily towards it: the three steps which above have been described as successive actually take place simultaneously.

### VI.4.1 c Equilibrium distributions for bosons and fermions

The kinetic equations with a collision integral which simulates either the Pauli principle of particles with half-integer spin [Eq. (VI.16)] or the “gregariousness” of particles with integer spin [Eq. (VI.17)] also obey an  $H$ -theorem, and thus tend at large time to respective stationary equilibrium distributions  $\bar{f}_{\text{eq.}}^{\text{F}}$  and  $\bar{f}_{\text{eq.}}^{\text{B}}$ .

Repeating a reasoning analogous to that in paragraph VI.4.1 a, with  $\bar{f}_{\text{eq.}}$  replaced by  $\bar{f}_{\text{eq.}}^{\text{F}}/(1-\bar{f}_{\text{eq.}}^{\text{F}})$  or  $\bar{f}_{\text{eq.}}^{\text{B}}/(1+\bar{f}_{\text{eq.}}^{\text{B}})$ , one finds after some straightforward algebra

$$\bar{f}_{\text{eq.}}^{\text{F}}(\vec{p}) = \frac{1}{C^{\text{F}} \exp(\vec{p}^2/2mk_{\text{B}}T) + 1} \quad (\text{VI.39a})$$

and

$$\bar{f}_{\text{eq.}}^{\text{B}}(\vec{p}) = \frac{1}{C^{\text{B}} \exp(\vec{p}^2/2mk_{\text{B}}T) - 1}, \quad (\text{VI.39b})$$

with  $C^{\text{F}}, C^{\text{B}}$  two constants ensuring the proper normalization of the distributions. These are respectively the Fermi–Dirac and Bose–Einstein distributions as found within equilibrium statistical mechanics for ideal quantum gases.

## VI.4.2 Approximate solutions

The Boltzmann equation (VI.15) identifies the rate of change of the single-particle distribution  $\bar{f}(t, \vec{r}, \vec{p})$  to a collision integral, which accounts for microscopic two-body elastic collisions. It is a non-linear partial integro-differential equation, and as thus especially difficult to solve. Accordingly, few exact solutions—apart from the equilibrium Maxwell–Boltzmann distribution (VI.35)–(VI.36)—are known analytically. Instead, one has to resort to approximate solutions, based on various schemes, two of which are shortly presented here.

### VI.4.2 a Orders of magnitude

As already mentioned in § V.2.2 b–V.2.2 c, the various terms of the Boltzmann equation a priori involve different time or length scales. Introducing the typical size  $\bar{f}_c$  of  $\bar{f}$  (at a given momentum), one may introduce characteristic times  $\tau_{\text{evol.}}, \tau_s, \tau_e$ , and  $\tau_{\text{coll.}}$  such that the dimensionless terms

$$\frac{\tau_{\text{evol.}}}{\bar{f}_c} \frac{\partial \bar{f}}{\partial t}, \quad \frac{\tau_s}{\bar{f}_c} \vec{v} \cdot \vec{\nabla}_{\vec{r}} \bar{f}, \quad \frac{\tau_e}{\bar{f}_c} \vec{F} \cdot \vec{\nabla}_{\vec{p}} \bar{f}, \quad \frac{\tau_{\text{coll.}}}{\bar{f}_c} \left( \frac{\partial \bar{f}}{\partial t} \right)_{\text{coll.}}$$

are all of order unity.  $\tau_e$ —imposed by the external force—is generally of the order of  $\tau_s$  or larger, and will no longer be considered in the following discussion.

- By construction,  $\tau_{\text{evol.}}$  is of the order of the inverse rate of change of  $\bar{f}/\bar{f}_c$ .

- $\tau_s$  is of the order of the characteristic length  $L$  over which the macroscopic system properties vary, divided by the typical particle velocity  $v_c$  in the system:

$$\tau_s \sim \frac{L}{v_c}. \quad (\text{VI.40})$$

- Starting from expression (VI.13b) of the collision integral in terms of the differential cross section, one finds that the collision term is of order

$$\left(\frac{\partial \bar{f}}{\partial t}\right)_{\text{coll}} \sim n \bar{f}_c v_c \sigma_{\text{tot}}, \quad (\text{VI.41})$$

where the number density  $n$  comes from integrating  $\bar{f}$  over momentum, while the total cross section  $\sigma_{\text{tot}}$  results from the integration of the differential cross section over all possible directions. Accordingly, one finds

$$\tau_{\text{coll}} \sim \frac{1}{n v_c \sigma_{\text{tot}}}. \quad (\text{VI.42a})$$

As could have been argued on physical grounds, this time scale associated with the collision term is of the same order as the mean free time between two successive scatterings of a given particle. In terms of the mean free path  $\ell_{\text{mfp}} \sim 1/n\sigma_{\text{tot}}$ , one may write

$$\tau_{\text{coll}} \sim \frac{\ell_{\text{mfp}}}{v_c} \quad (\text{VI.42b})$$

With the three above time scales one may construct two independent dimensionless numbers, namely on the one hand the *Knudsen* <sup>(cg)</sup> number  $\text{Kn}$

$$\text{Kn} \equiv \frac{\ell_{\text{mfp}}}{L} \quad (\text{VI.43})$$

or equivalently  $\text{Kn} \sim \tau_{\text{coll}}/\tau_s$ , which compares the collision integral and the drift term; and on the other hand the ratio

$$\frac{\tau_{\text{coll}}}{\tau_{\text{evol}}} \sim \frac{\ell_{\text{mfp}}}{v_c \tau_{\text{evol}}} = \frac{L}{v_c \tau_{\text{evol}}} \text{Kn},$$

in which the prefactor multiplying the Knudsen number is sometimes referred to as *Strouhal* <sup>(ch)</sup> number.

The two dimensionless parameters can take a priori (almost) any value. A small Knudsen number corresponds to a rather dense system, in which collision occur relatively often; while a large value of  $\text{Kn}$  corresponds to the free-streaming limit. In turn, small values of  $\tau_{\text{coll}}/\tau_{\text{evol}}$  correspond to slow evolutions of  $\bar{f}$ —thus  $\tau_{\text{evol}} \rightarrow \infty$ , yielding  $\tau_{\text{coll}}/\tau_{\text{evol}} \rightarrow 0$ , in stationary systems. On the other hand, on physical grounds  $\tau_{\text{coll}}/\tau_{\text{evol}}$  cannot be much larger than  $\text{Kn}$ , since this would imply the existence of an emergent “macroscopic” velocity scale  $L/\tau_{\text{evol}}$  much larger than the characteristic velocity  $v_c$  of the particles, which is hard to conceive.

Taking for simplicity  $\tau_{\text{coll}}/\tau_{\text{evol}}$  of order  $\text{Kn}$ , so has to have a single dimensionless quantity in the system, approximate solutions can be found in a systematic way when the Knudsen number is either very small,  $\text{Kn} \ll 1$ , or very large ( $\text{Kn} \gg 1$ ), since in either case there is a small parameter in the problem—either  $\text{Kn}$  or  $\text{Kn}^{-1}$ —, suggesting the use of perturbative expansions in that parameter.

**Remark:** The case of very small Knudsen numbers has to be taken with a grain of salt: to remain within the region of validity of the Boltzmann equation as a physically motivated model, the mean free path cannot become infinitely small, but has to remain (much) larger than the interaction range (§ VI.1.1).  $\text{Kn}$  can thus only be very small by letting the system size become large—which is perfectly acceptable.

<sup>(cg)</sup>M. KNUDSEN, 1871–1949    <sup>(ch)</sup>V. STROUHAL, 1850–1922

### VI.4.2b Hilbert expansion

Let us discuss a first instance of perturbative expansion for the solution of the Boltzmann equation, based on the assumption that the Knudsen number (VI.43) is small—and that the ratio  $\tau_{\text{coll.}}/\tau_{\text{evol.}}$  is of the same order of magnitude.

Throughout this paragraph, we shall omit the  $(t, \vec{r}, \vec{p})$  variables. In addition, we introduce two new notations: the collision integral on the right hand side of the Boltzmann equation will be denoted by

$$\mathcal{C}_{\text{coll.}}[\bar{f}, \bar{f}] \equiv \left( \frac{\partial \bar{f}(1)}{\partial t} \right)_{\text{coll.}} = \int_{\vec{p}_2} \int_{\vec{p}_3} \int_{\vec{p}_4} [\bar{f}(3)\bar{f}(4) - \bar{f}(1)\bar{f}(2)] \tilde{w}(\vec{p}_1, \vec{p}_2 \rightarrow \vec{p}_3, \vec{p}_4). \quad (\text{VI.44a})$$

Besides, we introduce the bilinear operator

$$\mathcal{C}_{\text{coll.}}[\bar{f}, \bar{g}] \equiv \frac{1}{2} \int_{\vec{p}_2} \int_{\vec{p}_3} \int_{\vec{p}_4} [\bar{f}(3)\bar{g}(4) + \bar{g}(3)\bar{f}(4) - \bar{f}(1)\bar{g}(2) - \bar{g}(1)\bar{f}(2)] \tilde{w}(\vec{p}_1, \vec{p}_2 \rightarrow \vec{p}_3, \vec{p}_4). \quad (\text{VI.44b})$$

The functional space on which  $\mathcal{C}_{\text{coll.}}$  is operating will be specified below.

To account for the physical interpretation according to which the left hand side of the Boltzmann equation is governed by a larger time scale than the right hand side—which corresponds to having a small Knudsen number—, we rewrite the equation with a dimensionless parameter  $\varepsilon$ , which will be treated as much smaller than 1, in front of the former:

$$\varepsilon \left( \frac{\partial \bar{f}}{\partial t} + \vec{v} \cdot \vec{\nabla}_{\vec{r}} \bar{f} + \vec{F} \cdot \vec{\nabla}_{\vec{p}} \bar{f} \right) = \mathcal{C}_{\text{coll.}}[\bar{f}, \bar{f}]. \quad (\text{VI.45a})$$

In turn, the phase space distribution  $\bar{f}$  solution to this equation is written as an expansion in powers of  $\varepsilon$ :

$$\bar{f} = \sum_{n=0}^{\infty} \varepsilon^n \bar{f}^{(n)}. \quad (\text{VI.45b})$$

The reader should view this expansion as a formal one. The underlying idea is clearly that by knowing enough of the first terms, one can approximate the true solution (for a given problem, e.g. with specified initial and boundary conditions) with a high precision. Yet the mathematical problem of the convergence of the expansion—for which values of  $\varepsilon$ ? in which phase-space region? what type of convergence?—depends on the inter-particle interactions, and is beyond the scope of these notes.

With the ansatz (VI.45b), known as the *Hilbert expansion*, the collision term (VI.44a) reads

$$\mathcal{C}_{\text{coll.}}[\bar{f}, \bar{f}] = \sum_{n=0}^{\infty} \varepsilon^n \mathcal{C}^{(n)} \quad \text{with} \quad \mathcal{C}^{(n)} \equiv \sum_{k=0}^n \mathcal{C}_{\text{coll.}}[\bar{f}^{(k)}, \bar{f}^{(n-k)}], \quad (\text{VI.45c})$$

where the expression of  $\mathcal{C}^{(n)}$  follows from some straightforward algebra. If the “leading term”  $\bar{f}^{(0)}$  is an equilibrium distribution, global or local, it cancels the collision integral, yielding

$$\mathcal{C}^{(0)} = \mathcal{C}_{\text{coll.}}[\bar{f}^{(0)}, \bar{f}^{(0)}] = 0. \quad (\text{VI.45d})$$

In turn, the higher-order  $\mathcal{C}^{(n)}$  are conveniently rewritten as

$$\mathcal{C}^{(n)} = 2 \mathcal{C}_{\text{coll.}}[\bar{f}^{(0)}, \bar{f}^{(n)}] + \sum_{k=1}^{n-1} \mathcal{C}_{\text{coll.}}[\bar{f}^{(k)}, \bar{f}^{(n-k)}] \quad \text{for } n \geq 1. \quad (\text{VI.45e})$$

Inserting now the Hilbert expansion (VI.45b) and the collision term (VI.45c) in the Boltzmann equation leads at once to

$$\sum_{n=0}^{\infty} \varepsilon^{n+1} \left( \frac{\partial \bar{f}^{(n)}}{\partial t} + \vec{v} \cdot \vec{\nabla}_{\vec{r}} \bar{f}^{(n)} + \vec{F} \cdot \vec{\nabla}_{\vec{p}} \bar{f}^{(n)} \right) = \sum_{n=0}^{\infty} \varepsilon^n \mathcal{C}^{(n)},$$

i.e. after identifying the factors multiplying the term  $\epsilon^n$

$$\frac{\partial \bar{f}^{(n-1)}}{\partial t} + \vec{v} \cdot \vec{\nabla}_{\vec{r}} \bar{f}^{(n-1)} + \vec{F} \cdot \vec{\nabla}_{\vec{p}} \bar{f}^{(n-1)} = \mathcal{C}^{(n)} \quad \text{for } n \geq 1. \quad (\text{VI.46})$$

Invoking relation (VI.45e) to rewrite the right member of this equation and reorganizing the terms, one eventually obtains

$$\mathcal{C}_{\text{coll.}}[\bar{f}^{(0)}, \bar{f}^{(n)}] = \frac{1}{2} \left( \frac{\partial \bar{f}^{(n-1)}}{\partial t} + \vec{v} \cdot \vec{\nabla}_{\vec{r}} \bar{f}^{(n-1)} + \vec{F} \cdot \vec{\nabla}_{\vec{p}} \bar{f}^{(n-1)} - \sum_{k=1}^{n-1} \mathcal{C}_{\text{coll.}}[\bar{f}^{(k)}, \bar{f}^{(n-k)}] \right) \quad \text{for } n \geq 1. \quad (\text{VI.47})$$

The right member only involves the functions  $\bar{f}^{(k)}$  with  $k < n$ , and accordingly is entirely determined when these functions are known. In turn, the term on the left hand side of Eq. (VI.47) is at fixed  $\bar{f}^{(0)}$  a linear functional in  $\bar{f}^{(n)}$ . Inverting the corresponding operator thus allows one in principle to obtain  $\bar{f}^{(n)}$  building on the previous determination of the functions  $\{\bar{f}^{(k)}\}$  with  $k < n$ , thereby yielding a systematic, sequential method.

More precisely, it is customary to rewrite the successive unknown function as  $\bar{f}^{(n)} = \bar{f}^{(0)} h^{(n)}$ . Inserting this form in the operator (VI.44b) then yields

$$\mathcal{C}_{\text{coll.}}[\bar{f}^{(0)}, \bar{f}^{(0)} h^{(n)}] = \frac{\bar{f}^{(0)}(1)}{2} \int_{\vec{p}_2} \int_{\vec{p}_3} \int_{\vec{p}_4} \bar{f}^{(0)}(2) [h^{(n)}(3) + h^{(n)}(4) - h^{(n)}(1) - h^{(n)}(2)] \tilde{\omega}(\vec{p}_1, \vec{p}_2 \rightarrow \vec{p}_3, \vec{p}_4),$$

where the identity  $\bar{f}^{(0)}(1) \bar{f}^{(0)}(2) = \bar{f}^{(0)}(3) \bar{f}^{(0)}(4)$  has been used. Since both sides of Eq. (VI.47) can without difficulty be divided by the factor  $\bar{f}^{(0)}(1)$ , the interest now lies in the linearized collision operator

$$\mathcal{C}_{\text{coll.}}^{\text{lin.}}[h] \equiv \frac{1}{2} \int_{\vec{p}_2} \int_{\vec{p}_3} \int_{\vec{p}_4} \bar{f}^{(0)}(2) [h(3) + h(4) - h(1) - h(2)] \tilde{\omega}(\vec{p}_1, \vec{p}_2 \rightarrow \vec{p}_3, \vec{p}_4), \quad (\text{VI.48})$$

where the functions  $h$  are assumed to be sufficiently regular to ensure the existence of the integral. By introducing an inner product

$$(h_1, h_2) \equiv \int_{\vec{p}} \bar{f}^{(0)} h_1 h_2,$$

the space of functions  $h$  becomes a Hilbert space  $\mathcal{H}$ . This simplifies the discussion of the properties of the operator  $\mathcal{C}_{\text{coll.}}^{\text{lin.}}$ , which is easily shown to be self-adjoint.

A problem with inverting the linearized collision operator so as to determine  $\bar{f}^{(n)}$ , or equivalently  $h^{(n)}$ , from the  $\{\bar{f}^{(k)}\}$  (or  $\{h^{(k)}\}$ ) with  $k < n$  is that 0 is actually a fivefold degenerate eigenvalue—the corresponding eigenfunctions being the collisional invariant functions  $\chi = 1, \vec{p}$  and  $\vec{p}^2/2m$ . The solution is to invert  $\mathcal{C}_{\text{coll.}}^{\text{lin.}}$  in the subspace of  $\mathcal{H}$  orthogonal to that spanned by these eigenfunctions, which however means that each  $h^{(n)}$  is only known up to a linear combination, with coefficients depending possibly on time and position, of the collisional invariants. The coefficients entering the expression of  $h^{(n)}$  are actually fixed at the following step, when requiring that the term on the right hand side of Eq. (VI.47) for the determination of  $\bar{f}^{(n+1)}$  is in the proper space.

### VI.4.2c Orthogonal polynomial solutions

will be added later.

### VI.4.3 Relaxation-time approximation

It may happen that the inter-particle interactions entering the collision integral of the Boltzmann equation are only imperfectly known, so that the approximation methods presented in the previous section [VI.4.2](#) cannot be used. In order to still be able to derive qualitative behaviors in that situation, it is customary to perform a rather drastic approximation of the collision term, based on the physical significance of the latter, whose role is to let the distribution  $\bar{f}(t, \vec{r}, \vec{p})$  relax to a local equilibrium distribution  $\bar{f}^{(0)}$ , given by expression [\(VI.38\)](#), with local density, temperature and average particle velocity which are specific to the conditions imposed on the system.

In the so-called *relaxation-time approximation*, it is assumed that the approach to the local equilibrium distribution is exponential, with a characteristic time scale  $\tau_r(\vec{r}, \vec{p})$ . In that case, the Boltzmann equation is approximated by the linear partial differential equation

$$\frac{\partial \bar{f}(t, \vec{r}, \vec{p})}{\partial t} + \vec{v} \cdot \vec{\nabla}_{\vec{r}} \bar{f}(t, \vec{r}, \vec{p}) + \vec{F} \cdot \vec{\nabla}_{\vec{p}} \bar{f}(t, \vec{r}, \vec{p}) = -\frac{\bar{f}(t, \vec{r}, \vec{p}) - \bar{f}^{(0)}(t, \vec{r}, \vec{p})}{\tau_r(\vec{r}, \vec{p})}. \quad (\text{VI.49})$$

This approximation thus leaves the integral aspect of the collision term, amounting to a “first linearization” of the Boltzmann equation.

#### Remarks:

- \* The *relaxation time*  $\tau_r$  models the typical duration for reaching local equilibrium (see §[VI.4.1 b](#)). In Eq. [\(VI.49\)](#) it is introduced as a free parameter. It should nevertheless naturally be larger than the *mean free time*, i.e. the average “free-flight” time between two successive scatterings of a given particle, since one expects that equilibration requires several collisions. On the other hand, it should remain (much) smaller than “external” time scales imposed by macroscopic conditions on the systems.

- \*  $\tau_r$  is often assumed to be uniform over the whole system—which is to a large extent justified as long as the spatial density does not vary too much—, yet still momentum dependent.

- \* When  $\tau_r$  is also assumed to be independent of  $\vec{p}$ —which is far less obvious—, the approximation is sometimes referred to as the Bhatnagar<sup>(ci)</sup>–Gross<sup>(cj)</sup>–Krook<sup>(ck)</sup> (BGK) approximation [\[63\]](#).

- \* The approximation [\(VI.49\)](#) may be seen as a truncation prescription of the BBGKY hierarchy [\(V.14\)](#), irrespective of the Boltzmann equation.

If the departure from equilibrium remains always small, then the true solution  $\bar{f}(t, \vec{r}, \vec{p})$  to the Boltzmann equation never deviates much from a local equilibrium distribution  $\bar{f}^{(0)}(t, \vec{r}, \vec{p})$ . One may then write

$$\bar{f}(t, \vec{r}, \vec{p}) = \bar{f}^{(0)}(t, \vec{r}, \vec{p}) + \bar{f}^{(1)}(t, \vec{r}, \vec{p}) \quad \text{with} \quad |\bar{f}^{(1)}(t, \vec{r}, \vec{p})| \ll \bar{f}^{(0)}(t, \vec{r}, \vec{p}) \quad \forall t, \vec{r}, \vec{p}. \quad (\text{VI.50})$$

The right-hand side of the simplified equation [\(VI.49\)](#) is then simply  $-\bar{f}^{(1)}(t, \vec{r}, \vec{p})/\tau_r(\vec{p})$ , while on the left-hand side one will account for the assumption  $|\bar{f}^{(1)}| \ll \bar{f}^{(0)}$  by linearizing the various terms with respect to the small perturbation  $\bar{f}^{(1)}$ . This constitutes the “second linearization” of the Boltzmann equation, and will be illustrated on a few examples in the following section.

<sup>(ci)</sup>P. L. BHATNAGAR, 1912–1976    <sup>(cj)</sup>E. P. GROSS, 1926–1991    <sup>(ck)</sup>M. KROOK, 1913–1985