

III.3.3 Newtonian fluid: Navier–Stokes equation

In a real moving fluid, there are *friction forces* that contribute to the transport of momentum between neighboring fluid layers when the latter are in relative motion. Accordingly, the momentum flux-density tensor is no longer given by Eq. (III.21b) or (III.22), but now contains extra terms, involving derivatives of the flow velocity. Accordingly, the Euler equation must be replaced by an alternative dynamical equation, including the friction forces.

III.3.3 a Momentum flux density in a Newtonian fluid

The momentum flux density (III.21b) in a perfect fluid only contains two terms—one proportional to the components g^{ij} of the inverse metric tensor, the other proportional to $v^i(t, \vec{r}) v^j(t, \vec{r})$. Since the coefficients in front of these two terms could a priori depend on \vec{v}^2 , this represents the most general symmetric tensor of degree 2 which can be constructed with the help of the flow velocity only.

If the use of terms that depend on the spatial derivatives of the velocity field is also allowed, the components of the momentum flux-density tensor can be of the following form, where for the sake of brevity the variables t and \vec{r} are omitted

$$\mathbf{T}^{ij} = \mathcal{P}g^{ij} + \rho v^i v^j + A \frac{\partial v^i}{\partial x_j} + B \frac{\partial v^j}{\partial x_i} + \mathcal{O}\left(\frac{\partial^2 v_i}{\partial x_j \partial x_k}\right) + \dots, \quad (\text{III.25})$$

with A, B coefficients that depend on i, j and on the fluid under consideration.

As discussed in Sec. I.1.3, the description of a system of particles as a continuous medium, and in particular as a fluid, in local thermodynamic equilibrium, rests on the assumption that the macroscopic quantities of relevance for the medium vary slowly both in space and time. Accordingly, (spatial) gradients should be small: the third and fourth terms in Eq. (III.25) should thus be on the one hand much smaller than the first two ones, on the other hand much larger than the rightmost term as well as those involving higher-order derivatives or of powers of the first derivatives. Neglecting these smaller terms, one obtains “first-order dissipative fluid dynamics”, which describes the motion of Newtonian fluids—this actually defines the latter.

Using the necessary symmetry of \mathbf{T}^{ij} , the third and fourth terms in Eq. (III.25) can be rewritten as the sum of a traceless symmetric contribution and a tensor proportional to the inverse metric tensor. This leads to the momentum flux-density tensor

$$\begin{aligned} \mathbf{T}^{ij}(t, \vec{r}) = & \mathcal{P}(t, \vec{r}) g^{ij}(t, \vec{r}) + \rho(t, \vec{r}) v^i(t, \vec{r}) v^j(t, \vec{r}) \\ & - \eta(t, \vec{r}) \left[\frac{\partial v^i(t, \vec{r})}{\partial x_j} + \frac{\partial v^j(t, \vec{r})}{\partial x_i} - \frac{2}{3} g^{ij}(t, \vec{r}) \vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right] \\ & - \zeta(t, \vec{r}) g^{ij}(t, \vec{r}) \vec{\nabla} \cdot \vec{v}(t, \vec{r}). \end{aligned} \quad (\text{III.26a})$$

In geometric formulation, this reads

$$\mathbf{T}(t, \vec{r}) = \mathcal{P}(t, \vec{r}) \mathbf{g}^{-1}(t, \vec{r}) + \rho(t, \vec{r}) \vec{v}(t, \vec{r}) \otimes \vec{v}(t, \vec{r}) + \boldsymbol{\pi}(t, \vec{r}) \quad (\text{III.26b})$$

where dissipative effects are encoded in the *viscous stress tensor*^(xxxvi)

$$\boldsymbol{\pi}(t, \vec{r}) \equiv -2\eta(t, \vec{r}) \left[\mathbf{D}(t, \vec{r}) - \frac{1}{3} [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] \mathbf{g}^{-1}(t, \vec{r}) \right] - \zeta(t, \vec{r}) [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] \mathbf{g}^{-1}(t, \vec{r}) \quad (\text{III.26c})$$

for a Newtonian fluid

with $\mathbf{D}(t, \vec{r})$ the strain rate tensor discussed in Sec. II.1.3. Component-wise

$$\pi^{ij}(t, \vec{r}) \equiv -2\eta(t, \vec{r}) \left[\mathbf{D}^{ij}(t, \vec{r}) - \frac{1}{3} [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] g^{ij}(t, \vec{r}) \right] - \zeta(t, \vec{r}) [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] g^{ij}(t, \vec{r}). \quad (\text{III.26d})$$

^(xxxvi) *viskoser Spannungstensor*

This viscous stress tensor involves two novel characteristics of the medium, so-called *transport coefficients*, namely:

- the (dynamical) *shear viscosity*^(xxxvii) η , which multiplies the traceless symmetric part of the velocity gradient tensor;
- the *bulk viscosity*, also called *second viscosity*,^(xxxviii) ζ , which multiplies the volume-expansion part of the velocity gradient tensor, i.e. the term proportional to $\vec{\nabla} \cdot \vec{v}(t, \vec{r})$.

The two corresponding contributions represent a *diffusive* transport of momentum in the fluid—representing a third type of transport besides the convective and thermal ones.

Remarks:

* In the case of a Newtonian fluid, the viscosity coefficients η and ζ are independent of the flow velocity. However, they still depend on the temperature and pressure of the fluid, so that they are not necessarily uniform and constant in a real flowing fluid.

* In an incompressible flow, $\vec{\nabla} \cdot \vec{v}(t, \vec{r}) = 0$, the last contribution to the momentum flux density (III.26) drops out. Thus, the bulk viscosity ζ only plays a role in compressible fluid motions.⁽⁶⁾

* Expression (III.26c) or (III.26d) of the viscous stress tensor assumes implicitly that the fluid is isotropic, since the coefficients η , ζ are independent of the directions i , j .

III.3.3b Surface forces in a Newtonian fluid

The Cauchy stress tensor corresponding to the momentum flux density (III.26) of a Newtonian fluid is

$$\boldsymbol{\sigma}(t, \vec{r}) = -\mathcal{P}(t, \vec{r}) \mathbf{g}^{-1}(t, \vec{r}) - \boldsymbol{\pi}(t, \vec{r}) \quad (\text{III.27a})$$

that is, using the form (III.26c) of the viscous stress tensor

$$\boldsymbol{\sigma}(t, \vec{r}) = -\mathcal{P}(t, \vec{r}) \mathbf{g}^{-1}(t, \vec{r}) + 2\eta(t, \vec{r}) \left[\mathbf{D}(t, \vec{r}) - \frac{1}{3} [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] \mathbf{g}^{-1}(t, \vec{r}) \right] + \zeta(t, \vec{r}) [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] \mathbf{g}^{-1}(t, \vec{r}). \quad (\text{III.27b})$$

Component-wise, this becomes

$$\sigma^{ij}(t, \vec{r}) = \left\{ -\mathcal{P}(t, \vec{r}) + \left[\zeta(t, \vec{r}) - \frac{2}{3} \eta(t, \vec{r}) \right] \vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right\} g^{ij}(t, \vec{r}) + \eta(t, \vec{r}) \left[\frac{\partial v^i(t, \vec{r})}{\partial x_j} + \frac{\partial v^j(t, \vec{r})}{\partial x_i} \right]. \quad (\text{III.27c})$$

Accordingly, the mechanical stress vector on an infinitesimally small surface element situated at point \vec{r} with unit normal vector $\vec{e}_n(\vec{r})$ reads

$$\vec{T}_s(t, \vec{r}) = \boldsymbol{\sigma}(t, \vec{r}) \cdot \vec{e}_n(\vec{r}) = \sum_{i,j=1}^3 \left\{ \left[-\mathcal{P}(t, \vec{r}) + \left(\zeta(t, \vec{r}) - \frac{2}{3} \eta(t, \vec{r}) \right) \vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right] g_j^i(t, \vec{r}) + \eta(t, \vec{r}) \left(\frac{\partial v^i(t, \vec{r})}{\partial x_j} + \frac{\partial v_j(t, \vec{r})}{\partial x_i} \right) \right\} n^j(\vec{r}) \vec{e}_i, \quad (\text{III.28})$$

with $n^j(\vec{r})$ the coordinate of $\vec{e}_n(\vec{r})$ along direction j . One identifies two components

- the term proportional to $\sum g_j^i n^j \vec{e}_i = \vec{e}_n$ is the *normal stress* on the surface element. It consists of the usual hydrostatic pressure term $-\mathcal{P} \vec{e}_n$, and a second one proportional to the local expansion rate $\vec{\nabla} \cdot \vec{v}$: in the compressible motion of a Newtonian—and more generally a dissipative—fluid, the normal stress is thus not only given by $-\mathcal{P} \vec{e}_n$, but includes additional contributions that vanish in the static case.

⁽⁶⁾As a consequence, the bulk viscosity is often hard to measure—one has to devise a compressible flow—so that it is actually not so well known for many substances, even well-studied ones [11].

^(xxxvii) *Scherviskosität* ^(xxxviii) *Dehnviskosität, Volumenviskosität, zweite Viskosität*

- the remaining term is the *tangential stress*, proportional to the shear viscosity η . Accordingly, the value of the latter can be deduced from a measurement of the tangential force acting on a surface element, see Sec. ??.

As in § III.3.2 a, the external contact forces acting on a fluid element delimited by a surface \mathcal{S} can easily be computed. Invoking the Stokes theorem yields

$$\begin{aligned} \oint_{\mathcal{S}} \vec{T}_s(t, \vec{r}) \, d^2\mathcal{S} &= -\oint_{\mathcal{S}} \mathcal{P}(t, \vec{r}) \vec{e}_n(\vec{r}) \, d^2\mathcal{S} - \oint_{\mathcal{S}} \boldsymbol{\pi}(t, \vec{r}) \cdot \vec{e}_n(\vec{r}) \, d^2\mathcal{S} \\ &= -\int_{\mathcal{V}} \vec{\nabla} \mathcal{P}(t, \vec{r}) \, d^3\mathcal{V} + \int_{\mathcal{V}} \vec{\nabla} \cdot \boldsymbol{\pi}(t, \vec{r}) \, d^3\mathcal{V} \\ &= -\int_{\mathcal{V}} \vec{\nabla} \mathcal{P}(t, \vec{r}) \, d^3\mathcal{V} + \int_{\mathcal{V}} \vec{f}_{\text{visc}}(t, \vec{r}) \, d^3\mathcal{V}, \end{aligned} \quad (\text{III.29a})$$

with the local viscous friction force density

$$\vec{f}_{\text{visc}}(t, \vec{r}) = \sum_{i,j=1}^3 \frac{\partial}{\partial x^i} \left\{ \eta(t, \vec{r}) \left[\frac{\partial v^i(t, \vec{r})}{\partial x_j} + \frac{\partial v^j(t, \vec{r})}{\partial x_i} \right] \right\} \vec{e}_j + \vec{\nabla} \left\{ \left[\zeta(t, \vec{r}) - \frac{2}{3} \eta(t, \vec{r}) \right] \vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right\}. \quad (\text{III.29b})$$

III.3.3 c Navier–Stokes equation

Combining the viscous force (III.29b) with the generic equations (III.12), (III.14) and (III.15), the application of Newton's second law to a volume \mathcal{V} of fluid leads to an identity between sums of volume integrals. Since this relation holds for any volume \mathcal{V} , it translates into an identity between the integrands, namely

$$\rho(t, \vec{r}) \left\{ \frac{\partial \vec{v}(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}] \vec{v}(t, \vec{r}) \right\} = -\vec{\nabla} \mathcal{P}(t, \vec{r}) + \vec{f}_{\text{visc}}(t, \vec{r}) + \vec{f}_V(t, \vec{r}) \quad (\text{III.30a})$$

or component-wise

$$\begin{aligned} \rho(t, \vec{r}) \left\{ \frac{\partial v^i(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}] v^i(t, \vec{r}) \right\} &= -\frac{\partial \mathcal{P}(t, \vec{r})}{\partial x_i} + \frac{\partial}{\partial x_i} \left\{ \left[\zeta(t, \vec{r}) - \frac{2}{3} \eta(t, \vec{r}) \right] \vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right\} \\ &\quad + \sum_{j=1}^3 \frac{\partial}{\partial x^j} \left\{ \eta(t, \vec{r}) \left[\frac{\partial v^i(t, \vec{r})}{\partial x_j} + \frac{\partial v^j(t, \vec{r})}{\partial x_i} \right] \right\} + [\vec{f}_V(t, \vec{r})]^i \end{aligned} \quad (\text{III.30b})$$

for $i = 1, 2, 3$.

If the implicit dependence of the viscosity coefficients on time and position is negligible, one may pull η and ζ outside of the spatial derivatives. As a result, one obtains the (compressible) *Navier–Stokes equation*^{(g),(h)}

$$\rho(t, \vec{r}) \left\{ \frac{\partial \vec{v}(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}] \vec{v}(t, \vec{r}) \right\} = -\vec{\nabla} \mathcal{P}(t, \vec{r}) + \eta \Delta \vec{v}(t, \vec{r}) + \left(\zeta + \frac{\eta}{3} \right) \vec{\nabla} [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] + \vec{f}_V(t, \vec{r}) \quad (\text{III.31})$$

with $\Delta = \vec{\nabla}^2$ the Laplacian. This is a non-linear partial differential equation of second order, while the Euler equation (III.18) is of first order.

The difference between the order of the equations is not a mere detail: while the Euler equation looks like the limit $\eta, \zeta \rightarrow 0$ of the Navier–Stokes equation, the corresponding is not necessarily true of their solutions. This is for instance due to the fact that their respective boundary conditions differ.

^(g)C.-L. NAVIER, 1785–1836 ^(h)G. G. STOKES, 1819–1903

In the case of an incompressible flow, the local expansion rate in the Navier–Stokes equation (III.31) vanishes, leading to the *incompressible Navier–Stokes equation*

$$\frac{\partial \vec{v}(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}] \vec{v}(t, \vec{r}) = -\frac{1}{\rho} \vec{\nabla} \mathcal{P}(t, \vec{r}) + \nu \Delta \vec{v}(t, \vec{r}), \quad (\text{III.32})$$

with $\nu \equiv \eta/\rho$ the *kinematic shear viscosity*.

Remark: The dimension of the dynamic viscosity coefficients η, ζ is $\text{ML}^{-1}\text{T}^{-1}$ and the corresponding unit in the SI system is the Poiseuille⁽ⁱ⁾, abbreviated Pa·s. In contrast, the kinematic viscosity has dimension L^2T^{-1} , i.e. depends only on space and time, hence its denomination.

III.3.3 d Boundary conditions

At the interface between a viscous fluid, in particular a Newtonian one, and another body—be it an obstacle in the flow, a wall containing the fluid, or even a second viscous fluid which is immiscible with the first one—the relative velocity between the fluid and the body must vanish. This holds not only for the normal component of the velocity, as in perfect fluids, but also for the tangential one, to account for the friction forces.

⁽ⁱ⁾J.-L.-M. Poiseuille, 1797–1869

III.4.2 Energy conservation in Newtonian fluids

In a real fluid, there exist further mechanisms for transporting energy besides the convective transport due to the fluid motion, namely *diffusion*, either of momentum or of energy:

- On the one hand, the viscous friction forces in the fluid, which already lead to the transport of momentum between neighboring fluid layers moving with different velocities, exert some work in the motion, which induces a diffusive transport of energy. This is accounted for by a contribution $\boldsymbol{\pi} \cdot \vec{v}$ to the energy flux density—component-wise, a contribution $\sum_j \pi_j^i v^j$ to the i -th component of the flux density—, with $\boldsymbol{\pi}$ the viscous stress tensor, given in the case of a Newtonian fluid by Eq. (III.26c).
- On the other hand, there is also *heat conduction* from the regions with higher temperatures towards those with lower temperatures. This transport is described by the introduction in the energy flux density of a heat current^(xxxix) $\vec{j}_Q(t, \vec{r}) = -\kappa(t, \vec{r}) \vec{\nabla} T(t, \vec{r})$ —in accordance with the local formulation of *Fourier's law*,^(j) see e.g. Sec. 1.2.1 in Ref. [2]—, with κ the *heat conductivity*^(xl) of the fluid.

Taking into account these additional contributions, the local formulation of energy conservation in a Newtonian fluid in the absence of external volume forces reads

$$\begin{aligned}
 & \frac{\partial}{\partial t} \left[\frac{1}{2} \rho(t, \vec{r}) \vec{v}(t, \vec{r})^2 + e(t, \vec{r}) \right] \\
 & + \vec{\nabla} \cdot \left\{ \left[\frac{1}{2} \rho(t, \vec{r}) \vec{v}(t, \vec{r})^2 + e(t, \vec{r}) + \mathcal{P}(t, \vec{r}) \right] \vec{v}(t, \vec{r}) \right. \\
 & \quad \left. - \eta(t, \vec{r}) \left[\left(\vec{v}(t, \vec{r}) \cdot \vec{\nabla} \right) \vec{v}(t, \vec{r}) + \vec{\nabla} \left(\frac{\vec{v}(t, \vec{r})^2}{2} \right) \right] \right. \\
 & \quad \left. - \left[\zeta(t, \vec{r}) - \frac{2\eta(t, \vec{r})}{3} \right] \left[\vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right] \vec{v}(t, \vec{r}) - \kappa(t, \vec{r}) \vec{\nabla} T(t, \vec{r}) \right\} = 0.
 \end{aligned} \tag{III.35}$$

If the three transport coefficients η , ζ and κ vanish, this equation simplifies to that for perfect fluids, Eq. (III.33).

III.4.3 Entropy balance in Newtonian fluids

In a real fluid, with viscous friction forces and heat conductivity, one can expect a priori that the transformation of mechanical energy into heat will lead in general to an increase in entropy, provided a closed system is being considered.

^(xxxix) Wärmestromvektor ^(xl) Wärmeleitfähigkeit

^(j) J. B. FOURIER, 1768–1830

Consider a volume \mathcal{V} of flowing Newtonian fluid, delimited by a surface \mathcal{S} at each point \vec{r} of which the boundary conditions $\vec{v}(t, \vec{r}) \cdot \vec{e}_n(\vec{r}) = 0$ and $\vec{j}_Q(t, \vec{r}) \cdot \vec{e}_n(\vec{r}) = 0$ hold, where $\vec{e}_n(\vec{r})$ denotes the unit normal vector to \mathcal{S} at \vec{r} . Physically, these boundary conditions mean that neither matter nor heat flows across the surface \mathcal{S} , so that the system inside \mathcal{S} is closed and isolated. To completely exclude energy exchanges with the exterior of \mathcal{S} , it is also assumed that there are no volume forces acting on the fluid inside volume \mathcal{V} . We shall investigate the implications of the continuity equation (III.10), the Navier–Stokes equation (III.31), and the energy conservation equation (III.35) for the total entropy S of the fluid inside \mathcal{V} . For the sake of brevity, the variables t, \vec{r} will be omitted in the remainder of this Section.

Starting with the energy conservation equation (III.35), the contribution

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho \vec{v}^2 \right) + \vec{\nabla} \cdot \left[\left(\frac{1}{2} \rho \vec{v}^2 \right) \vec{v} \right]$$

in its first two lines can be replaced by

$$\rho \vec{v} \cdot \frac{\partial \vec{v}}{\partial t} + \frac{1}{2} \frac{\partial \rho}{\partial t} \vec{v}^2 + \frac{1}{2} \left[\vec{\nabla} \cdot (\rho \vec{v}) \right] \vec{v}^2 + \sum_{i=1}^3 \rho v_i (\vec{v} \cdot \vec{\nabla}) v^i = \sum_{i=1}^3 \rho v_i \left[\frac{\partial v^i}{\partial t} + (\vec{v} \cdot \vec{\nabla}) v^i \right], \quad (\text{III.36a})$$

where the continuity equation (III.9) was used.

As recalled in Appendix A, the fundamental thermodynamic relation $U = TS - \mathcal{P}\mathcal{V} + \mu N$ gives on the one hand $e + \mathcal{P} = Ts + \mu n$, which leads to

$$\begin{aligned} \vec{\nabla} \cdot [(e + \mathcal{P})\vec{v}] &= T\vec{\nabla} \cdot (s\vec{v}) + \mu\vec{\nabla} \cdot (n\vec{v}) + \vec{v} \cdot (s\vec{\nabla}T + n\vec{\nabla}\mu) \\ &= T\vec{\nabla} \cdot (s\vec{v}) + \mu\vec{\nabla} \cdot (n\vec{v}) + \vec{v} \cdot \vec{\nabla}\mathcal{P}, \end{aligned} \quad (\text{III.36b})$$

where the second line follows from the Gibbs–Duhem relation $d\mathcal{P} = s dT + n d\mu$. On the other hand, it leads to

$$de = T ds + \mu dn,$$

which under consideration of the continuity equation for particle number yields

$$\frac{\partial e}{\partial t} = T \frac{\partial s}{\partial t} + \mu \frac{\partial n}{\partial t} = T \frac{\partial s}{\partial t} - \mu \vec{\nabla} \cdot (n\vec{v}). \quad (\text{III.36c})$$

With the help of relations (III.36a)–(III.36c), the energy conservation equation (III.35) can be rewritten as

$$\begin{aligned} \sum_{i=1}^3 \rho v_i \left[\frac{\partial v^i}{\partial t} + (\vec{v} \cdot \vec{\nabla}) v^i \right] + T \frac{\partial s}{\partial t} + T \vec{\nabla} \cdot (s\vec{v}) + \vec{v} \cdot \vec{\nabla}\mathcal{P} = \\ \sum_{i,j=1}^3 \frac{\partial}{\partial x^j} \left[\eta \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) v_i \right] + \sum_{i=1}^3 \frac{\partial}{\partial x_i} [\zeta (\vec{\nabla} \cdot \vec{v}) v_i] + \vec{\nabla} \cdot (\kappa \vec{\nabla} T). \end{aligned} \quad (\text{III.36d})$$

Multiplying the i -th component of Eq. (III.30b) by v_i gives

$$\rho v_i \left[\frac{\partial v^i}{\partial t} + (\vec{v} \cdot \vec{\nabla}) v^i \right] + v_i \frac{\partial \mathcal{P}}{\partial x_i} = \sum_{j=1}^3 v_i \frac{\partial}{\partial x^j} \left[\eta \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \right] + v_i \frac{\partial}{\partial x^i} (\zeta \vec{\nabla} \cdot \vec{v}).$$

Subtracting this identity, summed over $i = 1, 2, 3$, from Eq. (III.36d), yields

$$T \frac{\partial s}{\partial t} + T \vec{\nabla} \cdot (s\vec{v}) = \eta \sum_{i,j=1}^3 \frac{\partial v_i}{\partial x^j} \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) + \zeta (\vec{\nabla} \cdot \vec{v})^2 + \vec{\nabla} \cdot (\kappa \vec{\nabla} T). \quad (\text{III.37})$$

On the right hand side of this equation, one may use the identity

$$\frac{1}{2} \sum_{i,j=1}^3 \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} - \frac{2}{3} g_{ij} \vec{\nabla} \cdot \vec{v} \right) = \sum_{i,j=1}^3 \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \frac{\partial v_j}{\partial x^i}, \quad (\text{III.38a})$$

which follows from the fact that both symmetric terms $\partial v_i/\partial x^j$ and $\partial v_j/\partial x^i$ from the left member give the same contribution, while the term in g_{ij} yields a zero contribution, since it multiplies a traceless term.

Additionally, one has

$$\vec{\nabla} \cdot (\kappa \vec{\nabla} T) = T \vec{\nabla} \cdot \left(\frac{\kappa \vec{\nabla} T}{T} \right) + \frac{\kappa}{T} (\vec{\nabla} T)^2. \quad (\text{III.38b})$$

All in all, Eqs. (III.37) and (III.38) lead to

$$\begin{aligned} \frac{\partial s}{\partial t} + \vec{\nabla} \cdot (s \vec{v}) - \vec{\nabla} \cdot \left(\frac{\kappa \vec{\nabla} T}{T} \right) &= \frac{\eta}{2T} \sum_{i,j=1}^3 \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} - \frac{2}{3} g_{ij} \vec{\nabla} \cdot \vec{v} \right) \\ &\quad + \frac{\zeta}{T} (\vec{\nabla} \cdot \vec{v})^2 + \kappa \frac{(\vec{\nabla} T)^2}{T^2}. \end{aligned} \quad (\text{III.39})$$

This equation can then be integrated over the \mathcal{V} occupied by the fluid:

- When computing the integrals of the two divergence terms on the left hand side with the Stokes theorem, they vanish thanks to the boundary conditions imposed at the surface \mathcal{S} ;
- the remaining term in the left member is simply the time derivative dS/dt of the total entropy of the closed system;
- if all three transport coefficients η , ζ , κ are *positive*, then it is also the case of the three terms on the right hand side.

One thus finds $\frac{dS}{dt} \geq 0$, in agreement with the second law of thermodynamics.

Remarks:

* The previous derivation may be seen as a proof that the transport coefficients must be positive to ensure that the second law of thermodynamics holds.

* If all three transport coefficients η , ζ , κ vanish, i.e. in the case of a non-dissipative fluid, Eq. (III.39) simply reduces to the entropy conservation equation in perfect fluids (III.34).