Examples of relativistic perfect flows

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Dynamics of relativistic perfect fluids

Local conservation equations

$$\mathrm{d}_\mu N^\mu(\mathsf{x}) = 0 \qquad \mathrm{d}_\mu T^{\mu\nu}(\mathsf{x}) = 0$$

with $N^\mu(\mathsf{x}) = n(\mathsf{x}) u^\mu(\mathsf{x})$ for each conserved quantum number,

and $T^{\mu\nu}(\mathsf{x}) = \mathcal{P}(\mathsf{x}) g^{\mu\nu}(\mathsf{x}) + \left[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\right] \dfrac{u^\mu(\mathsf{x}) u^\nu(\mathsf{x})}{c^2}$

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Projecting energy-momentum conservation along the 4-velocity:

$$u^\mu(\mathsf{x}) \mathrm{d}_\mu \epsilon(\mathsf{x}) + \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big] \mathrm{d}_\mu u^\mu(\mathsf{x}) = 0$$

and perpendicular to the 4-velocity:

$$\frac{\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})}{c^2} u^\mu(\mathsf{x}) \mathrm{d}_\mu u^\nu(\mathsf{x}) + \left[g^{\mu\nu}(\mathsf{x}) + \frac{u^\mu(\mathsf{x}) u^\nu(\mathsf{x})}{c^2}\right] \mathrm{d}_\mu \mathcal{P}(\mathsf{x}) = 0$$

Hydrostatic solutions

4-velocity $u^\mu(\mathsf{x}) = \begin{pmatrix} u^0(\mathsf{x}) \\ \vec{0} \end{pmatrix}$, with $\left[\mathsf{u}(\mathsf{x})\right]^2 = g_{00}(\mathsf{x})[u^0(\mathsf{x})]^2 = -1$ (c = 1)

while all partial time derivatives ∂_0 vanish.

Equations of motion: $$\mathrm{d}_\mu T^{\mu\nu}(\mathsf{x}) = 0$$

with $$T^{\mu\nu}(\mathsf{x}) = \mathcal{P}(\mathsf{x})g^{\mu\nu}(\mathsf{x}) + \left[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\right]u^\mu(\mathsf{x})u^\nu(\mathsf{x})$$

Covariant derivatives...

For a scalar: $$\mathrm{d}_\mu \phi = \partial_\mu \phi$$

For the components of a vector: $$\mathrm{d}_\mu c^\nu = \partial_\mu c^\nu + \Gamma^\nu_{\rho\mu} c^\rho$$

For the components of a tensor:

$$\mathrm{d}_\mu (a^\lambda c^\nu) = \partial_\mu (a^\lambda c^\nu) + \Gamma^\lambda_{\rho\mu} a^\rho c^\nu + a^\lambda \Gamma^\nu_{\rho\mu} c^\rho$$

Setting $a = c$, $\lambda = \mu$:

$$\mathrm{d}_\mu (c^\mu c^\nu) = \partial_\mu (c^\mu c^\nu) + \Gamma^\mu_{\rho\mu} c^\rho c^\nu + \Gamma^\nu_{\rho\mu} c^\mu c^\rho$$

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yields $\partial^\nu \mathcal{P} + \partial_\mu[(\epsilon + \mathcal{P}) u^\mu u^\nu] = -(\epsilon + \mathcal{P})(\Gamma^\mu_{\mu\rho} u^\rho u^\nu + \Gamma^\nu_{\mu\rho} u^\rho u^\mu)$

where we also used $\mathrm{d}_\mu g^{\mu\nu} = 0$.

Now, only u^0 is non-zero...: taking ν = i, only Γ^i_{00} is needed.

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Invoking $\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\lambda} \left[\frac{\partial g_{\lambda\nu}}{\partial x^\rho} + \frac{\partial g_{\lambda\rho}}{\partial x^\nu} - \frac{\partial g_{\rho\nu}}{\partial x^\lambda} \right]$ with $\partial_0 = 0$, one finds

$$\Gamma^i_{00} = -\frac{1}{2} g^{i\lambda} \partial_\lambda g_{00} = -\frac{1}{2} \partial^i g_{00}$$

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Now, only u^0 is non-zero...: taking $\nu = i$, only Γ^i_{00} is needed.

$$\partial^i \mathcal{P} = -(\epsilon + \mathcal{P}) \Gamma^i_{00} (u^0)^2 = -\frac{\epsilon + \mathcal{P}}{2 g_{00}} \partial^i g_{00} = -(\epsilon + \mathcal{P}) \partial^i \ln \sqrt{-g_{00}}$$

Hydrostatic solutions

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$$\frac{\partial^i \mathcal{P}}{\epsilon + \mathcal{P}} = -\partial^i \ln\sqrt{-g_{00}}$$

Introducing f such that $\epsilon = f(n)$, $\mathcal{P} = n\frac{\mathrm{d}f}{\mathrm{d}n} - f$ one has

$$\partial^i \mathcal{P} = \frac{\mathrm{d}f}{\mathrm{d}n}\partial^i n + n\frac{\mathrm{d}^2 f}{\mathrm{d}n^2}\partial^i n - \frac{\mathrm{d}f}{\mathrm{d}n}\partial^i n = n\frac{\mathrm{d}^2 f}{\mathrm{d}n^2}\partial^i n$$

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$$-g_{00}\left(\frac{\mathrm{d}f}{\mathrm{d}n}\right)^2 = \text{constant}$$

With the Ansatz $f(n) = \alpha n^{1+c_s^2}$:

$$n^{2c_s^2} = \frac{\text{const}}{-g_{00}}$$

$$\mathcal{P} = c_s^2\epsilon = \text{const}\,(-g_{00})^{-\frac{1+c_s^2}{2c_s^2}}$$

Dust ($c_s = 0$): only if $-g_{00}$ is constant.

$c_s^2 = -1$ ("cosmological constant"): ϵ, $\mathcal{P}$ are uniform

$c_s^2 = \dfrac{1}{3}$ (ultrarelativistic matter): $\mathcal{P} \propto (-g_{00})^{-2}$

An example of relativistic flow: "Bjorken flow"

One-dimensional flow along the z-axis with the 3-velocity $v_z = \frac{z}{t}$ for $z < ct$ and $t > t_0$.

First(?) discussed by R. Hwa (1974); made popular by J.D. Bjorken (1983)

PHYSICAL REVIEW D VOLUME 27, NUMBER 1 1 JANUARY 1983

Highly relativistic nucleus-nucleus collisions: The central rapidity region

J. D. Bjorken
Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510*
(Received 13 August 1982)

The space-time evolution of the hadronic matter produced in the central rapidity region in extreme relativistic nucleus-nucleus collisions is described. We find, in agreement with previous studies, that quark-gluon plasma is produced at a temperature $\gtrsim 200-300$ MeV, and that it should survive over a time scale $\gtrsim 5$ fm/c. Our description relies on the existence of a flat central plateau and on the applicability of hydrodynamics.

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J.D. Bjorken (Fermilab). Jul 1982. 50 pp.
Published in **Phys.Rev. D27 (1983) 140-151**
FERMILAB-PUB-82-044-THY, FERMILAB-PUB-82-044-T
DOI: 10.1103/PhysRevD.27.140

References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
KEK scanned document ; ADS Abstract Service; Fermilab Library Server (fulltext available); Phys. Rev. D Server; Link to Fulltext

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as of June 7, 2015

An example of relativistic flow: "Bjorken flow"

One-dimensional flow along the z-axis with the 3-velocity $v_z = \frac{z}{t}$ for $z < ct$ and $t > t_0$.

☞ introduced to model the boost-invariant* longitudinal expansion of the hot & dense medium created in high-energy nuclear collisions.

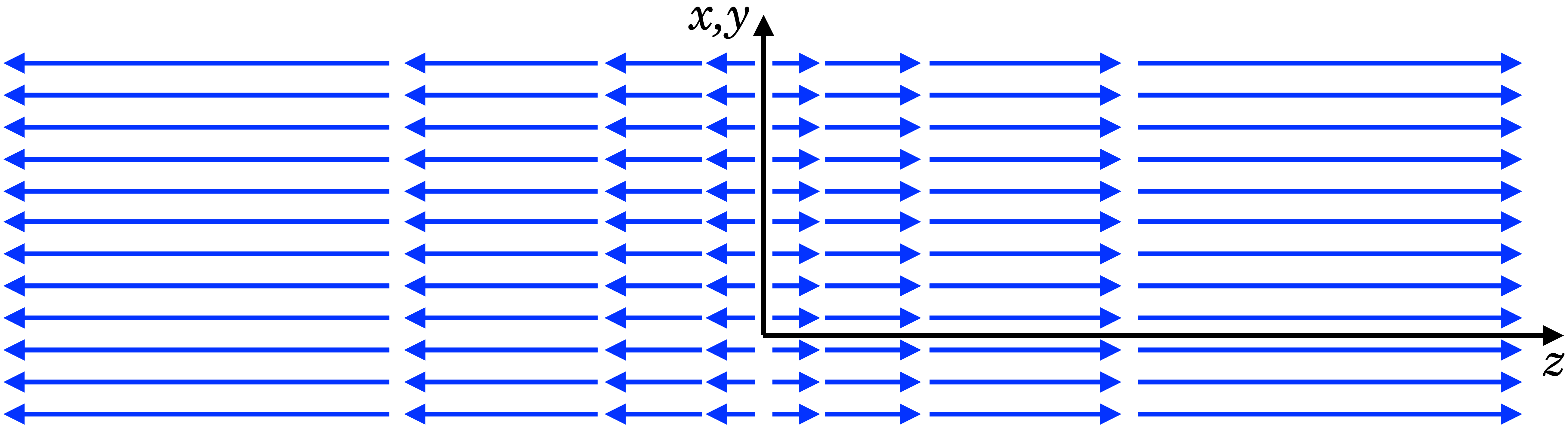
- Analytically tractable in a finite amount of time in case the medium, modeled as a fluid, has a constant speed of sound c_s
- in both cases of a perfect and a dissipative fluid.

*more later on this topic!

An example of relativistic flow: "Bjorken flow"

One-dimensional flow along the z-axis with the 3-velocity $v_z = \dfrac{z}{t}$ for $z < ct$ and $t > t_0$.

at fixed t:



Milne coordinates

Consider a one-dimensional problem along the z-axis.
A convenient choice of coordinates is often*

$$(x^0, x^1) = (t, z) \;\rightarrow\; (x^{0'}, x^{1'}) = (\tau, \varsigma)$$

with

$$\begin{cases} \tau \equiv \sqrt{t^2 - z^2} & \text{"proper time"} \\ \varsigma \equiv \dfrac{1}{2} \log \dfrac{t+z}{t-z} & \text{"spacetime rapidity"} \end{cases}$$

Note: from now on, c=1...

*Other possibility (for particles with $v \approx c$): light-cone coordinates

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†with $\quad \dfrac{1}{2}\log\dfrac{t+z}{t-z} = \mathrm{Artanh}\dfrac{z}{t}$

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What do we need?

- transformation of vector / tensor coordinates
- covariant derivatives

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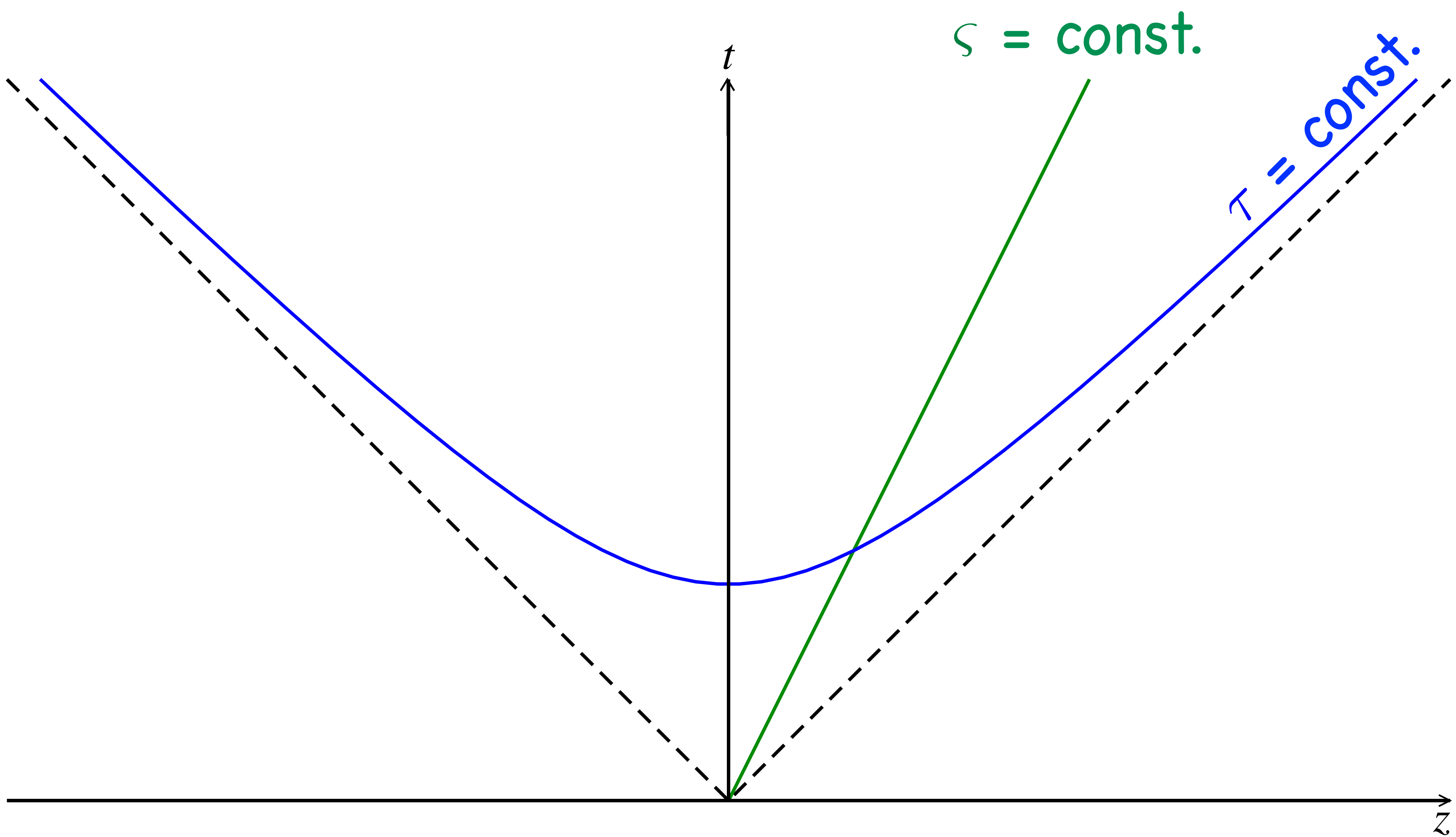

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Corresponding change of basis: $\{\mathsf{e}_\nu\} \;\to\; \{\mathsf{e}_{\mu'}\}$ with $\mathsf{e}_{\mu'} = \mathsf{e}_\nu \Lambda^\nu{}_{\mu'}$

where $\Lambda^\nu{}_{\mu'} \equiv \dfrac{\partial x^\nu}{\partial x^{\mu'}}$

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$$\Lambda^0{}_{0'} = \Lambda^t{}_\tau = \frac{\partial t}{\partial \tau} = \cosh \varsigma \qquad \Lambda^1{}_{0'} = \Lambda^z{}_\tau = \frac{\partial z}{\partial \tau} = \sinh \varsigma$$

$$\Lambda^0{}_{1'} = \Lambda^t{}_\varsigma = \frac{\partial t}{\partial \varsigma} = \tau \sinh \varsigma \qquad \Lambda^1{}_{1'} = \Lambda^z{}_\varsigma = \frac{\partial z}{\partial \varsigma} = \tau \cosh \varsigma$$

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- transformation of vector / tensor coordinates

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☞ metric tensor: $g_{\mu'\nu'} = \mathsf{e}_{\mu'} \cdot \mathsf{e}_{\nu'}$

using $g_{tt} = -g_{zz} = -1$ one finds $g_{\tau\tau} = -1$, $g_{\varsigma\varsigma} = \tau^2$

$g_{tz} = g_{zt} = 0$ $\qquad$ $g_{\tau\varsigma} = g_{\varsigma\tau} = 0$

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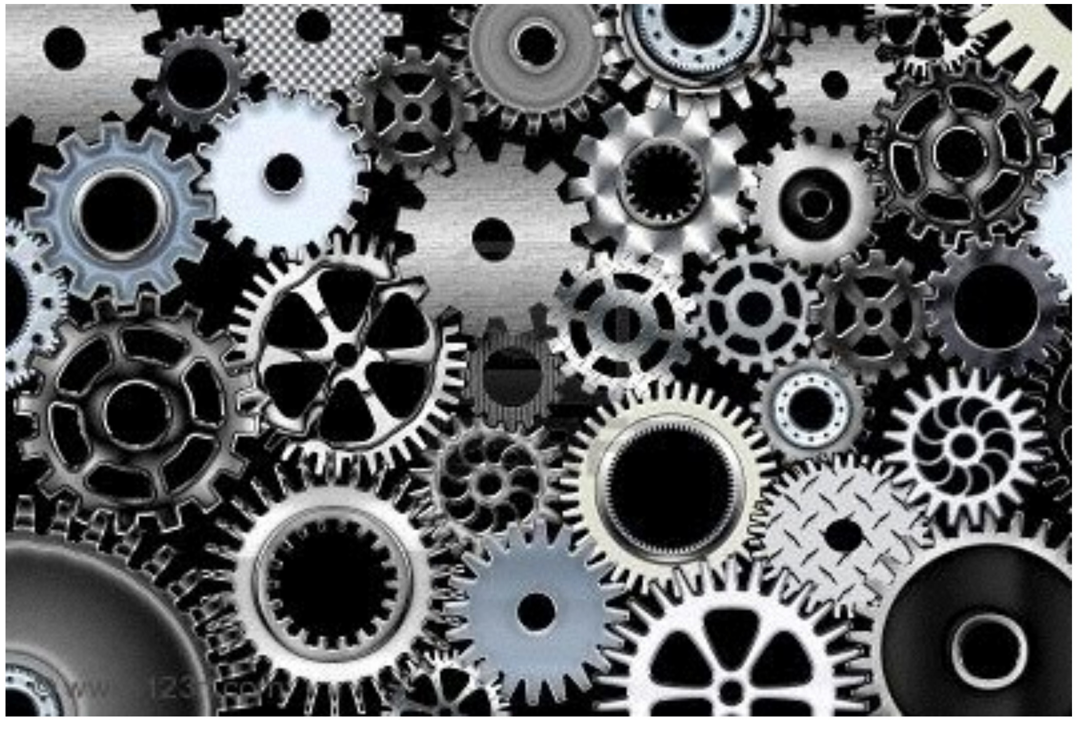

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$\Gamma^\tau_{\varsigma\varsigma} = \tau$, $\quad \Gamma^\varsigma_{\tau\varsigma} = \Gamma^\varsigma_{\varsigma\tau} = \frac{1}{\tau}$, all other coefficients are 0.

- covariant derivatives

Milne coordinates

$$\begin{cases} \tau \equiv \sqrt{t^2 - z^2} \\ \varsigma \equiv \frac{1}{2}\log\frac{t+z}{t-z} \end{cases} \Leftrightarrow \begin{cases} t = \tau\cosh\varsigma \\ z = \tau\sinh\varsigma \end{cases}$$

$$\begin{cases} \mathrm{e}_\tau = \cosh\varsigma\ \mathrm{e}_t + \sinh\varsigma\ \mathrm{e}_z \\ \mathrm{e}_\varsigma = \tau\sinh\varsigma\ \mathrm{e}_t + \tau\cosh\varsigma\ \mathrm{e}_z \end{cases} \qquad \{g_{\mu'\nu'}\} = \{\mathrm{e}_{\mu'}\cdot\mathrm{e}_{\nu'}\} = \begin{pmatrix} -1 & 0 \\ 0 & \tau^2 \end{pmatrix}$$

☞ Christoffel symbols: back to the definition $\frac{\partial \mathrm{e}_{\nu'}}{\partial x^{\rho'}} = \Gamma^{\mu'}_{\nu'\rho'}\mathrm{e}_{\mu'}$

$$\frac{\partial \mathrm{e}_\tau}{\partial\tau} = 0 \qquad \frac{\partial \mathrm{e}_\varsigma}{\partial\tau} = \frac{1}{\tau}\mathrm{e}_\varsigma \equiv \Gamma^\varsigma_{\varsigma\tau}\mathrm{e}_\varsigma$$

$$\frac{\partial \mathrm{e}_\tau}{\partial\varsigma} = \frac{1}{\tau}\mathrm{e}_\varsigma \equiv \Gamma^\varsigma_{\tau\varsigma}\mathrm{e}_\varsigma \qquad \frac{\partial \mathrm{e}_\varsigma}{\partial\varsigma} = \tau\mathrm{e}_\tau \equiv \Gamma^\tau_{\varsigma\varsigma}\mathrm{e}_\tau$$

$\Gamma^\tau_{\varsigma\varsigma} = \tau$, $\Gamma^\varsigma_{\tau\varsigma} = \Gamma^\varsigma_{\varsigma\tau} = \frac{1}{\tau}$, all other coefficients are 0.

Milne coordinates

$$\begin{cases} \tau \equiv \sqrt{t^2 - z^2} \\ \varsigma \equiv \frac{1}{2}\log\frac{t+z}{t-z} \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} t = \tau\cosh\varsigma \\ z = \tau\sinh\varsigma \end{cases}$$

- transformation of vector / tensor coordinates

$$\begin{cases} \mathsf{e}_\tau = \cosh\varsigma\,\mathsf{e}_t + \sinh\varsigma\,\mathsf{e}_z \\ \mathsf{e}_\varsigma = \tau\sinh\varsigma\,\mathsf{e}_t + \tau\cosh\varsigma\,\mathsf{e}_z \end{cases} \qquad \begin{cases} c^\tau = \cosh\varsigma\,c^t - \sinh\varsigma\,c^z \\ c^\varsigma = -\frac{1}{\tau}\sinh\varsigma\,c^t + \frac{1}{\tau}\cosh\varsigma\,c^z \end{cases}$$

- covariant derivatives $\frac{\mathrm{d}c^{\mu'}}{\mathrm{d}x^{\rho'}} = \frac{\partial c^{\mu'}}{\partial x^{\rho'}} + \Gamma^{\mu'}_{\nu'\rho'}c^{\nu'}$...

  ☞ involve the Christoffel symbols $\Gamma^{\tau}_{\varsigma\varsigma} = \tau$, $\Gamma^{\varsigma}_{\tau\varsigma} = \Gamma^{\varsigma}_{\varsigma\tau} = \frac{1}{\tau}$

Bjorken flow

One-dimensional flow along the z-axis with the 3-velocity $v_z = \frac{z}{t}$ for $z < ct$ and $t > t_0$.

☞ 4-velocity* $u^\mu(\mathrm{x}) = \begin{pmatrix} \gamma(\mathrm{x}) \\ \gamma(\mathrm{x}) v_z(\mathrm{x}) \end{pmatrix}$ with $\gamma(\mathrm{x}) = \frac{1}{\sqrt{1 - v_z(\mathrm{x})^2}}$

*only the non-trivial components are shown

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$$= \frac{t}{\sqrt{t^2 - z^2}}$$

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Change from Minkowski to Milne coordinates:

$$\begin{cases} c^\tau = \cosh\varsigma\, c^t - \sinh\varsigma\, c^z \\ c^\varsigma = -\dfrac{1}{\tau}\sinh\varsigma\, c^t + \dfrac{1}{\tau}\cosh\varsigma\, c^z \end{cases}$$

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Change from Minkowski to Milne coordinates:

$$\begin{cases} u^\tau = \cosh\varsigma\, u^t - \sinh\varsigma\, u^z = 1 \\ u^\varsigma = -\frac{1}{\tau}\sinh\varsigma\, u^t + \frac{1}{\tau}\cosh\varsigma\, u^z = 0 \end{cases}$$

*only the non-trivial components are shown

Bjorken flow

One-dimensional flow along the z-axis with the 3-velocity $v_z = \dfrac{z}{t}$ for $z < ct$ and $t > t_0$.

☞ 4-velocity* $u^{\mu'}(\mathrm{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in Milne coordinates $(x^{0'}, x^{1'}) = (\tau, \varsigma)$

*only the non-trivial components are shown

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☞ 4-velocity* $u^{\mu'}(\mathsf{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in Milne coordinates $(x^{0'}, x^{1'}) = (\tau, \varsigma)$

Assumption: no conserved quantum number in the game.

☞ only two equations of motion

$$u^{\mu'}(\mathsf{x})\mathrm{d}_{\mu'}\epsilon(\mathsf{x}) + \left[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\right]\mathrm{d}_{\mu'}u^{\mu'}(\mathsf{x}) = 0$$

$$\left[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\right]u^{\mu'}(\mathsf{x})\mathrm{d}_{\mu'}u^{\nu'}(\mathsf{x}) + \left[g^{\mu'\nu'}(\mathsf{x}) + u^{\mu'}(\mathsf{x})u^{\nu'}(\mathsf{x})\right]\mathrm{d}_{\mu'}\mathcal{P}(\mathsf{x}) = 0$$

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*only the non-trivial components are shown

Milne coordinates

Covariant derivatives: $$\frac{\mathrm{d}c^{\mu'}}{\mathrm{d}x^{\rho'}} = \frac{\partial c^{\mu'}}{\partial x^{\rho'}} + \Gamma^{\mu'}_{\nu'\rho'} c^{\nu'}$$

Milne coordinates

Covariant derivatives: $\mathrm{d}_{\rho'} c^{\mu'} = \partial_{\rho'} c^{\mu'} + \Gamma^{\mu'}_{\nu'\rho'} c^{\nu'}$

Milne coordinates

Covariant derivatives: $\mathrm{d}_{\rho'} c^{\mu'} = \partial_{\rho'} c^{\mu'} + \Gamma^{\mu'}_{\nu'\rho'} c^{\nu'}$

with only the Christoffel symbols $\Gamma^{\tau}_{\varsigma\varsigma} = \tau$, $\Gamma^{\varsigma}_{\tau\varsigma} = \Gamma^{\varsigma}_{\varsigma\tau} = \dfrac{1}{\tau}$

$$\mathrm{d}_{\tau} c^{\mu'} = \partial_{\tau} c^{\mu'} + \Gamma^{\mu'}_{\nu'\tau} c^{\nu'}$$

$$\left\{\begin{array}{l} \mathrm{d}_{\tau} c^{\tau} = \partial_{\tau} c^{\tau} + \Gamma^{\tau}_{\nu'\tau} c^{\nu'} = \partial_{\tau} c^{\tau} \\ \mathrm{d}_{\tau} c^{\varsigma} = \partial_{\tau} c^{\varsigma} + \Gamma^{\varsigma}_{\nu'\tau} c^{\nu'} = \partial_{\tau} c^{\varsigma} + \Gamma^{\varsigma}_{\varsigma\tau} c^{\varsigma} = \partial_{\tau} c^{\varsigma} + \dfrac{c^{\varsigma}}{\tau} \end{array}\right.$$

$$\mathrm{d}_{\varsigma} c^{\mu'} = \partial_{\varsigma} c^{\mu'} + \Gamma^{\mu'}_{\nu'\varsigma} c^{\nu'}$$

$$\left\{\begin{array}{l} \mathrm{d}_{\varsigma} c^{\tau} = \partial_{\varsigma} c^{\tau} + \Gamma^{\tau}_{\nu'\varsigma} c^{\nu'} = \partial_{\varsigma} c^{\tau} + \Gamma^{\tau}_{\varsigma\varsigma} c^{\varsigma} = \partial_{\varsigma} c^{\tau} + \tau c^{\varsigma} \\ \mathrm{d}_{\varsigma} c^{\varsigma} = \partial_{\varsigma} c^{\varsigma} + \Gamma^{\varsigma}_{\nu'\varsigma} c^{\nu'} = \partial_{\varsigma} c^{\varsigma} + \Gamma^{\varsigma}_{\tau\varsigma} c^{\tau} = \partial_{\varsigma} c^{\varsigma} + \dfrac{c^{\tau}}{\tau} \end{array}\right.$$

Milne coordinates

Covariant derivatives: $\mathrm{d}_{\rho'} c^{\mu'} = \partial_{\rho'} c^{\mu'} + \Gamma^{\mu'}_{\nu'\rho'} c^{\nu'}$

with only the Christoffel symbols $\Gamma^{\tau}_{\varsigma\varsigma} = \tau$, $\Gamma^{\varsigma}_{\tau\varsigma} = \Gamma^{\varsigma}_{\varsigma\tau} = \dfrac{1}{\tau}$

$$\begin{cases} \mathrm{d}_\tau c^\tau = \partial_\tau c^\tau + \Gamma^{\tau}_{\nu'\tau} c^{\nu'} = \partial_\tau c^\tau \\ \mathrm{d}_\tau c^\varsigma = \partial_\tau c^\varsigma + \Gamma^{\varsigma}_{\nu'\tau} c^{\nu'} = \partial_\tau c^\varsigma + \Gamma^{\varsigma}_{\varsigma\tau} c^\varsigma = \partial_\tau c^\varsigma + \dfrac{c^\varsigma}{\tau} \end{cases}$$

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☞ 4-divergence: $\mathrm{d}_{\mu'} c^{\mu'} = \mathrm{d}_\tau c^\tau + \mathrm{d}_\varsigma c^\varsigma = \partial_\tau c^\tau + \partial_\varsigma c^\varsigma + \dfrac{c^\tau}{\tau}$

Bjorken flow

4-velocity $u^{\mu'}(\mathsf{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in Milne coordinates $(x^{0'}, x^{1'}) = (\tau, \varsigma)$

4-divergence: $$\mathrm{d}_{\mu'} c^{\mu'} = \mathrm{d}_\tau c^\tau + \mathrm{d}_\varsigma c^\varsigma = \partial_\tau c^\tau + \partial_\varsigma c^\varsigma + \frac{c^\tau}{\tau}$$

☞ $$\mathrm{d}_{\mu'} u^{\mu'}(\mathsf{x}) = \frac{1}{\tau}$$

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☞ $$\mathrm{d}_{\mu'} u^{\mu'}(\mathsf{x}) = \frac{1}{\tau}$$

Projection of the 4-gradient on the velocity: $u^{\mu'}(\mathsf{x})\mathrm{d}_{\mu'} = u^\tau(\mathsf{x})\mathrm{d}_\tau = \mathrm{d}_\tau$

Equations of motion:

$$u^{\mu'}(\mathsf{x})\mathrm{d}_{\mu'}\epsilon(\mathsf{x}) + [\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})]\mathrm{d}_{\mu'} u^{\mu'}(\mathsf{x}) = 0$$

$$[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})] u^{\mu'}(\mathsf{x})\mathrm{d}_{\mu'} u^{\nu'}(\mathsf{x}) + [g^{\mu'\nu'}(\mathsf{x}) + u^{\mu'}(\mathsf{x})u^{\nu'}(\mathsf{x})]\mathrm{d}_{\mu'}\mathcal{P}(\mathsf{x}) = 0$$

+ an equation of state $\mathcal{P}(\mathsf{x}) = c_s(\mathsf{x})^2\epsilon(\mathsf{x})$ involving the speed of sound

Bjorken flow

$$u^{\mu'}(\mathsf{x})\mathrm{d}_{\mu'}\epsilon(\mathsf{x}) + \left[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\right]\mathrm{d}_{\mu'}u^{\mu'}(\mathsf{x}) = 0$$

☞ $$\mathrm{d}_\tau\epsilon(\mathsf{x}) + \frac{\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})}{\tau} = 0$$

Bjorken flow

$$u^{\mu'}(\mathrm{x})\mathrm{d}_{\mu'}\epsilon(\mathrm{x}) + \big[\epsilon(\mathrm{x}) + \mathcal{P}(\mathrm{x})\big]\mathrm{d}_{\mu'}u^{\mu'}(\mathrm{x}) = 0$$

☞ $$\mathrm{d}_\tau\epsilon(\mathrm{x}) + \frac{\epsilon(\mathrm{x}) + \mathcal{P}(\mathrm{x})}{\tau} = 0$$

or equivalently $$\mathrm{d}_\tau\big[\tau\epsilon(\mathrm{x})\big] = -\mathcal{P}(\mathrm{x})$$

which simply relates the change in the energy in a comoving volume (proportional to τ) to the work of pressure forces...

Bjorken flow

$$u^{\mu'}(\mathsf{x})\mathrm{d}_{\mu'}\epsilon(\mathsf{x}) + \left[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\right]\mathrm{d}_{\mu'}u^{\mu'}(\mathsf{x}) = 0$$

$$\mathrm{d}_\tau\epsilon(\mathsf{x}) + \frac{\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})}{\tau} = 0$$

$$\left[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\right]u^{\mu'}(\mathsf{x})\mathrm{d}_{\mu'}u^{\nu'}(\mathsf{x}) + \left[g^{\mu'\nu'}(\mathsf{x}) + u^{\mu'}(\mathsf{x})u^{\nu'}(\mathsf{x})\right]\mathrm{d}_{\mu'}\mathcal{P}(\mathsf{x}) = 0$$

$$\left[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\right]\mathrm{d}_\tau u^{\nu'}(\mathsf{x}) + \left[g^{\mu'\nu'}(\mathsf{x}) + u^{\mu'}(\mathsf{x})u^{\nu'}(\mathsf{x})\right]\mathrm{d}_{\mu'}\mathcal{P}(\mathsf{x}) = 0$$

$\nu' = \tau :$ $\quad [\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})]\mathrm{d}_\tau u^\tau(\mathsf{x}) = [\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})]\partial_\tau u^\tau(\mathsf{x}) = 0$ already known

$\nu' = \varsigma :$ $\quad [\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})]\mathrm{d}_\tau u^\varsigma(\mathsf{x}) + \dfrac{1}{\tau^2}\mathrm{d}_\varsigma\mathcal{P}(\mathsf{x}) = \dfrac{1}{\tau^2}\partial_\varsigma\mathcal{P}(\mathsf{x}) = 0$

pressure is independent of space-time rapidity

Bjorken flow

$$u^{\mu'}(\mathsf{x})\mathrm{d}_{\mu'}\epsilon(\mathsf{x}) + \left[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\right]\mathrm{d}_{\mu'}u^{\mu'}(\mathsf{x}) = 0$$

$$\mathrm{d}_{\tau}\epsilon(\mathsf{x}) + \frac{\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})}{\tau} = 0$$

Invoking the equation of state $\mathcal{P}(\mathsf{x}) = c_s(\mathsf{x})^2\epsilon(\mathsf{x})$

and using $\mathrm{d}_\tau = \partial_\tau$ when acting on scalar fields, one obtains

$$\partial_\tau\epsilon(\mathsf{x}) + \left[1 + c_s(\mathsf{x})^2\right]\frac{\epsilon(\mathsf{x})}{\tau} = 0$$

Bjorken flow

$$u^{\mu'}(\mathsf{x})\mathrm{d}_{\mu'}\epsilon(\mathsf{x}) + \left[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\right]\mathrm{d}_{\mu'}u^{\mu'}(\mathsf{x}) = 0$$

☞ $$\mathrm{d}_\tau \epsilon(\mathsf{x}) + \frac{\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})}{\tau} = 0$$

Invoking the equation of state $\mathcal{P}(\mathsf{x}) = c_s(\mathsf{x})^2\epsilon(\mathsf{x})$

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$$\partial_\tau \epsilon(\mathsf{x}) + \left[1 + c_s(\mathsf{x})^2\right]\frac{\epsilon(\mathsf{x})}{\tau} = 0$$

Assuming from now on that the speed of sound is constant:

$$\epsilon(\mathsf{x}) \propto \frac{1}{\tau^{1+c_s^2}} \qquad \mathcal{P}(\mathsf{x}) \propto \frac{1}{\tau^{1+c_s^2}}$$

Both are independent of space-time rapidity

Bjorken flow

Entropy conservation: $\mathrm{d}_\mu[s(\mathrm{x})u^\mu(\mathrm{x})] = 0$

☞ $\mathrm{d}_\tau s(\mathrm{x}) + \frac{s(\mathrm{x})}{\tau} = 0$

which leads at once to $s(\mathrm{x}) \propto \frac{1}{\tau}$

Together with $\epsilon(\mathrm{x}) \propto \frac{1}{\tau^{1+c_s^2}}$, $\mathcal{P}(\mathrm{x}) \propto \frac{1}{\tau^{1+c_s^2}}$ and $\epsilon + \mathcal{P} = Ts$

one obtains for temperature

$$T(\mathrm{x}) \propto \frac{1}{\tau^{c_s^2}}$$

Everything is independent of space-time rapidity!

Bjorken flow

We have found $\epsilon(\mathsf{x}) \propto \frac{1}{\tau^{1+c_s^2}}$, $\mathcal{P}(\mathsf{x}) \propto \frac{1}{\tau^{1+c_s^2}}$, $s(\mathsf{x}) \propto \frac{1}{\tau}$

and indirectly $T(\mathsf{x}) \propto \frac{1}{\tau^{c_s^2}}$

For an ultrarelativistic gas $\epsilon \propto T^4$ (Stefan–Boltzmann!), $\mathcal{P} \propto T^4$

$s \propto T^3$ (remember $\epsilon + \mathcal{P} = Ts$) and $c_s^2 = \frac{1}{3}$ ... Everything is OK!