VI.3 Perfect relativistic fluid

By definition, a fluid is perfect when there is no dissipative current in it, see definition (III.16a). As a consequence, one can at each point x of the fluid find a reference frame in which the local properties in the neighborhood of x are spatially *isotropic* [cf. definition (III.23)]. This reference frame represents the natural choice for the local rest frame at point x, LR(x).

The forms of the particle-number 4-current and the energy-momentum tensor of a perfect fluid are first introduced in Sec. VI.3.1. It is then shown that the postulated absence of dissipative current automatically leads to the conservation of entropy in the motion (Sec. VI.3.2). Eventually, the low-velocity limit of the dynamical equations is investigated in Sec. VI.3.3.

VI.3.1 Particle four-current and energy-momentum tensor of a perfect fluid

To express the defining feature of the local rest frame LR(x), namely the spatial isotropy of the local fluid properties, it is convenient to adopt a Cartesian coordinate system for the space-like directions in LR(x): since the fluid characteristics are the same in all spatial directions, this in particular holds along the three mutually perpendicular axes defining Cartesian coordinates.

Adopting momentarily such a system—and accordingly Minkowski coordinates on space-time—, the local-rest-frame values of the particle number flux density $\vec{\jmath}(\mathsf{x})$, the j-th component $cT^{0j}(\mathsf{x})$ of the energy flux density, and the density $c^{-1}T^{i0}(\mathsf{x})$ of the i-th component of momentum should all vanish. In addition, the momentum flux-density 3-tensor $\mathbf{T}(\mathsf{x})$ should also be diagonal in LR(x). All in all, one thus necessarily has

$$N^0(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = c\,n(\mathsf{x}), \qquad \vec{\jmath}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = \vec{0}, \tag{VI.16a}$$

and

$$\begin{aligned} T^{00}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} &= \epsilon(\mathsf{x}), \\ T^{ij}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} &= \mathcal{P}(\mathsf{x})\delta^{ij}, \qquad \forall i,j = 1,2,3 \\ T^{i0}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} &= T^{0j}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = 0, \qquad \forall i,j = 1,2,3 \end{aligned} \tag{VI.16b}$$

where the definitions (VI.12) were taken into account, while $\mathcal{P}(\mathsf{x})$ denotes the pressure. In matrix form, the energy-momentum tensor (VI.16b) becomes

$$T^{\mu\nu}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = \begin{pmatrix} \epsilon(\mathsf{x}) & 0 & 0 & 0 \\ 0 & \mathcal{P}(\mathsf{x}) & 0 & 0 \\ 0 & 0 & \mathcal{P}(\mathsf{x}) & 0 \\ 0 & 0 & 0 & \mathcal{P}(\mathsf{x}) \end{pmatrix}. \tag{VI.16c}$$

Remark: The identification of the diagonal spatial components with a "pressure" term is warranted by the physical interpretation of the $T^{ii}(\mathsf{x})$. Referring to it as "the" pressure anticipates the fact that it behaves as the thermodynamic quantity that is related to energy density and particle number by the mechanical equation of state of the fluid.

In an arbitrary reference frame and allowing for the possible use of curvilinear coordinates, the components of the particle number 4-current and the energy-momentum tensor of a perfect fluid are

$$\boxed{N^\mu(\mathsf{x}) = n(\mathsf{x})u^\mu(\mathsf{x})} \tag{VI.17a}$$

and

$$\boxed{T^{\mu\nu}(\mathsf{x}) = \mathcal{P}(\mathsf{x})g^{\mu\nu}(\mathsf{x}) + \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\frac{u^\mu(\mathsf{x})\,u^\nu(\mathsf{x})}{c^2}} \tag{VI.17b}$$

respectively, with $u^\mu(\mathsf{x})$ the components of the fluid 4-velocity.

Relation (VI.17a) resp. (VI.17b) is an identity between the components of two 4-vectors resp. two 4-tensors, which transform identically under Lorentz transformations—i.e. changes of reference frame—and coordinate basis changes. Since the components of these 4-vectors resp. 4-tensors are equal in a given reference frame—the local rest frame—and a given basis—that of Minkowski coordinates—, they remain equal in any coordinate system in any reference frame. □

In geometric formulation, the particle number 4-current and energy-momentum tensor respectively read

$$\boxed{\mathsf{N}(\mathsf{x}) = n(\mathsf{x})\mathsf{u}(\mathsf{x})} \tag{VI.18a}$$

and

$$\boxed{\mathbf{T}(\mathsf{x}) = \mathcal{P}(\mathsf{x})\mathbf{g}^{-1}(\mathsf{x}) + \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\frac{\mathsf{u}(\mathsf{x})\otimes\mathsf{u}(\mathsf{x})}{c^2}.} \tag{VI.18b}$$

The latter is very reminiscent of the 3-dimensional non-relativistic momentum flux density (III.22); similarly, the reader may also compare the component-wise formulations (III.21b) and (VI.17a).

Remarks:

∗ The energy-momentum tensor is obviously symmetric—which is a non-trivial physical statement. For instance, the identity $T^{i0} = T^{0i}$ means that ($1/c$ times) the energy flux density in direction i equals (c times) the density of the i-th component of momentum—where one may rightly argue that the factors of c are historical accidents in the choice of units. This is possible in a relativistic theory only because the energy density also contains the mass energy.

∗ In Eq. (VI.17b) or (VI.18b), the sum $\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})$ is equivalently the *enthalpy density* $w(\mathsf{x})$.

∗ Equation (VI.17b), (VI.18b) or (VI.19a) below represents the most general symmetric $\binom{2}{0}$-tensor that can be constructed using only the metric tensor and the 4-velocity.

The component form (VI.17b) of the energy-momentum tensor can trivially be recast as

$$T^{\mu\nu}(\mathsf{x}) = \epsilon(\mathsf{x})\frac{u^\mu(\mathsf{x})\,u^\nu(\mathsf{x})}{c^2} + \mathcal{P}(\mathsf{x})\Delta^{\mu\nu}(\mathsf{x}) \tag{VI.19a}$$

with

$$\Delta^{\mu\nu}(\mathsf{x}) \equiv g^{\mu\nu}(\mathsf{x}) + \frac{u^\mu(\mathsf{x})\,u^\nu(\mathsf{x})}{c^2} \tag{VI.19b}$$

the components of a tensor $\mathbf{\Delta}$ which—in its $\binom{1}{1}$-form—is actually a projector on the 3-dimensional vector space orthogonal to the 4-velocity $\mathsf{u}(\mathsf{x})$, while $u^\mu(\mathsf{x})\,u^\nu(\mathsf{x})/c^2$ projects on the time-like direction of the 4-velocity.

One easily checks the identities $\Delta^\mu{}_\nu(\mathsf{x})\Delta^\nu{}_\rho(\mathsf{x}) = \Delta^\mu{}_\rho(\mathsf{x})$ and $\Delta^\mu{}_\nu(\mathsf{x})u^\nu(\mathsf{x}) = 0$.

From Eq. (VI.19a) follows at once that the comoving pressure $\mathcal{P}(\mathsf{x})$ can be found in any reference frame as

$$\mathcal{P}(\mathsf{x}) = \frac{1}{3}\Delta_{\mu\nu}(\mathsf{x})T^{\mu\nu}(\mathsf{x}). \tag{VI.20}$$

which complements relations (VI.14) and (VI.15) for the particle number density and energy density, respectively.

Remark: Contracting the energy-momentum tensor $\mathbf{T}$ with the metric tensor twice yields a scalar, the so-called *trace* of $\mathbf{T}$

$$\mathbf{T}(\mathsf{x}) : \mathbf{g}(\mathsf{x}) = T^{\mu\nu}(\mathsf{x})g_{\mu\nu}(\mathsf{x}) = T^\mu{}_\mu(\mathsf{x}) = 3\mathcal{P}(\mathsf{x}) - \epsilon(\mathsf{x}). \tag{VI.21}$$

VI.3.2 Entropy in a perfect fluid

Let $s(\mathsf{x})$ denote the (comoving) entropy density of the fluid, as defined in the local rest frame $\mathrm{LR}(\mathsf{x})$ at point x.

VI.3.2 a Entropy conservation

For a perfect fluid, the fundamental equations of motion (VI.2) and (VI.7) lead automatically to the *local conservation of entropy*

$$\boxed{\mathrm{d}_\mu\big[s(\mathsf{x})u^\mu(\mathsf{x})\big] = 0} \tag{VI.22}$$

with $s(\mathsf{x})u^\mu(\mathsf{x})$ the *entropy four-current*.

Proof: The relation $U = TS - \mathcal{P}\mathcal{V} + \mu N$ with U resp. μ the internal energy resp. the chemical potential gives for the local thermodynamic densities $\epsilon = Ts - \mathcal{P} + \mu n$. Inserting this expression of the energy density in Eq. (VI.17b) yields (dropping the x variable for the sake of brevity):

$$T^{\mu\nu} = \mathcal{P}g^{\mu\nu} + (Ts+\mu n)\frac{u^\mu u^\nu}{c^2} = \mathcal{P}g^{\mu\nu} + \big[T(su^\mu)+\mu(nu^\mu)\big]\frac{u^\nu}{c^2}.$$

Taking the 4-gradient d_μ of this identity gives

$$\mathrm{d}_\mu T^{\mu\nu} = \mathrm{d}^\nu\mathcal{P} + \big[T(su^\mu)+\mu(nu^\mu)\big]\frac{\mathrm{d}_\mu u^\nu}{c^2} + \big[s\,\mathrm{d}_\mu T + n\,\mathrm{d}_\mu\mu\big]\frac{u^\mu u^\nu}{c^2} + \big[T\,\mathrm{d}_\mu(su^\mu)+\mu\,\mathrm{d}_\mu(nu^\mu)\big]\frac{u^\nu}{c^2}.$$

Invoking the energy-momentum conservation equation (VI.7), the leftmost member of this identity vanishes. The second term between square brackets on the right hand side can be rewritten with the help of the Gibbs–Duhem relation as $s\,\mathrm{d}_\mu T + n\,\mathrm{d}_\mu\mu = \mathrm{d}_\mu\mathcal{P}$. Eventually, the particle number conservation formulation (VI.7) can be used in the rightmost term. Multiplying everything by u_ν yields

$$0 = u_\nu\,\mathrm{d}^\nu\mathcal{P} + \big[T(su^\mu)+\mu(nu^\mu)\big]\frac{u_\nu\,\mathrm{d}_\mu u^\nu}{c^2} + (\mathrm{d}_\mu\mathcal{P})\frac{u^\mu u^\nu u_\nu}{c^2} + \big[T\,\mathrm{d}_\mu(su^\mu)\big]\frac{u_\nu u^\nu}{c^2}.$$

The constant normalization $u_\nu u^\nu = -c^2$ of the 4-velocity implies $u_\nu\,\mathrm{d}_\mu u^\nu = 0$ for $\mu = 0,\dots,3$, so that the equation becomes

$$0 = u_\nu\,\mathrm{d}^\nu\mathcal{P} - (\mathrm{d}_\mu\mathcal{P})u^\mu - T\,\mathrm{d}_\mu(su^\mu),$$

leading to $\mathrm{d}_\mu(su^\mu) = 0$. □

VI.3.2 b Isentropic distribution

The local conservation of entropy (VI.22) implies the conservation of the entropy per particle $s(\mathsf{x})/n(\mathsf{x})$ along the motion, where $n(\mathsf{x})$ denotes the comoving particle number density.

Proof: the total time derivative of the entropy per particle reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{s}{n}\right) = \frac{\partial}{\partial t}\left(\frac{s}{n}\right) + \vec{\mathrm{v}}\cdot\vec{\nabla}\left(\frac{s}{n}\right) = \frac{1}{\gamma}\mathsf{u}\cdot\mathsf{d}\left(\frac{s}{n}\right),$$

where the second identity makes use of Eq. (VI.11), with γ the Lorentz factor. The rightmost term is then

$$\mathsf{u}\cdot\mathsf{d}\left(\frac{s}{n}\right) = \frac{1}{n}\mathsf{u}\cdot\mathsf{d}s - \frac{s}{n^2}\mathsf{u}\cdot\mathsf{d}n = \frac{1}{n}\left(\mathsf{u}\cdot\mathsf{d}s - \frac{s}{n}\mathsf{u}\cdot\mathsf{d}n\right).$$

The continuity equation $\mathsf{d}\cdot(n\mathsf{u}) = 0$ gives $\mathsf{u}\cdot\mathsf{d}n = -n\,\mathsf{d}\cdot\mathsf{u}$, implying

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{s}{n}\right) = \frac{1}{\gamma}\mathsf{u}\cdot\mathsf{d}\left(\frac{s}{n}\right) = \frac{1}{\gamma n}(\mathsf{u}\cdot\mathsf{d}s + s\,\mathsf{d}\cdot\mathsf{u}) = \frac{1}{\gamma n}\mathsf{d}\cdot(s\mathsf{u}) = 0,$$

where the last identity expresses the conservation of entropy. □

VI.3.3 Non-relativistic limit

We shall now consider the low-velocity limit $|\vec{\mathsf{v}}| \ll c$ of the relativistic equations of motion (VI.2) and (VI.7), in the case when the conserved currents are those of perfect fluids, namely as given by relations (VI.17a) and (VI.17b). Anticipating on the result, we shall recover the equations governing the dynamics of non-relativistic perfect fluids presented in Chapter III, as could be expected for the sake of consistency.

In the small-velocity limit, the typical velocity of the atoms forming the fluid is also much smaller than the speed of light, which has two consequences. On the one hand, the available energies are too low to allow the creation of particle–antiparticle pairs—while their annihilation remains possible—, so that the fluid consists of either particles or antiparticles. Accordingly, the "net" particle number density $n(\mathsf{x})$, difference of the amounts of particles and antiparticles in a unit volume, actually coincides with the "true" particle number density.

On the other hand, the relativistic energy density ϵ can then be expressed as the sum of the contribution from the (rest) masses of the particles and of a kinetic energy term. By definition, the latter is the local internal energy density e of the fluid, while the former is simply the number density of particles multiplied by their mass energy:

$$\epsilon(\mathsf{x}) = n(\mathsf{x})mc^2 + e(\mathsf{x}) = \rho(\mathsf{x})c^2 + e(\mathsf{x}), \tag{VI.23}$$

with $\rho(\mathsf{x})$ the mass density of the fluid constituents. It is important to note that the internal energy density e is of order $\vec{\mathsf{v}}^2/c^2$ with respect to the mass-energy term. The same holds for the pressure $\mathcal{P}$, which is of the same order of magnitude as e.[31]

Eventually, Taylor expanding the Lorentz factor associated with the flow velocity yields

$$\gamma(\mathsf{x}) \underset{|\vec{\mathsf{v}}|\ll c}{\sim} 1 + \frac{1}{2}\frac{\vec{\mathsf{v}}(\mathsf{x})^2}{c^2} + \mathcal{O}\left(\frac{\vec{\mathsf{v}}(\mathsf{x})^4}{c^4}\right). \tag{VI.24}$$

Accordingly, to leading order in $\vec{\mathsf{v}}^2/c^2$, the components (VI.11) of the flow 4-velocity read

$$u^\mu(\mathsf{x}) \underset{|\vec{\mathsf{v}}|\ll c}{\sim} \begin{pmatrix} c \\ \vec{\mathsf{v}}(\mathsf{x}) \end{pmatrix}. \tag{VI.25}$$

Throughout the Section, we shall omit for the sake of brevity the variables x resp. $(t, \vec{r})$ of the various fields. In addition, we adopt for simplicity a system of Minkowski coordinates.

VI.3.3 a Particle number conservation

The 4-velocity components (VI.25) give for those of the particle number 4-current (VI.17a)

$$N^\mu \underset{|\vec{\mathsf{v}}|\ll c}{\sim} \begin{pmatrix} nc \\ n\vec{\mathsf{v}} \end{pmatrix}.$$

Accordingly, the particle number conservation equation (VI.2) becomes

$$0 = \partial_\mu N^\mu \approx \frac{1}{c}\frac{\partial(nc)}{\partial t} + \sum_{i=1}^{3}\frac{\partial(n\mathsf{v}^i)}{\partial x^i} = \frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\,\vec{\mathsf{v}}). \tag{VI.26}$$

That is, one recovers the non-relativistic continuity equation (III.10).

VI.3.3 b Momentum and energy conservation

The (components of the) energy-momentum tensor of a perfect fluid are given by Eq. (VI.17b). Performing a Taylor expansion including the leading and next-to-leading terms in $|\vec{\mathsf{v}}|/c$ yields, under consideration of relation (VI.23)

[31] This is exemplified for instance by the non-relativistic classical ideal gas, in which the internal energy density is $e = nc_{\mathcal{V}}k_{\mathrm{B}}T$ with $c_{\mathcal{V}}$ a number of order 1—this results e.g. from the *equipartition theorem*—while its pressure is $\mathcal{P} = nk_{\mathrm{B}}T$.

$$T^{00} = -\mathcal{P} + \gamma^2(\rho c^2 + e + \mathcal{P}) \underset{|\vec{\mathrm{v}}|\ll c}{\sim} \rho c^2 + e + \rho\,\vec{\mathrm{v}}^2 + \mathcal{O}\!\left(\frac{\vec{\mathrm{v}}^2}{c^2}\right); \tag{VI.27a}$$

$$T^{0j} = T^{j0} = \gamma^2(\rho c^2 + e + \mathcal{P})\frac{\mathrm{v}^j}{c} \underset{|\vec{\mathrm{v}}|\ll c}{\sim} \rho c\mathrm{v}^j + \big(e + \mathcal{P} + \rho\,\vec{\mathrm{v}}^2\big)\frac{\mathrm{v}^j}{c} + \mathcal{O}\!\left(\frac{|\vec{\mathrm{v}}|^3}{c^3}\right); \tag{VI.27b}$$

$$T^{ij} = \mathcal{P}g^{ij} + \gamma^2(\rho c^2 + e + \mathcal{P})\frac{\mathrm{v}^i\mathrm{v}^j}{c^2} \underset{|\vec{\mathrm{v}}|\ll c}{\sim} \mathcal{P}g^{ij} + \rho\,\mathrm{v}^i\mathrm{v}^j + \mathcal{O}\!\left(\frac{\vec{\mathrm{v}}^2}{c^2}\right) = \mathbf{T}^{ij} + \mathcal{O}\!\left(\frac{\vec{\mathrm{v}}^2}{c^2}\right). \tag{VI.27c}$$

In the last line, we have introduced the components $\mathbf{T}^{ij}$, defined in Eq. (III.21b), of the three-dimensional momentum flux-density tensor for a perfect non-relativistic fluid. As emphasized below Eq. (VI.23), the internal energy density and pressure in the rightmost terms of the first or second equations are of the same order of magnitude as the term $\rho\,\vec{\mathrm{v}}^2$ with which they appear, i.e. they are always part of the highest-order term.

Momentum conservation

Considering first the components (VI.27b), (VI.27c), the low-velocity limit of the relativistic momentum-conservation equation $\partial_\mu T^{\mu j} = 0$ for $j = 1,2,3$ reads

$$0 = \frac{1}{c}\frac{\partial(\rho c\mathrm{v}^j)}{\partial t} + \sum_{i=1}^{3}\frac{\partial\mathbf{T}^{ij}}{\partial x^i} + \mathcal{O}\!\left(\frac{\vec{\mathrm{v}}^2}{c^2}\right) = \frac{\partial(\rho\mathrm{v}^j)}{\partial t} + \sum_{i=1}^{3}\frac{\partial\mathbf{T}^{ij}}{\partial x^i} + \mathcal{O}\!\left(\frac{\vec{\mathrm{v}}^2}{c^2}\right). \tag{VI.28}$$

This is precisely the conservation-equation formulation (III.24a) of the Euler equation in absence of external volume forces.

Energy conservation

Given the physical interpretation of the components T^{00}, T^{i0} with $i = 1,2,3$, the component $\nu = 0$ of the energy-momentum conservation equation (VI.7), $\partial_\mu T^{\mu 0} = 0$, should represent the conservation of energy.

As was mentioned several times, the relativistic energy density and flux density actually also contain a term from the rest mass of the fluid constituents. Thus, the leading order contribution to $\partial_\mu T^{\mu 0} = 0$, coming from the first terms in the right members of Eqs. (VI.27a) and (VI.27b), is

$$0 = \frac{\partial(\rho c)}{\partial t} + \sum_{i=1}^{3}\frac{\partial(\rho c\mathrm{v}^i)}{\partial x^i} + \mathcal{O}\!\left(\frac{\vec{\mathrm{v}}^2}{c^2}\right),$$

that is, up to a factor c, exactly the continuity equation (III.9), which was already shown to be the low-velocity limit of the conservation of the particle-number 4-current.

To isolate the internal energy contribution, it is thus necessary to subtract that of mass energy. In the fluid local rest frame, relation (VI.23) shows that one must subtract ρc^2 from ϵ. The former simply equals $\rho c u^0|_{\mathrm{LR}}$, while the latter is the component $\mu = 0$ of $T^{\mu 0}|_{\mathrm{LR}}$, whose space-like components vanish in the local rest frame. To fully subtract the mass energy contribution in any frame from both the energy density and flux density, one should thus consider the 4-vector $T^{\mu 0} - \rho c u^\mu$.

Accordingly, instead of simply using $\partial_\mu T^{\mu 0} = 0$, one should start from the equivalent—thanks to Eq. (VI.2) and the relation $\rho = mn$—equation $\partial_\mu(T^{\mu 0} - \rho c u^\mu) = 0$. With the approximations

$$\rho c u^0 = \gamma\rho c^2 = \rho c^2 + \frac{1}{2}\rho\,\vec{\mathrm{v}}^2 + \mathcal{O}\!\left(\frac{\vec{\mathrm{v}}^2}{c^2}\right)$$

and

$$\rho c u^j = \gamma\rho c\mathrm{v}^j = \rho c\mathrm{v}^j + \left(\frac{1}{2}\rho\,\vec{\mathrm{v}}^2\right)\frac{v^j}{c} + \mathcal{O}\!\left(\frac{|\vec{\mathrm{v}}|^5}{c^3}\right)$$

one finds

$$0 = \partial_\mu\big(T^{\mu0} - \rho c u^\mu\big) = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\,\vec{\mathsf{v}}^2 + e\right) + \sum_{j=1}^{3}\frac{\partial}{\partial x^j}\left[\left(\frac{1}{2}\rho\,\vec{\mathsf{v}}^2 + e + \mathcal{P}\right)\frac{\mathsf{v}^j}{c}\right] + \mathcal{O}\!\left(\frac{\vec{\mathsf{v}}^2}{c^2}\right),$$

that is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\,\vec{\mathsf{v}}^2 + e\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho\,\vec{\mathsf{v}}^2 + e + \mathcal{P}\right)\vec{\mathsf{v}}\right] \approx 0. \tag{VI.29}$$

This is the non-relativistic local formulation of energy conservation (III.25) for a perfect fluid in absence of external volume forces. Since that equation had been postulated in Section III.4.1, the above derivation may be seen as its belated proof.

VI.3.3 c Entropy conservation

Using the approximate 4-velocity components (VI.25), the entropy conservation equation (VI.22) becomes in the low-velocity limit

$$0 = \partial_\mu(s u^\mu) \approx \frac{1}{c}\frac{\partial(sc)}{\partial t} + \sum_{i=1}^{3}\frac{\partial(s\mathsf{v}^i)}{\partial x^i} = \frac{\partial s}{\partial t} + \vec{\nabla}\cdot(s\,\vec{\mathsf{v}}), \tag{VI.30}$$

i.e. gives the non-relativistic equation (III.26).