Tutorial sheet 14

35. A family of solutions of the dynamical equations for perfect relativistic fluids

Let $\{x^\mu\}$ denote Minkowski coordinates and $\tau^2 \equiv -x^\mu x_\mu$, where the "mostly plus" metric is used. Show that the following four-velocity, pressure and charge density constitute a solution of the equations describing the motion of a perfect relativistic fluid with equation of state $\mathcal{P} = K\varepsilon$ and a single conserved charge:

$$u^\mu(\mathsf{x}) = \frac{x^\mu}{\tau} \quad , \quad \mathcal{P}(\mathsf{x}) = \mathcal{P}_0\left(\frac{\tau_0}{\tau}\right)^{3(1+K)} \quad , \quad n(\mathsf{x}) = n_0\left(\frac{\tau_0}{\tau}\right)^3 \mathcal{N}\big(\sigma(\mathsf{x})\big), \tag{1}$$

with τ_0, $\mathcal{P}_0$, n_0 arbitrary constants and $\mathcal{N}$ an arbitrary function of a single argument, while σ is a function of spacetime coordinates with vanishing comoving derivative: $u^\mu\partial_\mu\sigma(\mathsf{x}) = 0$.

36. Speed of sound in ultrarelativistic matter

Consider a perfect fluid with the usual energy-momentum tensor. $T^{\mu\nu} = \mathcal{P}g^{\mu\nu} + (\epsilon + \mathcal{P})u^\mu u^\nu/c^2$. It is assumed that there is no conserved quantum number relevant for thermodynamics, so that the energy density in the local rest frame ϵ is function of a single thermodynamic variable, for instance $\epsilon = \epsilon(\mathcal{P})$. Throughout the exercise, Minkowski coordinates are used.

A background "flow" with uniform local-rest-frame energy density and pressure ϵ_0 and $\mathcal{P}_0$ is submitted to a small perturbation resulting in $\epsilon = \epsilon_0 + \delta\epsilon$, $\mathcal{P} = \mathcal{P}_0 + \delta\mathcal{P}$, and $\vec{\mathsf{v}} = \vec{0} + \delta\vec{\mathsf{v}}$.

i. Starting from the energy-momentum conservation equation $\partial_\mu T^{\mu\nu} = 0$, show that linearization to first order in the perturbations leads to the two equations of motion $\partial_t\delta\epsilon = -(\epsilon_0 + \mathcal{P}_0)\vec{\nabla}\cdot\delta\vec{\mathsf{v}}$ and $(\epsilon_0 + \mathcal{P}_0)\partial_t\delta\vec{\mathsf{v}} = -c^2\vec{\nabla}\delta\mathcal{P}$.

ii. Show that the speed of sound is given by the expression $c_s^2 = \dfrac{c^2}{\mathrm{d}\epsilon/\mathrm{d}\mathcal{P}}$.

iii. Compute c_s for a fluid obeying the Stefan–Boltzmann law[1] $\mathcal{P} = \dfrac{g\pi^2}{90}\dfrac{(k_\mathrm{B}T)^4}{(\hbar c)^3}$, with g the number of degrees of freedom (e.g. $g = 2$ for blackbody radiation).

Hint: You may find the Gibbs–Duhem relation useful...

37. Vorticity in a perfect relativistic fluid

Consider the *kinematic vorticity tensor* defined by its components

$$\omega_{\mu\nu} \equiv \frac{1}{2}\Delta^\alpha{}_\mu\Delta^\beta{}_\nu\big(\mathrm{d}_\beta u_\alpha - \mathrm{d}_\alpha u_\beta\big), \tag{2}$$

where $\Delta^\alpha{}_\mu \equiv g^\alpha{}_\mu + u^\alpha u_\mu$.

a) Why is no calculation necessary to prove the identity $\Delta^{\mu\nu}\omega_{\mu\nu} = 0$? Show the identity

$$\omega_{\mu\nu} = \frac{1}{2}\big(\mathrm{d}_\nu u_\mu - \mathrm{d}_\mu u_\nu + a_\mu u_\nu - a_\nu u_\mu\big), \tag{3}$$

where $a^\mu \equiv u^\nu\mathrm{d}_\nu u^\mu$.

b) Let $\epsilon^{\mu\nu\rho\sigma}$ denote the totally antisymmetric (Levi-Civita) tensor, with the convention $\epsilon^{0123} = -1$. Define a four-vector by $\omega^\mu \equiv -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\omega_{\rho\sigma}u_\nu$ (assuming Minkowski coordinates). What are its components in the local rest frame? What do you recognize?

c) Show that if $a^\mu = 0$, then the vorticity 4-vector obeys the evolution equation

$$u^\nu\mathrm{d}_\nu\omega^\mu = -\frac{2}{3}(\mathrm{d}_\rho u^\rho)\omega^\mu + S^\mu{}_\nu\omega^\nu, \tag{4}$$

where $S^{\mu\nu}$ is the rate-of-shear tensor.

[1] This is a good opportunity to refresh your knowledge on the statistical physics of relativistic systems. Can you give a physical argument why quantum effects always play a role in such systems, as signaled by the presence of $\hbar$ in the equation of state?