

Examples of relativistic perfect flows

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- Hydrostatic solutions

- Bjorken flow

Dynamics of relativistic perfect fluids

Local conservation equations

$$d_\mu N^\mu(\mathbf{x}) = 0$$

$$d_\mu T^{\mu\nu}(\mathbf{x}) = 0$$

with $N^\mu(\mathbf{x}) = n(\mathbf{x})u^\mu(\mathbf{x})$ for each conserved quantum number,

and $T^{\mu\nu}(\mathbf{x}) = \mathcal{P}(\mathbf{x})g^{\mu\nu}(\mathbf{x}) + [\epsilon(\mathbf{x}) + \mathcal{P}(\mathbf{x})] \frac{u^\mu(\mathbf{x})u^\nu(\mathbf{x})}{c^2}$

Projecting energy-momentum conservation along the 4-velocity:

$$u^\mu(\mathbf{x})d_\mu\epsilon(\mathbf{x}) + [\epsilon(\mathbf{x}) + \mathcal{P}(\mathbf{x})]d_\mu u^\mu(\mathbf{x}) = 0$$

and perpendicular to the 4-velocity:

$$\frac{\epsilon(\mathbf{x}) + \mathcal{P}(\mathbf{x})}{c^2} u^\mu(\mathbf{x})d_\mu u^\nu(\mathbf{x}) + \left[g^{\mu\nu}(\mathbf{x}) + \frac{u^\mu(\mathbf{x})u^\nu(\mathbf{x})}{c^2} \right] d_\mu \mathcal{P}(\mathbf{x}) = 0$$

Hydrostatic solutions

4-velocity $u^\mu(\mathbf{x}) = \begin{pmatrix} u^0(\mathbf{x}) \\ \vec{0} \end{pmatrix}$, with $[u(\mathbf{x})]^2 = g_{00}(\mathbf{x})[u^0(\mathbf{x})]^2 = -1$ ($c=1$)

while all partial time derivatives ∂_0 vanish.

Equations of motion: $d_\mu T^{\mu\nu}(\mathbf{x}) = 0$

with $T^{\mu\nu}(\mathbf{x}) = \mathcal{P}(\mathbf{x})g^{\mu\nu}(\mathbf{x}) + [\epsilon(\mathbf{x}) + \mathcal{P}(\mathbf{x})]u^\mu(\mathbf{x})u^\nu(\mathbf{x})$

Covariant derivatives...

For a scalar:

$$d_{\mu}\phi = \partial_{\mu}\phi$$

For the components of a vector: $d_{\mu}c^{\nu} = \partial_{\mu}c^{\nu} + \Gamma_{\rho\mu}^{\nu}c^{\rho}$

For the components of a tensor:

$$d_{\mu}(a^{\lambda}c^{\nu}) = \partial_{\mu}(a^{\lambda}c^{\nu}) + \Gamma_{\rho\mu}^{\lambda}a^{\rho}c^{\nu} + a^{\lambda}\Gamma_{\rho\mu}^{\nu}c^{\rho}$$

Setting $a = c$, $\lambda = \mu$:

$$d_{\mu}(c^{\mu}c^{\nu}) = \partial_{\mu}(c^{\mu}c^{\nu}) + \Gamma_{\rho\mu}^{\mu}c^{\rho}c^{\nu} + \Gamma_{\rho\mu}^{\nu}c^{\mu}c^{\rho}$$

Hydrostatic solutions

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yields $\partial^\nu \mathcal{P} + \partial_\mu [(\epsilon + \mathcal{P})u^\mu u^\nu] = -(\epsilon + \mathcal{P})(\Gamma_{\mu\rho}^\mu u^\rho u^\nu + \Gamma_{\mu\rho}^\nu u^\rho u^\mu)$

where we also used $d_\mu g^{\mu\nu} = 0$.

Now, only u^0 is non-zero...: taking $\nu = i$, only Γ_{00}^i is needed.

Covariant derivatives...

For a scalar: $d_\mu \phi = \partial_\mu \phi$

For the components of a vector: $d_\mu c^\nu = \partial_\mu c^\nu + \Gamma_{\rho\mu}^\nu c^\rho$

For the components of a tensor:

$$d_\mu (a^\lambda c^\nu) = \partial_\mu (a^\lambda c^\nu) + \Gamma_{\rho\mu}^\lambda a^\rho c^\nu + a^\lambda \Gamma_{\rho\mu}^\nu c^\rho$$

Setting $a = c$, $\lambda = \mu$:

$$d_\mu (c^\mu c^\nu) = \partial_\mu (c^\mu c^\nu) + \Gamma_{\rho\mu}^\mu c^\rho c^\nu + \Gamma_{\rho\mu}^\nu c^\mu c^\rho$$

Invoking $\Gamma_{\nu\rho}^\mu = \frac{1}{2} g^{\mu\lambda} \left[\frac{\partial g_{\lambda\nu}}{\partial x^\rho} + \frac{\partial g_{\lambda\rho}}{\partial x^\nu} - \frac{\partial g_{\rho\nu}}{\partial x^\lambda} \right]$ with $\partial_0 = 0$, one finds

$$\Gamma_{00}^i = -\frac{1}{2} g^{i\lambda} \partial_\lambda g_{00} = -\frac{1}{2} \partial^i g_{00}$$

Hydrostatic solutions

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yields $\partial^\nu \mathcal{P} + \partial_\mu [(\epsilon + \mathcal{P})u^\mu u^\nu] = -(\epsilon + \mathcal{P})(\Gamma_{\mu\rho}^\mu u^\rho u^\nu + \Gamma_{\mu\rho}^\nu u^\rho u^\mu)$

where we also used $d_\mu g^{\mu\nu} = 0$.

Now, only u^0 is non-zero...: taking $\nu = i$, only Γ_{00}^i is needed.

$$\partial^i \mathcal{P} = -(\epsilon + \mathcal{P})\Gamma_{00}^i (u^0)^2 = -\frac{\epsilon + \mathcal{P}}{2g_{00}} \partial^i g_{00} = -(\epsilon + \mathcal{P}) \partial^i \ln \sqrt{-g_{00}}$$

Hydrostatic solutions

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$$\frac{\partial^i \mathcal{P}}{\epsilon + \mathcal{P}} = -\partial^i \ln \sqrt{-g_{00}}$$

Introducing f such that* $\epsilon = f(n)$, $\mathcal{P} = n \frac{df}{dn} - f$ one has

$$\partial^i \mathcal{P} = \frac{df}{dn} \partial^i n + n \frac{d^2 f}{dn^2} \partial^i n - \frac{df}{dn} \partial^i n = n \frac{d^2 f}{dn^2} \partial^i n$$

and
$$\epsilon + \mathcal{P} = n \frac{df}{dn} .$$

👉
$$\frac{\partial^i \mathcal{P}}{\epsilon + \mathcal{P}} = \frac{d^2 f / dn^2}{df / dn} \partial^i n = \partial^i \ln \frac{df}{dn}$$

* valid for single-component perfect fluids

Hydrostatic solutions

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$$\frac{\partial^i \mathcal{P}}{\epsilon + \mathcal{P}} = \partial^i \ln \frac{df}{dn} = -\partial^i \ln \sqrt{-g_{00}}$$

$$-g_{00} \left(\frac{df}{dn} \right)^2 = \text{constant}$$

With the Ansatz $f(n) = \alpha n^{1+c_s^2}$: $n^{2c_s^2} = \frac{\text{const}}{-g_{00}}$

$$\mathcal{P} = c_s^2 \epsilon = \text{const} (-g_{00})^{-\frac{1+c_s^2}{2c_s^2}}$$

Dust ($c_s = 0$): only if $-g_{00}$ is constant.

$c_s^2 = -1$ ("cosmological constant"): ϵ, \mathcal{P} are uniform

$c_s^2 = \frac{1}{3}$ (ultrarelativistic matter): $\mathcal{P} \propto (-g_{00})^{-2}$

Examples of relativistic perfect flows

- Hydrostatic solutions

- Bjorken flow

An example of relativistic flow: “Bjorken flow”

One-dimensional flow along the z -axis with the 3-velocity $v_z = \frac{z}{t}$ for $z < ct$ and $t > t_0$.

First(?) discussed by R.Hwa (1974); made popular by J.D.Bjorken (1983)

PHYSICAL REVIEW D

VOLUME 27, NUMBER 1

1 JANUARY 1983

Highly relativistic nucleus-nucleus collisions: The central rapidity region

J. D. Bjorken

*Fermi National Accelerator Laboratory, * P.O. Box 500, Batavia, Illinois 60510*

(Received 13 August 1982)

The space-time evolution of the hadronic matter produced in the central rapidity region in extreme relativistic nucleus-nucleus collisions is described. We find, in agreement with previous studies, that quark-gluon plasma is produced at a temperature $\gtrsim 200\text{--}300$ MeV, and that it should survive over a time scale $\gtrsim 5$ fm/c. Our description relies on the existence of a flat central plateau and on the applicability of hydrodynamics.

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J.D. Bjorken (Fermilab) (Jul, 1982)

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An example of relativistic flow: “Bjorken flow”

One-dimensional flow along the z -axis with the 3-velocity $v_z = \frac{z}{t}$ for $z < ct$ and $t > t_0$.

👉 introduced to model the boost-invariant* longitudinal expansion of the hot & dense medium created in high-energy nuclear collisions.

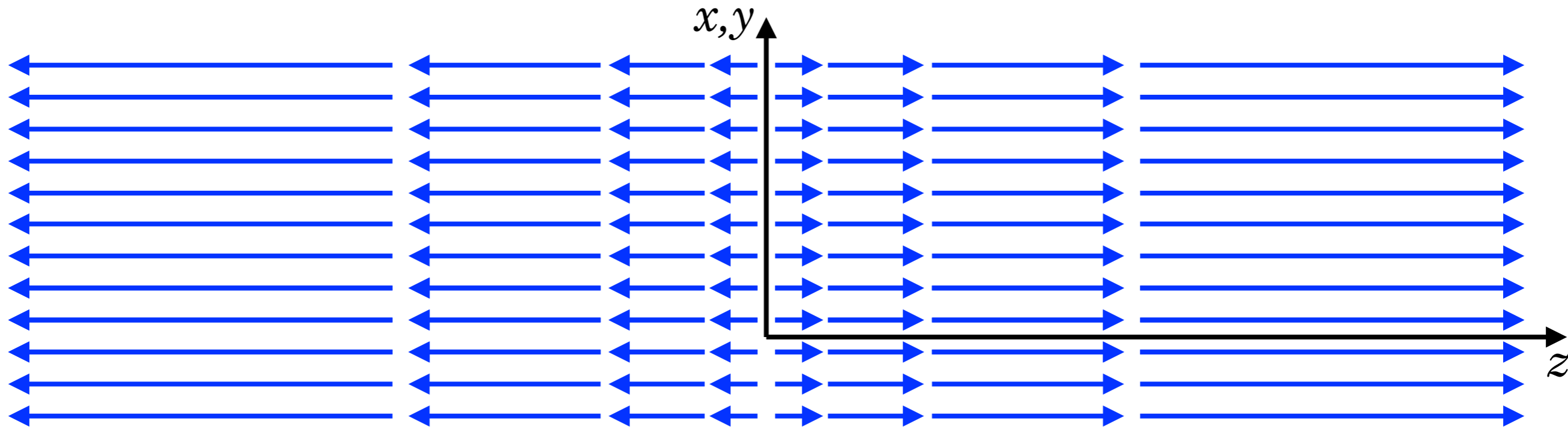
- 🌐 Analytically tractable in a finite amount of time in case the system, modeled as a fluid, has a constant speed of sound c_s
- 🌐 in both cases of a perfect and a dissipative fluid.

*more on this topic later!

An example of relativistic flow: "Bjorken flow"

One-dimensional flow along the z -axis with the 3-velocity $v_z = \frac{z}{t}$ for $z < ct$ and $t > t_0$.

at fixed t :



Milne coordinates

Consider a one-dimensional problem along the z -axis.
A convenient choice of coordinates is often*

$$(x^{0'}, x^{1'}) = (t, z) \quad \rightarrow \quad (x^0, x^1) = (\tau, \varsigma)$$

with

$$\begin{cases} \tau \equiv \sqrt{t^2 - z^2} & \text{"proper time"} \\ \varsigma \equiv \frac{1}{2} \log \frac{t+z}{t-z} & \text{"spacetime rapidity"} \end{cases}$$

Note: from now on, $c=1$...

*Other possibility (for particles with $v \approx c$): light-cone coordinates

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$$\begin{cases} \tau \equiv \sqrt{t^2 - z^2} & \text{"proper time"} \\ \varsigma \equiv \frac{1}{2} \log \frac{t+z}{t-z} & \text{"spatial rapidity"}^\dagger \end{cases} \Leftrightarrow \begin{cases} t = \tau \cosh \varsigma \\ z = \tau \sinh \varsigma \end{cases}$$

Note: from now on, $c=1$...

† with $\frac{1}{2} \log \frac{t+z}{t-z} = \text{Artanh} \frac{z}{t}$

Milne coordinates

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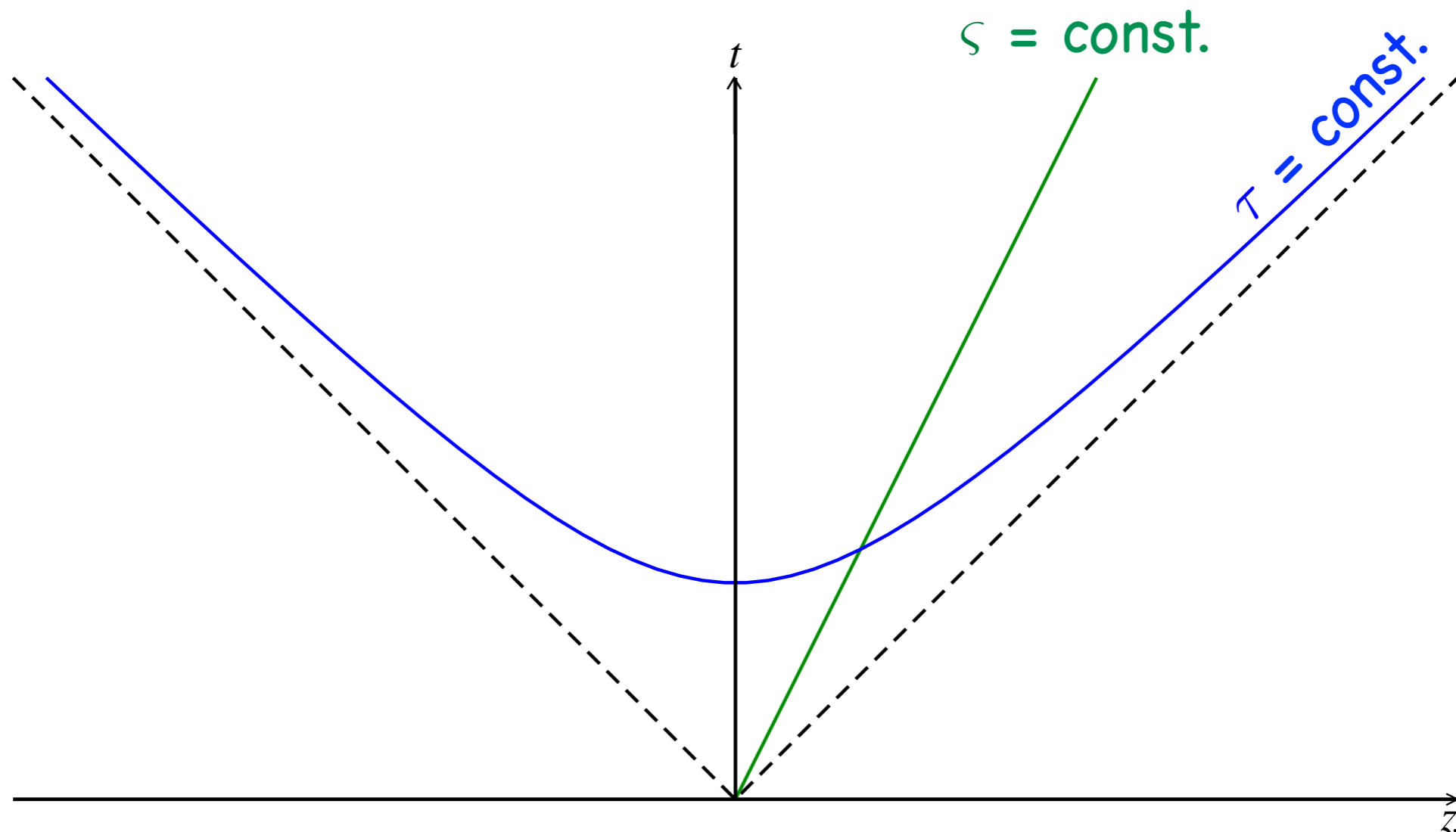
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What do we need?

- transformation of vector / tensor coordinates
- covariant derivatives

Milne coordinates

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Corresponding change of basis: $\{e_{\nu'}\} \rightarrow \{e_{\mu}\}$ with $e_{\mu} = e_{\nu'} \Lambda^{\nu'}_{\mu}$

where $\Lambda^{\nu'}_{\mu} \equiv \frac{\partial x^{\nu'}}{\partial x^{\mu}}$

$$\Lambda^t_{\tau} = \frac{\partial t}{\partial \tau} = \cosh \varsigma$$

$$\Lambda^z_{\tau} = \frac{\partial z}{\partial \tau} = \sinh \varsigma$$

$$\Lambda^t_{\varsigma} = \frac{\partial t}{\partial \varsigma} = \tau \sinh \varsigma$$

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$$\{c^{\nu'}\} \rightarrow \{c^{\mu}\} \quad \text{with} \quad c^{\mu} = \Lambda^{\mu}_{\nu'} c^{\nu'} \quad \text{where} \quad \Lambda^{\mu}_{\nu'} \equiv \frac{\partial x^{\mu}}{\partial x^{\nu'}}$$

$$\begin{cases} e_{\tau} = \cosh \varsigma e_t + \sinh \varsigma e_z \\ e_{\varsigma} = \tau \sinh \varsigma e_t + \tau \cosh \varsigma e_z \end{cases} \quad \begin{cases} c^{\tau} = \cosh \varsigma c^t - \sinh \varsigma c^z \\ c^{\varsigma} = -\frac{1}{\tau} \sinh \varsigma c^t + \frac{1}{\tau} \cosh \varsigma c^z \end{cases}$$

● transformation of vector / tensor coordinates

Milne coordinates

$$\begin{cases} \tau \equiv \sqrt{t^2 - z^2} \\ \varsigma \equiv \frac{1}{2} \log \frac{t+z}{t-z} \end{cases} \Leftrightarrow \begin{cases} t = \tau \cosh \varsigma \\ z = \tau \sinh \varsigma \end{cases}$$

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👉 metric tensor: $g_{\mu\nu} = \mathbf{e}_\mu \cdot \mathbf{e}_\nu$

using $g_{tt} = -g_{zz} = -1$ one finds $g_{\tau\tau} = -1, \quad g_{\varsigma\varsigma} = \tau^2$

$$g_{tz} = g_{zt} = 0 \qquad g_{\tau\varsigma} = g_{\varsigma\tau} = 0$$

Milne coordinates

$$\begin{cases} \tau \equiv \sqrt{t^2 - z^2} \\ \varsigma \equiv \frac{1}{2} \log \frac{t+z}{t-z} \end{cases} \Leftrightarrow \begin{cases} t = \tau \cosh \varsigma \\ z = \tau \sinh \varsigma \end{cases}$$

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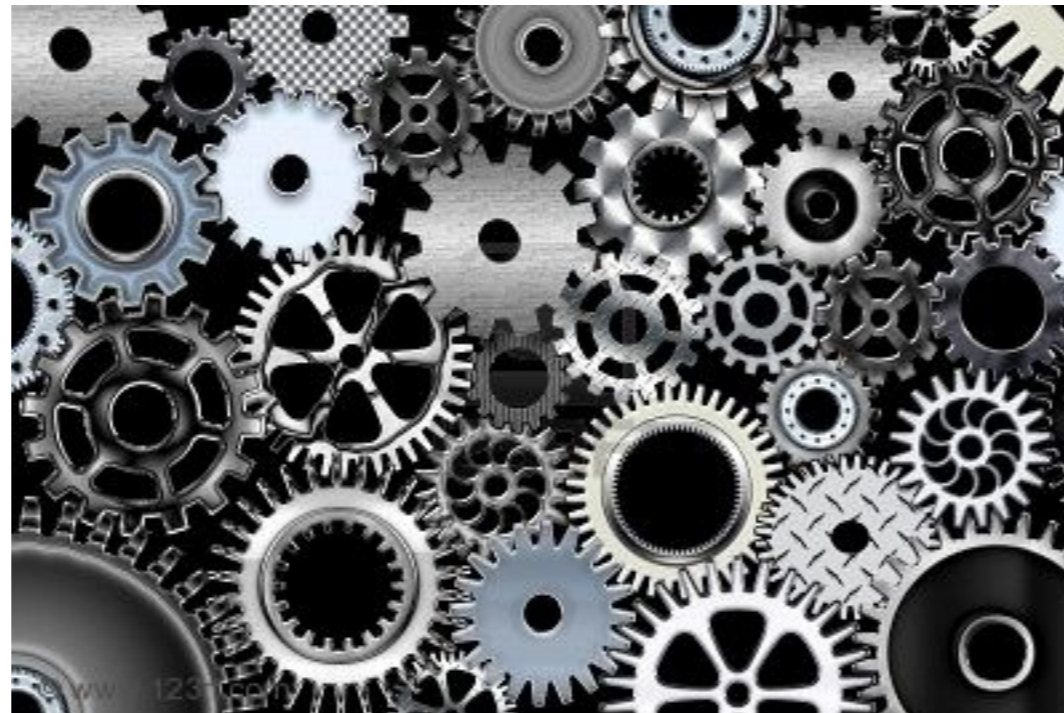
$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & \tau^2 \end{pmatrix}$$

Milne coordinates

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👉 Christoffel symbols: $\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\rho} \left[\frac{\partial g_{\nu\rho}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right]$



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$$\Gamma_{\varsigma\varsigma}^\tau = \tau, \quad \Gamma_{\tau\varsigma}^\varsigma = \Gamma_{\varsigma\tau}^\varsigma = \frac{1}{\tau}, \quad \text{all other coefficients are 0.}$$

🌐 covariant derivatives

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👉 Christoffel symbols: back to the definition $\frac{\partial \mathbf{e}_\nu}{\partial x^\rho} = \Gamma_{\nu\rho}^\mu \mathbf{e}_\mu$

$$\frac{\partial \mathbf{e}_\tau}{\partial \tau} = 0$$

$$\frac{\partial \mathbf{e}_\varsigma}{\partial \tau} = \frac{1}{\tau} \mathbf{e}_\varsigma \equiv \Gamma_{\varsigma\tau}^\varsigma \mathbf{e}_\varsigma$$

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● covariant derivatives $\frac{dc^\mu}{dx^\rho} = \frac{\partial c^\mu}{\partial x^\rho} + \Gamma_{\nu\rho}^\mu c^\nu \dots$

👉 involve the Christoffel symbols $\Gamma_{\varsigma\varsigma}^\tau = \tau$, $\Gamma_{\tau\varsigma}^\varsigma = \Gamma_{\varsigma\tau}^\varsigma = \frac{1}{\tau}$

Bjorken flow

One-dimensional flow along the z -axis with the 3-velocity $v_z = \frac{z}{t}$ for $z < ct$ and $t > t_0$.

👉 4-velocity* $u^{\mu'}(x) = \begin{pmatrix} \gamma(x) \\ \gamma(x)v_z(x) \end{pmatrix}$ with $\gamma(x) = \frac{1}{\sqrt{1 - v_z(x)^2}}$
 $= \frac{t}{\sqrt{t^2 - z^2}}$

*only the non-trivial components are shown

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👉 4-velocity* $u^{\mu'}(x) = \begin{pmatrix} t \\ \frac{t}{\sqrt{t^2 - z^2}} \\ z \\ \frac{z}{\sqrt{t^2 - z^2}} \end{pmatrix} = \begin{pmatrix} \cosh \varsigma \\ \sinh \varsigma \end{pmatrix}$

Milne coordinates $\begin{cases} \tau \equiv \sqrt{t^2 - z^2} \\ \varsigma \equiv \frac{1}{2} \log \frac{t+z}{t-z} \end{cases} \Leftrightarrow \begin{cases} t = \tau \cosh \varsigma \\ z = \tau \sinh \varsigma \end{cases}$

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Bjorken flow

One-dimensional flow along the z -axis with the 3-velocity $v_z = \frac{z}{t}$ for $z < ct$ and $t > t_0$.

👉 4-velocity* $u^{\mu'}(x) = \begin{pmatrix} t \\ \sqrt{t^2 - z^2} \\ z \\ \sqrt{t^2 - z^2} \end{pmatrix} = \begin{pmatrix} \cosh \varsigma \\ \sinh \varsigma \end{pmatrix}$

Change from Minkowski to Milne coordinates:

$$\begin{cases} c^\tau = \cosh \varsigma c^t - \sinh \varsigma c^z \\ c^s = -\frac{1}{\tau} \sinh \varsigma c^t + \frac{1}{\tau} \cosh \varsigma c^z \end{cases}$$

*only the non-trivial components are shown

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Change from Minkowski to Milne coordinates:

$$\begin{cases} u^\tau = \cosh \varsigma u^t - \sinh \varsigma u^z = 1 \\ u^\varsigma = -\frac{1}{\tau} \sinh \varsigma u^t + \frac{1}{\tau} \cosh \varsigma u^z = 0 \end{cases}$$

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Bjorken flow

One-dimensional flow along the z -axis with the 3-velocity $v_z = \frac{z}{t}$ for $z < ct$ and $t > t_0$.

👉 4-velocity* $u^\mu(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in Milne coordinates

Assumption: no conserved quantum number in the game.

👉 only two equations of motion

$$u^\mu(x) d_\mu \epsilon(x) + [\epsilon(x) + \mathcal{P}(x)] d_\mu u^\mu(x) = 0$$

$$[\epsilon(x) + \mathcal{P}(x)] u^\mu(x) d_\mu u^\nu(x) + [g^{\mu\nu}(x) + u^\mu(x) u^\nu(x)] d_\mu \mathcal{P}(x) = 0$$

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Milne coordinates

Covariant derivatives: $d_\rho c^\mu = \partial_\rho c^\mu + \Gamma_{\nu\rho}^\mu c^\nu$

with only the Christoffel symbols $\Gamma_{\varsigma\varsigma}^\tau = \tau$, $\Gamma_{\tau\varsigma}^\varsigma = \Gamma_{\varsigma\tau}^\varsigma = \frac{1}{\tau}$

$$d_\tau c^\mu = \partial_\tau c^\mu + \Gamma_{\nu\tau}^\mu c^\nu$$

$$\text{(no sum)} \begin{cases} d_\tau c^\tau = \partial_\tau c^\tau + \Gamma_{\nu\tau}^\tau c^\nu = \partial_\tau c^\tau \\ \text{☞} \quad d_\tau c^\varsigma = \partial_\tau c^\varsigma + \Gamma_{\nu\tau}^\varsigma c^\nu = \partial_\tau c^\varsigma + \Gamma_{\varsigma\tau}^\varsigma c^\varsigma = \partial_\tau c^\varsigma + \frac{c^\varsigma}{\tau} \end{cases}$$

$$d_\varsigma c^\mu = \partial_\varsigma c^\mu + \Gamma_{\nu\varsigma}^\mu c^\nu$$

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Milne coordinates

Covariant derivatives: $d_\rho c^\mu = \partial_\rho c^\mu + \Gamma_{\nu\rho}^\mu c^\nu$

with only the Christoffel symbols $\Gamma_{\zeta\zeta}^\tau = \tau$, $\Gamma_{\tau\zeta}^\zeta = \Gamma_{\zeta\tau}^\zeta = \frac{1}{\tau}$

$$\begin{cases} d_\tau c^\tau = \partial_\tau c^\tau + \Gamma_{\nu\tau}^\tau c^\nu = \partial_\tau c^\tau \\ d_\tau c^\zeta = \partial_\tau c^\zeta + \Gamma_{\nu\tau}^\zeta c^\nu = \partial_\tau c^\zeta + \Gamma_{\zeta\tau}^\zeta c^\zeta = \partial_\tau c^\zeta + \frac{c^\zeta}{\tau} \end{cases}$$

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👉 4-divergence: $d_\mu c^\mu = d_\tau c^\tau + d_\zeta c^\zeta = \partial_\tau c^\tau + \partial_\zeta c^\zeta + \frac{c^\tau}{\tau}$

Bjorken flow

4-velocity $u^\mu(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in Milne coordinates

4-divergence: $d_\mu c^\mu = d_\tau c^\tau + d_\varsigma c^\varsigma = \partial_\tau c^\tau + \partial_\varsigma c^\varsigma + \frac{c^\tau}{\tau}$

👉 $d_\mu u^\mu(x) = \frac{1}{\tau}$

Bjorken flow

4-velocity $u^\mu(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in Milne coordinates

4-divergence: $d_\mu c^\mu = d_\tau c^\tau + d_\varsigma c^\varsigma = \partial_\tau c^\tau + \partial_\varsigma c^\varsigma + \frac{c^\tau}{\tau}$

$$\Rightarrow d_\mu u^\mu(x) = \frac{1}{\tau}$$

Projection of the 4-gradient on the velocity: $u^\mu(x)d_\mu = u^\tau(x)d_\tau = d_\tau$

Equations of motion:

$$u^\mu(x)d_\mu \epsilon(x) + [\epsilon(x) + \mathcal{P}(x)]d_\mu u^\mu(x) = 0$$

$$[\epsilon(x) + \mathcal{P}(x)]u^\mu(x)d_\mu u^\nu(x) + [g^{\mu\nu}(x) + u^\mu(x)u^\nu(x)]d_\mu \mathcal{P}(x) = 0$$

+ an equation of state $\mathcal{P}(x) = c_s(x)^2 \epsilon(x)$ involving the speed of sound

Bjorken flow

$$u^\mu(\mathbf{x})d_\mu\epsilon(\mathbf{x}) + [\epsilon(\mathbf{x}) + \mathcal{P}(\mathbf{x})]d_\mu u^\mu(\mathbf{x}) = 0$$

$$\text{👉 } d_\tau\epsilon(\mathbf{x}) + \frac{\epsilon(\mathbf{x}) + \mathcal{P}(\mathbf{x})}{\tau} = 0$$

or equivalently

$$d_\tau[\tau\epsilon(\mathbf{x})] = -\mathcal{P}(\mathbf{x})$$

which simply relates the change in the energy in a comoving volume (proportional to τ) to the work of pressure forces...

Bjorken flow

$$u^\mu(x) d_\mu \epsilon(x) + [\epsilon(x) + \mathcal{P}(x)] d_\mu u^\mu(x) = 0$$

$$\Rightarrow d_\tau \epsilon(x) + \frac{\epsilon(x) + \mathcal{P}(x)}{\tau} = 0$$

$$[\epsilon(x) + \mathcal{P}(x)] u^\mu(x) d_\mu u^\nu(x) + [g^{\mu\nu}(x) + u^\mu(x) u^\nu(x)] d_\mu \mathcal{P}(x) = 0$$

$$\Rightarrow [\epsilon(x) + \mathcal{P}(x)] d_\tau u^\nu(x) + [g^{\mu\nu}(x) + u^\mu(x) u^\nu(x)] d_\mu \mathcal{P}(x) = 0$$

$$\nu = \tau : [\epsilon(x) + \mathcal{P}(x)] d_\tau u^\tau(x) = [\epsilon(x) + \mathcal{P}(x)] \partial_\tau u^\tau(x) = 0 \quad \text{already known}$$

$$\nu = \varsigma : [\epsilon(x) + \mathcal{P}(x)] d_\tau u^\varsigma(x) + \frac{1}{\tau^2} d_\varsigma \mathcal{P}(x) = \frac{1}{\tau^2} \partial_\varsigma \mathcal{P}(x) = 0$$

pressure is independent of space-time rapidity

Bjorken flow

$$u^\mu(\mathbf{x})d_\mu\epsilon(\mathbf{x}) + [\epsilon(\mathbf{x}) + \mathcal{P}(\mathbf{x})]d_\mu u^\mu(\mathbf{x}) = 0$$

$$\text{👉 } d_\tau\epsilon(\mathbf{x}) + \frac{\epsilon(\mathbf{x}) + \mathcal{P}(\mathbf{x})}{\tau} = 0$$

Invoking the equation of state $\mathcal{P}(\mathbf{x}) = c_s(\mathbf{x})^2\epsilon(\mathbf{x})$

and using $d_\tau = \partial_\tau$ when acting on scalar fields, one obtains

$$\partial_\tau\epsilon(\mathbf{x}) + [1 + c_s(\mathbf{x})^2]\frac{\epsilon(\mathbf{x})}{\tau} = 0$$

Assuming from now on that the speed of sound is constant:

$$\epsilon(\mathbf{x}) \propto \frac{1}{\tau^{1+c_s^2}} \qquad \mathcal{P}(\mathbf{x}) \propto \frac{1}{\tau^{1+c_s^2}}$$

Both are independent of space-time rapidity

Bjorken flow

Entropy conservation: $d_\mu [s(x)u^\mu(x)] = 0$

$$\text{👉 } d_\tau s(x) + \frac{s(x)}{\tau} = 0$$

which leads at once to $s(x) \propto \frac{1}{\tau}$

Together with $\epsilon(x) \propto \frac{1}{\tau^{1+c_s^2}}$, $\mathcal{P}(x) \propto \frac{1}{\tau^{1+c_s^2}}$ and $\epsilon + \mathcal{P} = Ts$

one obtains for temperature

$$T(x) \propto \frac{1}{\tau c_s^2}$$

Everything is independent of space-time rapidity!

Bjorken flow

We have found $\epsilon(x) \propto \frac{1}{\tau^{1+c_s^2}}$, $\mathcal{P}(x) \propto \frac{1}{\tau^{1+c_s^2}}$, $s(x) \propto \frac{1}{\tau}$

and indirectly $T(x) \propto \frac{1}{\tau^{c_s^2}}$

For an ultrarelativistic gas $\epsilon \propto T^4$ (Stefan-Boltzmann!), $\mathcal{P} \propto T^4$
 $s \propto T^3$ (remember $\epsilon + \mathcal{P} = T s$) and $c_s^2 = \frac{1}{3}$... Everything is OK!