Curvilinear coordinates

Curvilinear coordinates

- Changing coordinates
- Covariant differentiation of vector and tensor fields
- Application (next week): Milne coordinates

Recipes, without proof nor attempt at rigor!

Motivation:
an issue with polar coordinates

Consider the (2D) plane with

- Cartesian coordinates $(x,y) \stackrel{\text{def}}{=} (x^{1'},x^{2'})$ with associated basis vectors $(\vec{\mathrm{e}}_x,\vec{\mathrm{e}}_y) \stackrel{\text{def}}{=} (\vec{\mathrm{e}}_{1'},\vec{\mathrm{e}}_{2'})$ — note the at first bizarre primed indices as well as their positions; ☞ the reference!
- the usual polar coordinates $(r,\theta) \stackrel{\text{def}}{=} (x^1,x^2)$ with the associated basis vectors $(\vec{\mathrm{e}}_r,\vec{\mathrm{e}}_\theta) \stackrel{\text{def}}{=} (\vec{\mathrm{e}}_1,\vec{\mathrm{e}}_2)$ — here with unprimed indices.

The position $\vec{R}$ of a point P of the plane may be denoted

$$\vec{R} = x\,\vec{\mathrm{e}}_x + y\,\vec{\mathrm{e}}_y = \sum_{i'} x^{i'}\vec{\mathrm{e}}_{i'} = x^{i'}\vec{\mathrm{e}}_{i'}$$

But you know that you would NOT write $\vec{R} = r\,\vec{\mathrm{e}}_r + \theta\,\vec{\mathrm{e}}_\theta = x^i\,\vec{\mathrm{e}}_i\,.$ 🤔

Motivation:
an issue with polar coordinates

If you look at two neighboring points P and P + dP and call $\mathrm{d}\vec{R}$ their separation, you will also write (Cartesian coordinates)

$$\mathrm{d}\vec{R} = \mathrm{d}x\,\vec{\mathrm{e}}_x + \mathrm{d}y\,\vec{\mathrm{e}}_y = \sum_{i'} \mathrm{d}x^{i'}\,\vec{\mathrm{e}}_{i'} = \mathrm{d}x^{i'}\,\vec{\mathrm{e}}_{i'}$$

And perhaps are you then ready to write (in polar coordinates)

$$\mathrm{d}\vec{R} = \mathrm{d}r\,\vec{\mathrm{e}}_r + \mathrm{d}\theta\,\vec{\mathrm{e}}_\theta$$

so that at least one formula:

$$\boxed{\mathrm{d}\vec{R} = \mathrm{d}x^i\,\vec{\mathrm{e}}_i} \qquad (1)$$

is valid in every coordinate system.

Fine? Let us see where this is leading to...

Separation vector between neighboring points in arbitrary coordinates

It is tempting to write $\mathrm{d}\vec{R} = \dfrac{\partial\vec{R}}{\partial x^i}\,\mathrm{d}x^i$.

An issue is that we know how to express $\vec{R}$ as a function of the Cartesian coordinates $\{x^{i'}\}$, but in general not in terms of arbitrary coordinates $\{x^i\}$.

But... we generally know how to express the $\{x^i\}$ in terms of the $\{x^{i'}\}$ and reciprocally: functions $x^i(\{x^{i'}\})$ and $x^{i'}(\{x^i\})$.

For instance: $r(x,y) = \sqrt{x^2+y^2}\ ,\ \theta(x,y) = \arctan(y/x)$

and $x(r,\theta) = r\cos\theta\ ,\ y(r,\theta) = r\sin\theta$

Separation vector between neighboring points in arbitrary coordinates

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☞ We may write $\vec{R}(\{x^i\}) = x(\{x^i\})\,\vec{\mathrm{e}}_x + y(\{x^i\})\,\vec{\mathrm{e}}_y = x^{i'}(\{x^i\})\,\vec{\mathrm{e}}_{i'}$

and now differentiate w.r.t. x^i:

$$\frac{\partial\vec{R}}{\partial x^i} = \frac{\partial x^{i'}}{\partial x^i}\,\vec{\mathrm{e}}_{i'} \qquad\Rightarrow\qquad \mathrm{d}\vec{R} = \frac{\partial x^{i'}}{\partial x^i}\,\vec{\mathrm{e}}_{i'}\,\mathrm{d}x^i \tag{3}$$

Basis vectors of an arbitrary coordinate system

Identifying Eqs. (1) and (3), we find the “proper” basis vectors $\{\vec{\mathrm{e}}_i\}$:

$$\left.\begin{aligned} \mathrm{d}\vec{R} &= \mathrm{d}x^i\,\vec{\mathrm{e}}_i \\ \mathrm{d}\vec{R} &= \frac{\partial x^{i'}}{\partial x^i}\,\vec{\mathrm{e}}_{i'}\,\mathrm{d}x^i \end{aligned}\right\} \qquad \boxed{\vec{\mathrm{e}}_i = \frac{\partial x^{i'}}{\partial x^i}\vec{\mathrm{e}}_{i'}} \tag{4}$$

For the polar coordinates: $x(r,\theta) = r\cos\theta\ ,\ \ y(r,\theta) = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta\ ,\ \frac{\partial y}{\partial r} = \sin\theta \qquad \Rightarrow \qquad \vec{\mathrm{e}}_r = \cos\theta\,\vec{\mathrm{e}}_x + \sin\theta\,\vec{\mathrm{e}}_y$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta\ ,\ \frac{\partial y}{\partial \theta} = r\cos\theta \qquad \Rightarrow \qquad \vec{\mathrm{e}}_\theta = -r\sin\theta\,\vec{\mathrm{e}}_x + r\cos\theta\,\vec{\mathrm{e}}_y$$

Well $\vec{\mathrm{e}}_r$ is fine and I knew it, but… this $\vec{\mathrm{e}}_\theta$ is not normalized to 1! 😲

Motivation:
a problem with polar coordinates

For the polar coordinates: $x(r,\theta) = r\cos\theta\ ,\ y(r,\theta) = r\sin\theta$

$$\vec{\mathrm{e}}_r = \cos\theta\,\vec{\mathrm{e}}_x + \sin\theta\,\vec{\mathrm{e}}_y \qquad\qquad \vec{\mathrm{e}}_\theta = -r\sin\theta\,\vec{\mathrm{e}}_x + r\cos\theta\,\vec{\mathrm{e}}_y$$

Well $\vec{\mathrm{e}}_r$ is fine and I knew it, but… this $\vec{\mathrm{e}}_\theta$ is not normalized to 1! 😲

Indeed! But who said it should? 🤓

Now $\vec{\mathrm{e}}_\theta$ is well-behaved in the limit $r \to 0$, when θ is not defined: isn't this a nice feature, worth discarding the normalization? 😇

(the perplexed students): OK, we'll live with it… 🤨

Metric tensor

Start from eq.(1): $\mathrm{d}\vec{R} = \mathrm{d}x^i\,\vec{\mathrm{e}}_i$ and square it:

$$\mathrm{d}\vec{R}^{\,2} = (\mathrm{d}x^i\vec{\mathrm{e}}_i)\cdot(\mathrm{d}x^j\vec{\mathrm{e}}_j) = (\vec{\mathrm{e}}_i\cdot\vec{\mathrm{e}}_j)\mathrm{d}x^i\,\mathrm{d}x^j$$

Define $\boxed{g_{ij} \equiv \vec{\mathrm{e}}_i\cdot\vec{\mathrm{e}}_j}$ and replace: $\boxed{\mathrm{d}\vec{R}^{\,2} = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j}$

- Cartesian coordinates: $g_{i'j'} = \delta_{i'j'} \quad\Leftrightarrow\quad \mathrm{d}\vec{R}^{\,2} = (\mathrm{d}x)^2 + (\mathrm{d}y)^2$
- Polar coordinates:* $g_{11} = g_{rr} = 1\ ,\ g_{22} = g_{\theta\theta} = r^2\ ,\ g_{r\theta} = g_{\theta r} = 0$

$$\Leftrightarrow\quad \mathrm{d}\vec{R}^{\,2} = (\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2$$

looks familiar...

g_{ij} is called metric tensor.

(More precisely, these are its components, but let's not be picky today).

* Using the expressions of $\vec{\mathrm{e}}_r$, $\vec{\mathrm{e}}_\theta$ in terms of $\vec{\mathrm{e}}_x$, $\vec{\mathrm{e}}_y$, since we know how to compute the scalar product in Cartesian coordinates.

Metric tensor and its inverse

Define g^{ij} such that its "product" (more accurately: contraction) with the metric tensor equals the identity:

$$g^{ij}g_{jk} = \delta^i_k$$

g^{ij} is called the inverse metric tensor.

Cartesian coordinates: $g^{i'j'} = \delta^{i'j'}$; polar coordinates: $g^{rr} = 1$, $g^{\theta\theta} = \dfrac{1}{r^2}$.

g_{ij} and g^{ij} can be used to lower / raise indices:

- No impact on Cartesian components.
- In polar coordinates: no impact on r-components, but θ-components vary.

☞ What does it mean? Not much today...

Next step: generalize!

Starting from coordinates $\{x^{i'}\}$ with orthogonal* and normalized* basis vectors $\{\vec{\mathrm{e}}_{i'}\}$, one goes to alternative coordinates $\{x^i\}$:

- with the basis vectors $\vec{\mathrm{e}}_i = \dfrac{\partial x^{i'}}{\partial x^i}\vec{\mathrm{e}}_{i'}$ such that $\mathrm{d}\vec{R} = \mathrm{d}x^i\,\vec{\mathrm{e}}_i$ holds
- and the metric tensor $g_{ij} \equiv \vec{\mathrm{e}}_i \cdot \vec{\mathrm{e}}_j$ such that $\mathrm{d}\vec{R}^{\,2} = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$.

We may do that:

- ... in 2-dimensional Euclidean space: $g_{i'j'}$ = diag(+1, +1)
- ... in 3-dimensional Euclidean space: $g_{i'j'}$ = diag(+1, +1, +1)
- ... in 4-dimensional Minkowski space: $g_{i'j'}$ = diag(−1, +1, +1, +1)

* Using the appropriate (pseudo)scalar product encoded in $g_{i'j'}$

Second motivation: another issue with polar coordinates

How should one differentiate?

Consider a (smooth enough) function f.

Its derivatives are supposed to measure the variations of the function in the various respective directions:

$$f(x+\delta x, y) \simeq f(x,y) + \frac{\partial f}{\partial x}\delta x$$

and so on...

Second motivation: another issue with polar coordinates

How should one differentiate?

Consider a (smooth enough) function $f(r,\theta)$.

Its derivatives are supposed to measure the variations of the function in the various respective directions:

$$f(r,\theta+\delta\theta) \simeq f(r,\theta) + \frac{\partial f}{\partial\theta}\delta\theta$$

and so on...

Second motivation: another issue with polar coordinates

For instance: $\vec{f}(r,\theta) = \cos\theta\,\vec{\mathrm{e}}_r - \dfrac{\sin\theta}{r}\vec{\mathrm{e}}_\theta$

with (slide 6) $\vec{\mathrm{e}}_r = \cos\theta\,\vec{\mathrm{e}}_x + \sin\theta\,\vec{\mathrm{e}}_y$ and $\vec{\mathrm{e}}_\theta = -r\sin\theta\,\vec{\mathrm{e}}_x + r\cos\theta\,\vec{\mathrm{e}}_y$

Let us differentiate:

$$\frac{\partial\vec{f}(r,\theta)}{\partial\theta} = \frac{\partial\cos\theta}{\partial\theta}\,\vec{\mathrm{e}}_r - \frac{1}{r}\frac{\partial\sin\theta}{\partial\theta}\vec{\mathrm{e}}_\theta = -\sin\theta\,\vec{\mathrm{e}}_r - \frac{\cos\theta}{r}\vec{\mathrm{e}}_\theta$$

OK, fine!

Wait! $\vec{f}(r,\theta) = \cos\theta\,\vec{\mathrm{e}}_r - \dfrac{\sin\theta}{r}\vec{\mathrm{e}}_\theta$

$$= \cos\theta\big(\cos\theta\,\vec{\mathrm{e}}_x + \sin\theta\,\vec{\mathrm{e}}_y\big) - \frac{\sin\theta}{r}\big(-r\sin\theta\,\vec{\mathrm{e}}_x + r\cos\theta\,\vec{\mathrm{e}}_y\big)$$

$= \vec{\mathrm{e}}_x$ = $\overrightarrow{\text{constant}}$! How can its derivative be ≠ $\vec{0}$?

Second motivation: another issue with polar coordinates

For instance: $\vec{f}(r,\theta) = \cos\theta\,\vec{\mathrm{e}}_r - \dfrac{\sin\theta}{r}\vec{\mathrm{e}}_\theta$

with (slide 6) $\boxed{\vec{\mathrm{e}}_r = \cos\theta\,\vec{\mathrm{e}}_x + \sin\theta\,\vec{\mathrm{e}}_y}$ and $\boxed{\vec{\mathrm{e}}_\theta = -r\sin\theta\,\vec{\mathrm{e}}_x + r\cos\theta\,\vec{\mathrm{e}}_y}$

are also functions of (r,θ)!

Let us differentiate:

$$\xcancel{\frac{\partial\vec{f}(r,\theta)}{\partial\theta} = \frac{\partial\cos\theta}{\partial\theta}\vec{\mathrm{e}}_r - \frac{1}{r}\frac{\partial\sin\theta}{\partial\theta}\vec{\mathrm{e}}_\theta = -\sin\theta\,\vec{\mathrm{e}}_r - \frac{\cos\theta}{r}\vec{\mathrm{e}}_\theta}$$

Use the product rule!

OK, fine!

Wait! $\vec{f}(r,\theta) = \cos\theta\,\vec{\mathrm{e}}_r - \dfrac{\sin\theta}{r}\vec{\mathrm{e}}_\theta$

$$= \cos\theta\big(\cos\theta\,\vec{\mathrm{e}}_x + \sin\theta\,\vec{\mathrm{e}}_y\big) - \frac{\sin\theta}{r}\big(-r\sin\theta\,\vec{\mathrm{e}}_x + r\cos\theta\,\vec{\mathrm{e}}_y\big)$$

$= \vec{\mathrm{e}}_x$ = $\overrightarrow{\text{constant}}$! How can its derivative be ≠ $\vec{0}$

Second motivation: another issue with polar coordinates

For instance: vector field $\vec{c}(r,\theta) = c^r(r,\theta)\vec{\mathrm{e}}_r(r,\theta) + c^\theta(r,\theta)\vec{\mathrm{e}}_\theta(r,\theta)$

To differentiate w.r.t. r or θ, one must not forget to differentiate the basis vectors…

Every time? But that is time-consuming!

There is a nice trick!

Covariant derivative

Consider the (position dependent: P) basis vectors $\{\vec{\mathrm{e}}_i(P)\}$ associated with coordinates $\{x^i\}$.

The partial derivative $\dfrac{\partial\vec{\mathrm{e}}_i(P)}{\partial x^j}$ is itself a vector at P:

☞ can be written as linear combination of the $\{\vec{\mathrm{e}}_i(P)\}$:

$$\boxed{\frac{\partial\vec{\mathrm{e}}_i(P)}{\partial x^j} = \Gamma^k_{ij}(P)\vec{\mathrm{e}}_k(P)}$$

with coefficients Γ^k_{ij}: Christoffel symbols. (symmetric under $i\leftrightarrow j$)

Consider now a vector field $\vec{c}(P) = c^i(P)\vec{\mathrm{e}}_i(P)$

One has $\dfrac{\partial\vec{c}(P)}{\partial x^j} = \dfrac{\mathrm{d}c^i(P)}{\mathrm{d}x^j}\vec{\mathrm{e}}_i(P)$ with $\boxed{\dfrac{\mathrm{d}c^i(P)}{\mathrm{d}x^j} \equiv \dfrac{\partial c^i(P)}{\partial x^j} + \Gamma^i_{jk}(P)c^k(P)}$

Covariant derivative

For a vector field $\vec{c}(P) = c^i(P)\vec{\mathrm{e}}_i(P)$ one has $\dfrac{\partial\vec{c}(P)}{\partial x^j} = \dfrac{\mathrm{d}c^i(P)}{\mathrm{d}x^j}\vec{\mathrm{e}}_i(P)$ with the covariant derivatives

$$\boxed{\frac{\mathrm{d}c^i(P)}{\mathrm{d}x^j} \equiv \frac{\partial c^i(P)}{\partial x^j} + \Gamma^i_{jk}(P)\,c^k(P)}$$

Proof:

$$\frac{\partial\vec{c}(P)}{\partial x^j} = \frac{\partial c^i(P)}{\partial x^j}\vec{\mathrm{e}}_i(P) + c^i(P)\frac{\partial\vec{\mathrm{e}}_i(P)}{\partial x^j} = \frac{\partial c^i(P)}{\partial x^j}\vec{\mathrm{e}}_i(P) + \underbrace{c^i(P)\,\Gamma^k_{ij}(P)\,\vec{\mathrm{e}}_k(P)}_{c^i\,\Gamma^k_{ij}\vec{\mathrm{e}}_k \,=\, c^k\,\Gamma^i_{jk}\vec{\mathrm{e}}_i}$$

$$\frac{\partial\vec{c}(P)}{\partial x^j} = \frac{\partial c^i(P)}{\partial x^j}\vec{\mathrm{e}}_i(P) + c^k(P)\,\Gamma^i_{jk}(P)\vec{\mathrm{e}}_i(P) = \frac{\mathrm{d}c^i(P)}{\mathrm{d}x^j}\vec{\mathrm{e}}_i(P)$$

Covariant derivative: polar coordinates

From $\vec{\mathrm{e}}_r(r,\theta) = \cos\theta\,\vec{\mathrm{e}}_x + \sin\theta\,\vec{\mathrm{e}}_y \quad , \quad \vec{\mathrm{e}}_\theta(r,\theta) = -r\sin\theta\,\vec{\mathrm{e}}_x + r\cos\theta\,\vec{\mathrm{e}}_y$

one computes

$$\frac{\partial\vec{\mathrm{e}}_r(r,\theta)}{\partial r} = \vec{0}\ ,\quad \frac{\partial\vec{\mathrm{e}}_r(r,\theta)}{\partial\theta} = \frac{1}{r}\vec{\mathrm{e}}_\theta(r,\theta)\ ,\quad \frac{\partial\vec{\mathrm{e}}_\theta(r,\theta)}{\partial r} = \frac{1}{r}\vec{\mathrm{e}}_\theta(r,\theta)\ ,\quad \frac{\partial\vec{\mathrm{e}}_\theta(r,\theta)}{\partial\theta} = -r\vec{\mathrm{e}}_r(r,\theta)$$

These derivatives are linear combinations $\dfrac{\partial\vec{\mathrm{e}}_i(P)}{\partial x^j} = \Gamma^k_{ij}(P)\vec{\mathrm{e}}_k(P)$ with the Christoffel symbols

$$\Gamma^r_{rr} = \Gamma^\theta_{rr} = 0,\quad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r},\quad \Gamma^r_{\theta\theta} = -r,\quad \Gamma^r_{r\theta} = \Gamma^r_{\theta r} = 0,\quad \Gamma^\theta_{\theta\theta} = 0$$

☞ $$\left\{\begin{aligned} \frac{\mathrm{d}c^i}{\mathrm{d}r} &= \frac{\partial c^i}{\partial r} + \Gamma^i_{rk}c^k = \frac{\partial c^i}{\partial r} + \Gamma^i_{r\theta}c^\theta = \frac{\partial c^i}{\partial r} + \frac{c^\theta}{r}\delta^{i\theta} \\[2ex] \frac{\mathrm{d}c^i}{\mathrm{d}\theta} &= \frac{\partial c^i}{\partial\theta} + \Gamma^i_{\theta k}c^k = \frac{\partial c^i}{\partial\theta} + \Gamma^i_{\theta r}c^r + \Gamma^i_{\theta\theta}c^\theta = \frac{\partial c^i}{\partial\theta} + \frac{c^r}{r}\delta^{i\theta} - rc^\theta\delta^{ir} \end{aligned}\right.$$

Covariant derivative: polar coordinates

Consider again the example

$$\vec{c}(r,\theta) = \cos\theta\,\vec{\mathrm{e}}_r - \frac{\sin\theta}{r}\vec{\mathrm{e}}_\theta \quad (= \vec{\mathrm{e}}_x)$$

i.e. with components $c^r = \cos\theta\ ,\ c^\theta = -\dfrac{\sin\theta}{r}$

and let us compute the covariant derivatives w.r.t r and θ:

- w.r.t r: $\dfrac{\mathrm{d}c^i}{\mathrm{d}r} = \dfrac{\partial c^i}{\partial r} + \dfrac{c^\theta}{r}\delta^{i\theta}$ gives $\dfrac{\mathrm{d}c^r}{\mathrm{d}r} = \dfrac{\partial c^r}{\partial r} = 0$, $\dfrac{\mathrm{d}c^\theta}{\mathrm{d}r} = \dfrac{\partial c^\theta}{\partial r} + \dfrac{c^\theta}{r} = 0$
- w.r.t. θ: $\dfrac{\mathrm{d}c^i}{\mathrm{d}\theta} = \dfrac{\partial c^i}{\partial \theta} + \dfrac{c^r}{r}\delta^{i\theta} - c^\theta\delta^{ir}$ gives

  $\dfrac{\mathrm{d}c^r}{\mathrm{d}\theta} = \dfrac{\partial c^r}{\partial \theta} - rc^\theta = 0$ and $\dfrac{\mathrm{d}c^\theta}{\mathrm{d}\theta} = \dfrac{\partial c^\theta}{\partial \theta} + \dfrac{c^r}{r} = 0$

All covariant derivatives vanish! 😀

Covariant derivative: rules for arbitrary tensor fields

- Scalar field: $\dfrac{\mathrm{d}c(P)}{\mathrm{d}x^j} \equiv \dfrac{\partial c(P)}{\partial x^j}$

- "Contravariant vector": $\dfrac{\mathrm{d}c^i(P)}{\mathrm{d}x^j} \equiv \dfrac{\partial c^i(P)}{\partial x^j} + \Gamma^i_{jk}(P)\,c^k(P)$

- "Covariant vector": $\dfrac{\mathrm{d}c_i(P)}{\mathrm{d}x^j} \equiv \dfrac{\partial c_i(P)}{\partial x^j} - \Gamma^k_{ij}(P)\,c_k(P)$

- Arbitrary (m contra-, n covariant indices) tensor field:

$$\frac{\mathrm{d}\mathbf{T}^{i_1\cdots i_m}_{j_1\ldots j_n}(P)}{\mathrm{d}x^k} = \frac{\partial\mathbf{T}^{i_1\cdots i_m}_{j_1\ldots j_n}(P)}{\partial x^k} + \Gamma^{i_1}_{kl}(P)\,\mathbf{T}^{l i_2\cdots i_m}_{j_1\ldots j_n}(P) + \cdots + \Gamma^{i_m}_{kl}(P)\,\mathbf{T}^{i_1\cdots i_{m-1}l}_{j_1\ldots j_n}(P)$$

$$- \Gamma^{l}_{j_1 k}(P)\,\mathbf{T}^{i_1\cdots i_m}_{l j_2\ldots j_n}(P) - \cdots - \Gamma^{l}_{j_n k}(P)\,\mathbf{T}^{i_1\cdots i_m}_{j_1\ldots j_{n-1}l}(P)$$

A last formula

To compute the Christoffel symbols with the help of the metric tensor and its inverse ☞ no need to go back to the expression of the basis vectors $\{\vec{\mathrm{e}}_i\}$ in terms of those of the Cartesian / Minkowski basis.

$$\Gamma^k_{ij}(P) = \frac{1}{2} g^{kl}(P) \left[\frac{\partial g_{jl}(P)}{\partial x^i} + \frac{\partial g_{il}(P)}{\partial x^j} - \frac{\partial g_{ij}(P)}{\partial x^l} \right]$$

☞ easy to automatize (symbolic algebra programs)