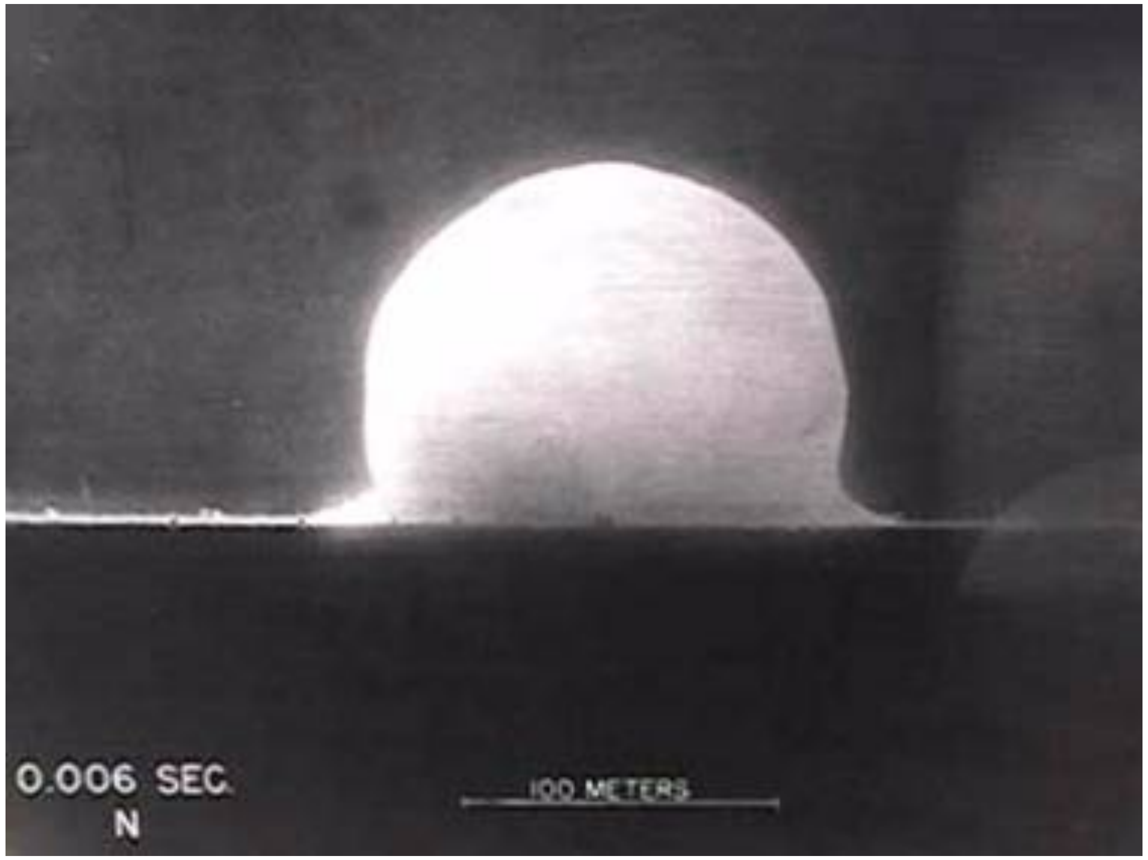
Dimensional analysis: a neat example

Expansion velocity of a shock front

How fast will a shock front arising from the sudden release of a high amount of energy expand in a fluid?

(Original reasoning: G.I.Taylor, Proc. Roy. Soc. A 201 (1950) 175)

Motivation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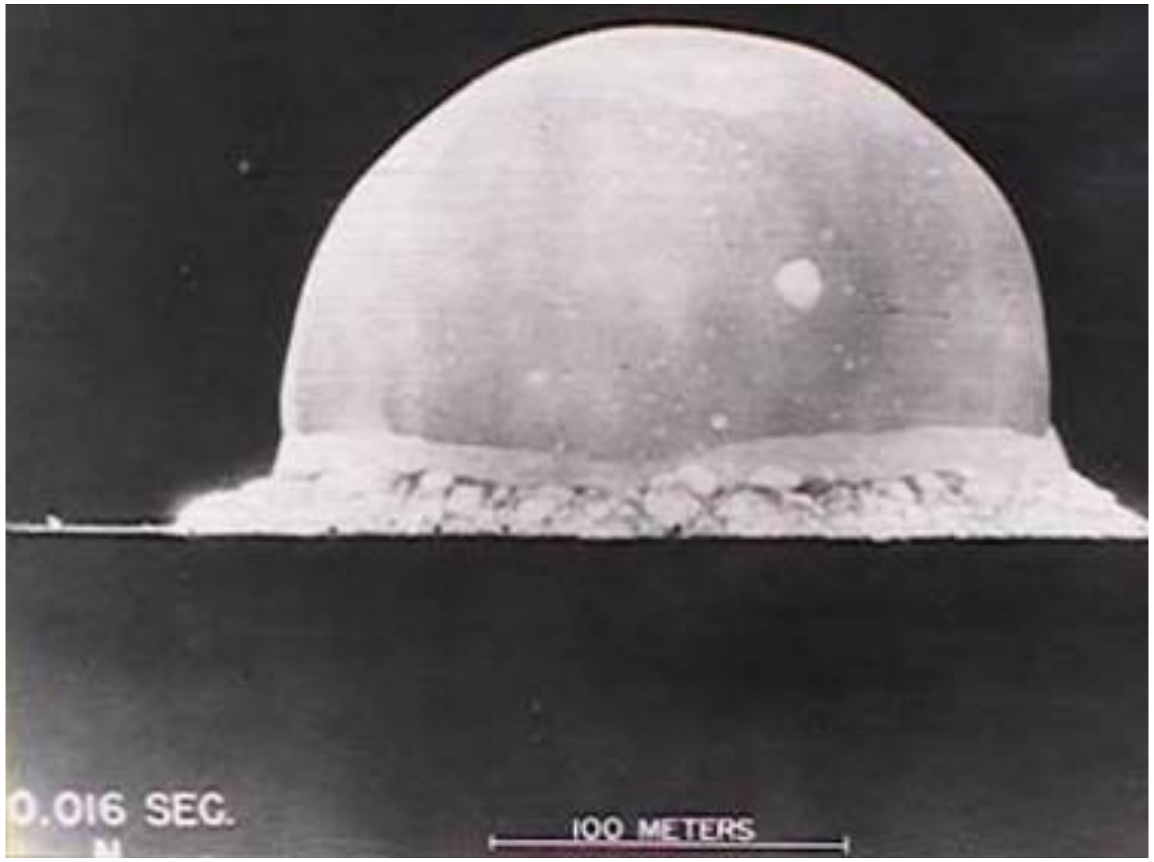
N
100 METERS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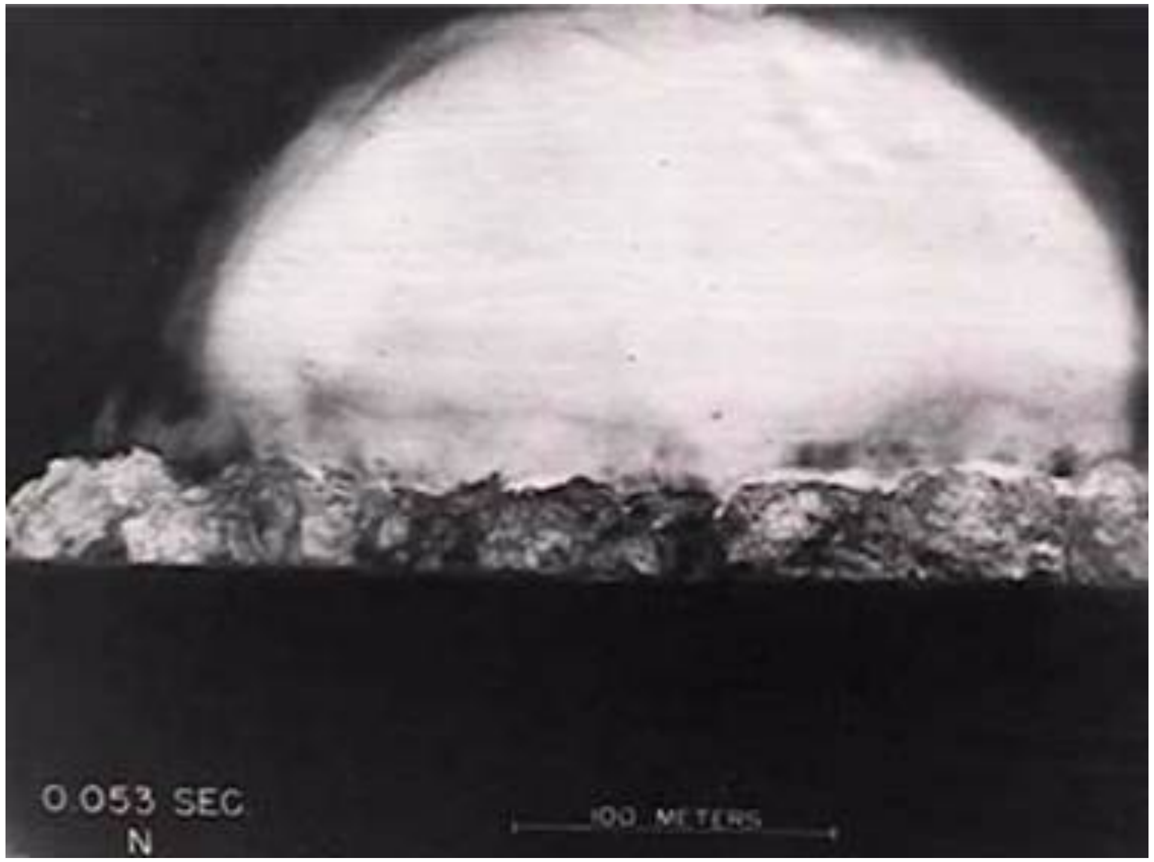
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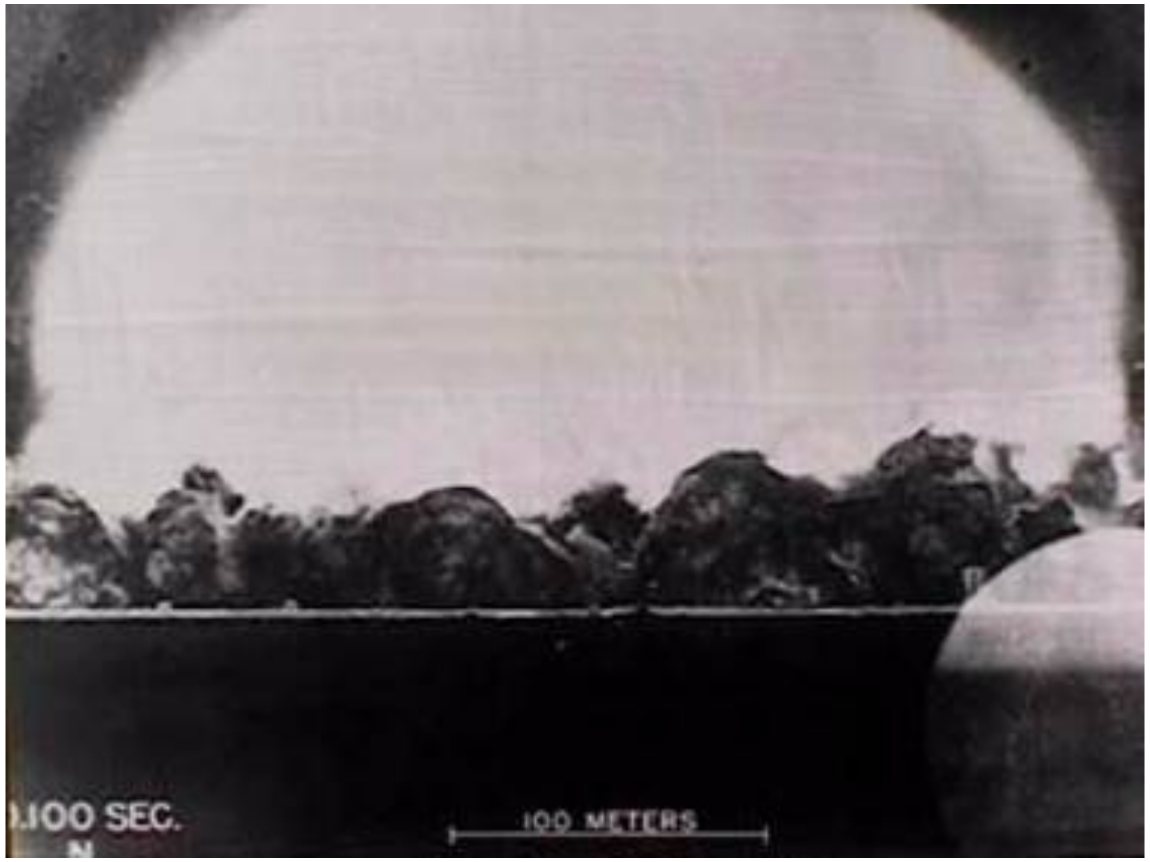
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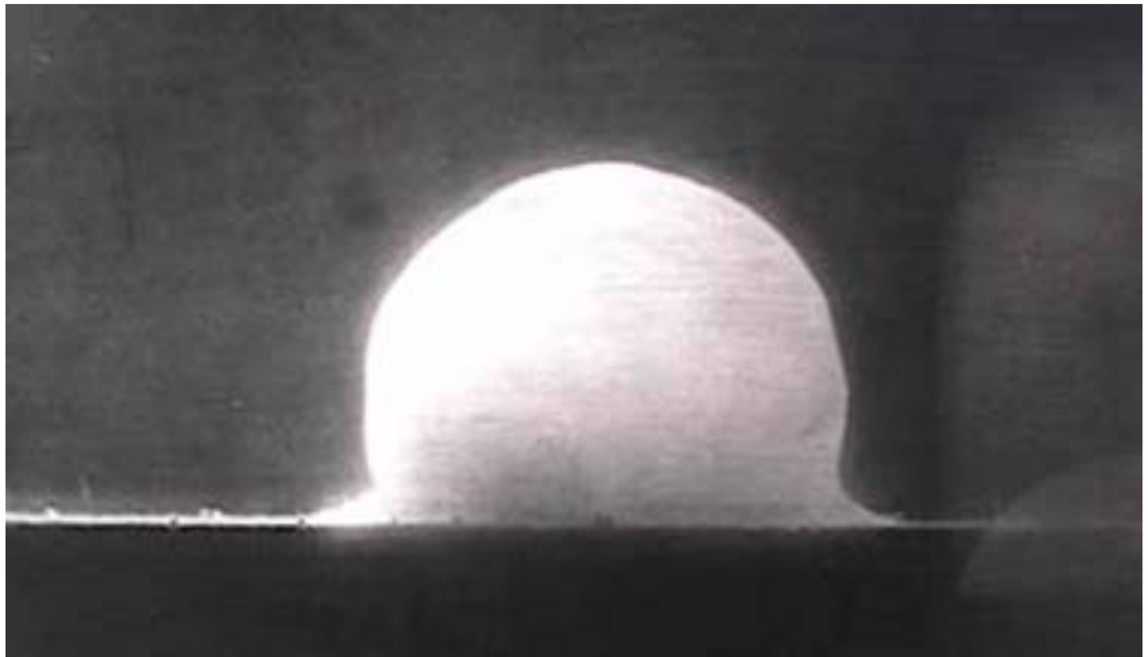
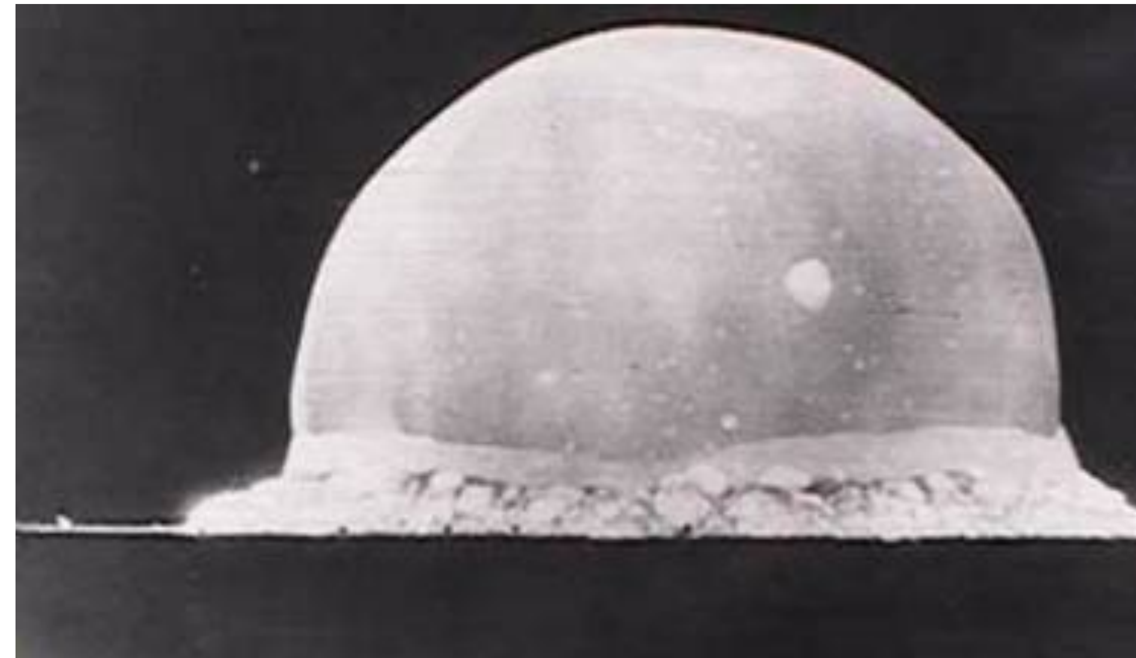
100 METERS

Motivation

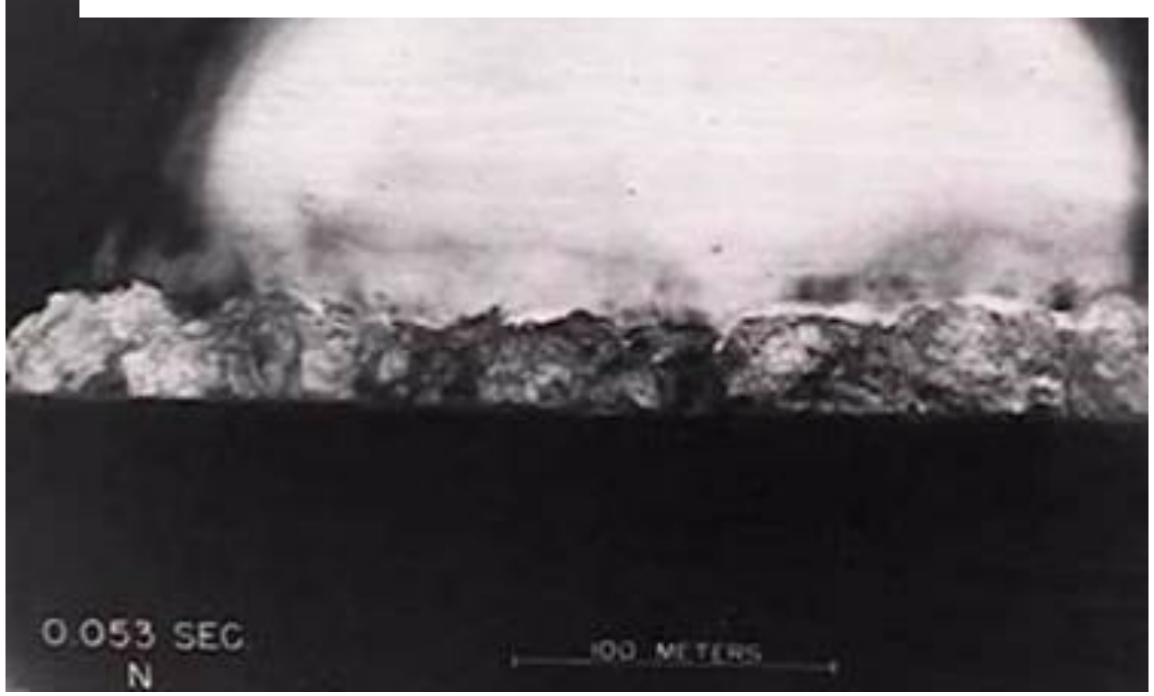


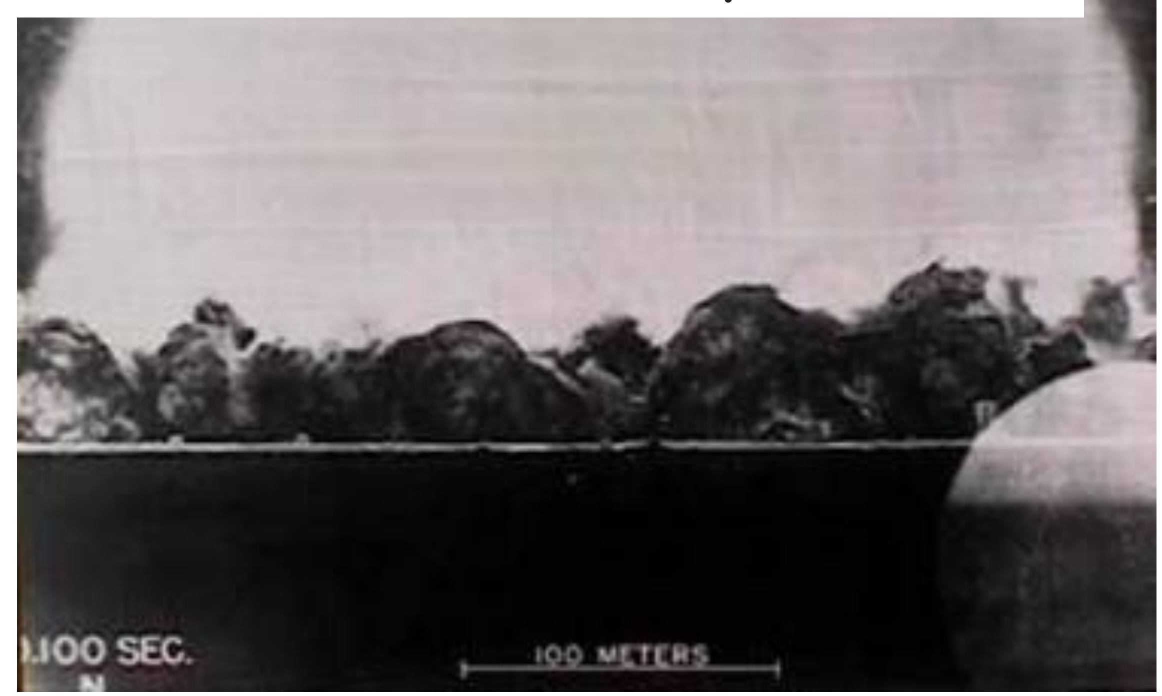


More pictures at

http://nuclearweaponarchive.org/Usa/Tests/Trinity.html

☞ the first test, July 15, 1945





Expansion velocity of a shock front

How fast will a shock front arising from the sudden release of a high amount of energy expand in a fluid?

Relevant physical quantities:

- Energy E: $[E]$ = $\mathrm{M L^2 T^{-2}}$
  assumed to be released instantaneously (no t_c) in an effectively pointlike (no L_c) region
- Radius $R(t)$ of the approximately spherical expansion: $[R]$ = L, $[t]$ = T
- Mass density ρ of the fluid in which the front expands: $[\rho]$ = $\mathrm{M L^{-3}}$
  here: air

t, E, and ρ are dimensionally independent: rank r = 3.

☞ search $R = f(t, E, \rho)$

Expansion velocity of a shock front

t, E, and ρ are dimensionally independent.

☞ Only possible characteristic length: $\dfrac{E^{1/5}\,t^{2/5}}{\rho^{1/5}}$

Check:

$$\left[\frac{E^{1/5}\,t^{2/5}}{\rho^{1/5}}\right] = \frac{\left(\mathsf{M}\,\mathsf{L}^2\,\mathsf{T}^{-2}\right)^{1/5}\ \mathsf{T}^{2/5}}{\left(\mathsf{M}\,\mathsf{L}^{-3}\right)^{1/5}} = \mathsf{M}^{1/5-1/5}\,\mathsf{L}^{2/5+3/5}\,\mathsf{T}^{-2/5+2/5} = \mathsf{L}$$

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☞ π-theorem: $$\frac{R}{E^{1/5}\, t^{2/5}/\rho^{1/5}} = \mathsf{f}^*() = \text{constant} \equiv C$$

i.e.

$$R(t) = C\,\frac{E^{1/5}\, t^{2/5}}{\rho^{1/5}}$$

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determined by initial condition (Taylor: $C \approx 1$)

Expansion velocity of a shock front

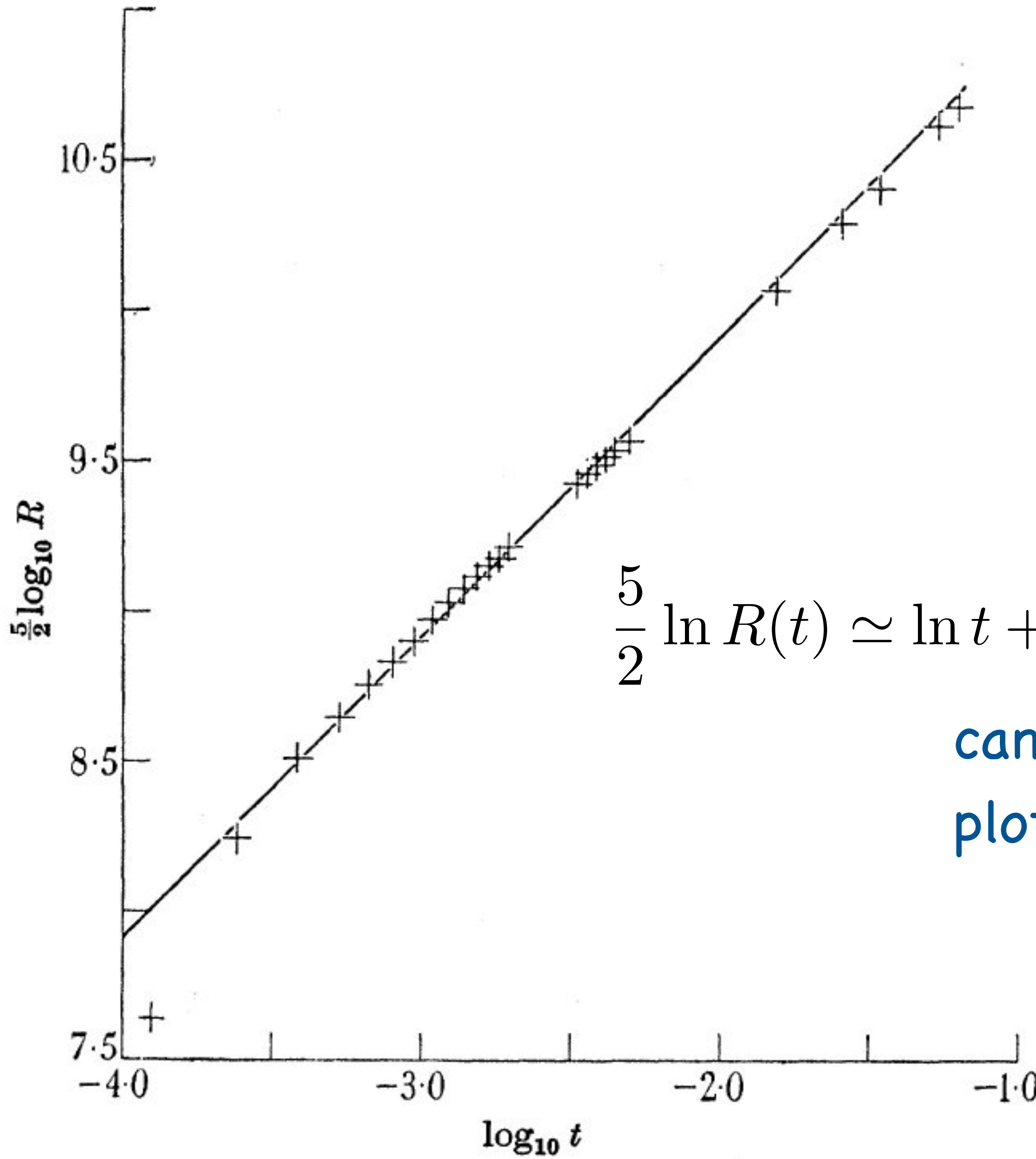

FIGURE 1. Logarithmic plot showing that $R^{\frac{5}{2}}$ is proportional to t.

$$\frac{5}{2}\ln R(t) \simeq \ln t + 2\ln E - 2\ln\rho$$

can be read off the plot at ln t = 0

Expansion velocity of a shock front

From his back-of-the-envelope calculation, Taylor estimated (with the help of photographs on the cover of the Life magazine) a released energy $E \approx 10^{14}$ J, which is the correct order of magnitude.

… and which was classified info at that time!