Elements of Hydrodynamics

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Foreword

The following pages were originally not designed to fall under your eyes. They grew up from handwritten notes for myself, listing the important points which I should not forget in the lecture room. As time went by, more and more remarks or developments were added, which is why I started to replace the growingly dirty sheets of paper by an electronic version—that could then also be easily uploaded on the web page of my lecture, for the benefit(?) of the students.

Again, additional results, calculations, comments, paragraphs or even whole chapters accumulated, leading to the temporary outcome which you are reading now: a not necessarily optimal overall outline; at times, unfinished sentences; not fully detailed proofs or calculations—because the missing steps are obvious to me—; insufficient discussions of the physics of some results—which I hopefully provide in the classroom—; not-so-good-looking figures; incomplete bibliography; etc. You may also expect a few solecisms, inconsistent notations, and the usual, unavoidable typos.[(*)] Eventually, you will have to cope with the many idiosyncrasies in my writing, as for instance my immoderate use of footnotes, dashes or parentheses, quotation marks, which are not considered as "good practice".

In short: the following chapters may barely be called "lecture notes"; they cannot replace a textbook[(†)] and the active participation in a course and in the corresponding tutorial/exercise sessions.

[(*)] Comments and corrections are welcome!

[(†)] ... which is one of several good reasons why you should think at least twice before printing a hard copy!

Contents

Introduction

General introduction and outline.
Notations, conventions, etc.

General references

(in alphabetical order)

- Faber, *Fluid dynamics for physicists* [1]
- Guyon, Hulin, Petit & Mitescu, *Physical hydrodynamics* [2]
- Greiner & Stock, *Hydrodynamik* [3]
- Landau & Lifshitz, *Course of theoretical physics. Vol. 6: Fluid mechanics* [4]
  = Landau & Lifschitz, *Lehrbuch der theoretischen Physik. Band VI: Hydrodynamik* [5]
- Rieutord, *Fluid dynamics — An introduction* [6]
- Sommerfeld, *Lectures on theoretical physics. Vol. II: Mechanics of deformable bodies* [7]
  = *Vorlesungen über theoretische Physik. Band II: Mechanik der deformierbaren Medien* [8]

CHAPTER I

Basic notions on continuous media

A system of many microscopic degrees of freedom is often more conveniently described as a material body that fills some region of space continuously, rather than as a collection of discrete point particles (Sec. I.1). This theoretical approach, which is especially suited to represent systems whose internal deformations are relevant, is an instance of physical *modeling*, originally motivated by the agreement of its predictions with experimental observations. Like every model, that of a continuous medium is valid only in some range of physical conditions, in particular on macroscopic scales.

Mathematically, a classical continuous medium at a given instant is described as a volume—or more generally a manifold—in usual Euclidean space. The infinitesimal elements of this volume constitute the elementary "material points", which are entirely characterized by their position.

To describe the time evolution of the physical system modeled as a continuous medium, two equivalent approaches are available. The first one consists in following the trajectories of the material points as time passes by (Sec. I.2). The physical picture of continuousness is then enforced by requesting that the mapping between the position of a given point at some reference initial time and its position at any later instant is continuous.

The second point of view, which will mostly be adopted in the remainder of these notes, focuses on the change in the various physical quantities at a fixed position as time elapses (Sec. I.3). The reference for the medium evolution between successive instants t and $t + \mathrm{d}t$ is the "current" configuration of the material points, i.e. at time t, instead of their positions in the (far) past. In that description, the spatial variables are no longer dynamical, but only labels for the position at which some observable is considered. Accordingly, the dynamical quantities in the system are now time-dependent fields; the desired continuousness of the medium translates into continuity conditions on those fields.

Eventually, the mathematical object that models internal forces in a continuous medium, i.e. the influence from neighboring material points on each other, is shortly introduced (Sec. I.4). This allows the classification of deformable continuous media into two traditional large classes, and in particular the definition of fluids.

I.1 Continuous medium: a model for many-body systems

In this Section, we first spell out a few arguments which lead to the introduction of the model of a continuous medium (§ I.1.1). The basic ingredients of the mathematical implementation of the model are then presented and a few notions are defined (§ I.1.2). Eventually, the physical assumptions underlying the modeling are reexamined in greater detail, and some more or less obvious limitations of the continuous description are indicated (§ I.1.3).

I.1.1 Basic ideas and concepts

The actual structure of matter at the microscopic scale is discrete, and involves finite "elementary" entities: electrons, atoms, ions, molecules, ..., which in the remainder of these notes will be collectively referred to as "atoms". Any macroscopic sample of matter contains a large amount of these atoms. For instance, the number density in an ideal gas under normal conditions is about 2.7×10^{25} m^{-3}, so that one cubic millimeter still contains 2.7×10^{16} atoms. Similarly, even though the number density in the interstellar medium might be as low as 10^2 m^{-3}, any volume relevant for astrophysics, i.e. with at least a kilometer-long linear size, involves a large number of atoms.

Additionally, these atoms are in constant chaotic motion, with individual velocities of order 10^2–10^3 m$\cdot$s^{-1} for a system at thermal equilibrium at temperature $T \simeq 300$ K. Given a mean free path[(i)] of order 10^{-7} m in a gas under normal conditions, each atom undergoes 10^9–10^{10} times per second, i.e. its trajectory changes direction constantly.

As in Statistical Mechanics, it is in general unnecessary to know the details of the motion of each atom in a macroscopic system: as a matter of fact, there emerge global characteristics, which can be predicted to a high degree of accuracy thanks to the large number of degrees of freedom involved in their definition (mostly as averages), despite the chaoticity of the individual atomic behaviors. The macroscopic properties of systems at (global) thermodynamic equilibrium are thus entirely determined by a handful of collective variables, either extensive—like entropy, internal energy, volume, particle number, total momentum...—, or intensive—as e.g. the respective densities of the various extensive variables, or temperature, pressure, chemical potential, average velocity...—, where the latter take the same value throughout the system.

When thermodynamic equilibrium does not hold globally in a system, there is still the possibility that one may consider that it is valid locally, "at each point" in space. In that situation—whose underlying assumptions will be specified in greater detail in § I.1.3—the intensive thermodynamic variables characterizing the system macroscopically become *fields*, which can vary from point to point. More generally, experience shows that it is fruitful to describe a large amount of characteristics—not only thermodynamic, but also of mechanical nature, like forces and the displacements or deformations they induce—of macroscopic bodies as fields. A "continuous medium" is then intuitively a system described by such fields, which should satisfy some (mathematical) continuity property with respect to the spatial variables that parameterize the representation of the physical system as a geometrical quantity. This picture will be better specified in Secs. I.2 and I.3.

Assuming the relevance of the model of a medium whose properties are described by continuous fields is often referred to as *continuum hypothesis*[(ii)].

The reader should keep in mind that the modeling of a given macroscopic system as a continuous medium does not invalidate the existence of its underlying discrete atomic structure. Specific

[(i)] *mittlere freie Weglänge* [(ii)] *Kontinuumshypothese*

phenomena will still directly probe the latter, as e.g. X-ray scattering experiments for the gases or liquids of our everyday life. That is, the model has limitations to its validity, especially at small wavelengths or high frequencies, where "small" or "high" implies a comparison to some microscopic physical scale characteristic of the system under consideration. Turning the reasoning the other way around, the continuous-medium picture is often referred to as a long-wavelength, low-frequency approximation to a more microscopic description—from which it can actually be shown to emerge in the corresponding limits.

> It is important to realize that the model itself is blind to its own limitations, i.e. there is no a priori criterion *within* the mathematical continuous-medium description that signals the breakdown of the relevance of the picture to actual physics. In practice, there might be hints that the equations of the continuous model are being applied in a regime where they should not, as for instance if they yield negative values for a quantity which should be positive, but such occurrences are not the general rule.

Remarks:

∗ The model of a continuous model is not only applicable—and applied—to obvious cases like gases, liquids or (deformable) solids, it may also be used to describe the behaviors of large crowds, fish schools, car traffic... provided the number of "elementary" constituents is large and the system is studied on a large enough scale.

∗ Even if the continuous description is valid on "long wavelengths", it remains obvious that any physical system, viewed on a scale much larger than its spatial extent, is to first approximation best described as pointlike.
Consider for instance a molecular cloud of interstellar medium with a 10 parsec radius and about 10^{10} H_2 molecules per cubic meter. For a star forming at its core, it behaves a continuous medium; 1 kpc away, however, the inner degrees of the cloud are most likely already irrelevant and it is best described as a mere point.

I.1.2 General mathematical framework

Consider a non-relativistic classical macroscopic physical system Σ, described by Newtonian physics. The positions of its individual atoms, viewed as pointlike, at a given instant t—which is the same for all observers—are points in a three-dimensional Euclidean space $\mathscr{E}_3$.

In the description as a continuous medium, the system Σ is represented by a geometrical manifold in $\mathscr{E}_3$, which for the sake of simplicity will be referred to as a "volume" and denoted by $\mathcal{V}$. The basic constituents of $\mathcal{V}$ are its infinitesimal elements $\mathrm{d}^3\mathcal{V}$, called *material points*[(iii)] or *continuous medium particles*[(iv)]—which explains a posteriori our designating the discrete constituents of matter as "atoms"—, or, in the specific case of the elementary subdivisions of a fluid, *fluid particles*[(v)]. As we shall state more explicitly in § I.1.3, these infinitesimal elements are assumed to have the same physical properties as a finite macroscopic piece.

Associated with the physical picture attached to the notion of continuousness is the requirement that neighboring material points in the medium remain close to each other throughout the system evolution. We shall see below how this picture is implemented in the mathematical description.

Remark: The volume $\mathcal{V}$—with the topology inherited from $\mathscr{E}_3$—need not be simply connected. For instance, one may want to describe the flow of a river around a bridge pier: the latter represents a physical region which water cannot penetrate, which is modeled as a hole throughout the volume $\mathcal{V}$ occupied by fluid particles.

To characterize the position of a given material point, as well as some of the observables relative to the physical system Σ, one still needs to specify the *reference frame* in which the system is

[(iii)] *Materielle Punkte* [(iv)] *Mediumteilchen* [(v)] *Fluidteilchen*

studied, corresponding to the point of view of a given observer, and to choose a *coordinate system* in that reference frame. This choice allows one to define vectors—like position vectors, velocities, or forces—and tensors.

The basis vectors of the coordinate system will generically be designated as $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, while the components of a given vector will be denoted with upper ("contravariant") indices, as e.g. $\vec{c} = c^i \vec{e}_i$, where the summation convention over repeated upper and lower indices was used.

Once the reference frame and coordinate system are determined, the macroscopic state of the physical system at time t is mapped onto a corresponding *configuration* κ_t of the medium, consisting of the continuous set of the position vectors $\vec{r} = x^i \vec{e}_i$ of its constituting material points. Since the volume occupied by the latter may also depends on time, it will also be labeled by t: $\mathcal{V}_t$.

To be able to formalize the necessary continuity conditions in the following Sections, one also introduces a reference time t_0—conveniently taken as the origin of the time axis, $t_0 = 0$—and the corresponding *reference configuration* κ_0 of the medium, which occupies a volume $\mathcal{V}_0$. The generic position vector of a material point in this reference configuration will be denoted as $\vec{R} = X^i \vec{e}_i$.

Remark: In so-called "classical" continuous media, as have been introduced here, the material points are entirely characterized by their position vector. In particular, they have no intrinsic angular momentum.

I.1.3 Local thermodynamic equilibrium

In a more bottom-up approach to the modeling of a system Σ of discrete constituents as a continuous medium, one should first divide Σ (in thought) into small cells of fixed—yet not necessarily universal—size fulfilling two conditions:

(i) each individual cell can meaningfully be treated as a thermodynamic system, i.e. it must be large enough that the relative fluctuations of the usual extensive thermodynamic quantities computed for the content of the cell are negligible;

(ii) the thermodynamic properties vary little over the cell scale, i.e. cells cannot be too large, so that (approximate) homogeneity is ensured.

The rationale behind these two requirements is illustrated by Fig. I.1, which represents schematically how the value of a local macroscopic quantity, e.g. a density, depends on the resolution of the apparatus with which it is measured, i.e. equivalently on the length scale on which it is defined. If the apparatus probes too small a length scale, so that the discrete degrees of freedom become relevant, the measured value strongly fluctuates from one observation to the next one, as hinted at by the displayed envelope of possible results of measurements: this is the issue addressed by condition (i). Simultaneously, a small change in the measurement resolution, even with the apparatus still centered on the same point in the system, can lead to a large variation in the measured value of the observable, corresponding to the erratic behavior of the curve at small scales shown in Fig. I.1. This fluctuating pattern decreases with increasing size of the observation scale, since this increase leads to a growth in the number of atoms inside the probed volume, and thus a drop in the size of relative fluctuations. At the other end of the curve, one reaches a regime where the low resolution of the observation leads to encompassing domains with enough atoms to be rid of fluctuations, yet with inhomogeneous macroscopic properties, in a single probed region—in violation of condition (ii). As a result, the measured value of the density under consideration slowly evolves with the observation scale.

In between these two domains of strong statistical fluctuations and slow macroscopic variations lies a regime where the value measured for an observable barely depends on the scale over which it is determined. This represents the appropriate regime for meaningfully defining—and measuring—a local density, and more general local quantities.

It is important to note that this intermediate "mesoscopic" interval may not always exist. There are physical systems in which strong macroscopic variations are already present in a range of scales

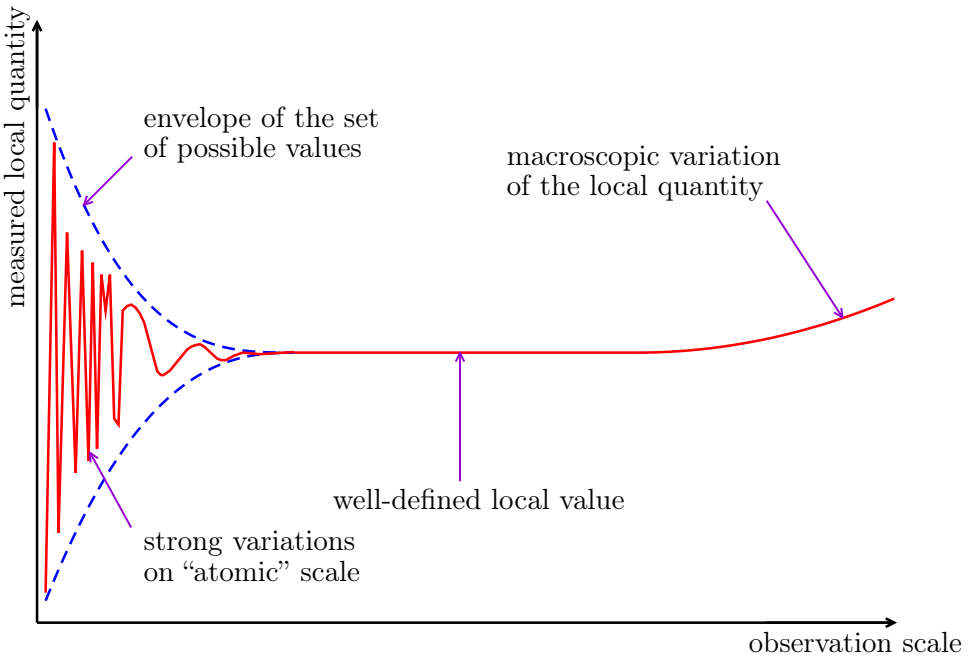

Figure I.1 – Typical variation of the measured value for a "local" macroscopic observable as a function of the size scale over which it is determined.

where microscopic fluctuations are still sizable. For such systems, one cannot find scale-independent local variables. That is, the proper definition of local quantities implicitly relies on the existence of a clear separation of scales in the physical system under consideration, which is what will be assumed in the remainder of these notes.

Remark: The smallest volume over which meaningful local quantities can be defined is sometimes called *representative volume element* (RVE), or *representative elementary volume*.

When conditions (i) and (ii) hold, one may in particular define *local thermodynamic variables*, corresponding to the values taken in each intermediate-size cell—labeled by its position $\vec{r}$—by the usual extensive parameters: internal energy, number of atoms... Since the separation between cells is immaterial, nothing prevents energy or matter from being transported from a cell to its neighbors, even if the global system is isolated. Accordingly, the local extensive variables in any given cell are actually time-dependent in the general case. In addition, it becomes important to add linear momentum—with respect to some reference frame—to the set of local extensive variables characterizing the content of a cell.

The size of each cell is physically irrelevant, as long as it satisfies the two key requirements; there is thus no meaningful local variable corresponding to volume. Similarly, the values of the extensive variables in a given cell, which are by definition proportional to the cell size, are as arbitrary as the latter. They are thus conveniently replaced by the respective local densities: internal energy density $e(t,\vec{r})$, number density $n(t,\vec{r})$, linear momentum density $\rho(t,\vec{r})\vec{v}(t,\vec{r})$, where ρ denotes the mass density, entropy density $s(t,\vec{r})$...

Remark: Rather than considering the densities of extensive quantities, some authors—in particular Landau & Lifshitz [4, 5]—prefer to work with *specific quantities*, i.e. their respective amounts per unit mass, instead of per unit volume. The relation between densities and specific quantities is trivial: denoting by $\mathcal{X}_j$ resp. $\mathcal{X}_{j,\mathrm{m}}$ a generic local density resp. specific amount for the same physical quantity, one has the identity

$$\mathcal{X}_j(t,\vec{r}) = \rho(t,\vec{r})\,\mathcal{X}_{j,\mathrm{m}}(t,\vec{r}) \tag{I.1}$$

in every cell—labeled by $\vec{r}$—and at every time t.

Once the local extensive variables have been meaningfully defined, one can develop the usual formalism of thermodynamics in each cell. In particular, one introduces the conjugate intensive variables, as e.g. local temperature $T(t,\vec{r})$ and pressure $\mathcal{P}(t,\vec{r})$. The underlying, important hypothesis is the assumption of a *local thermodynamic equilibrium*. According to the latter, the equation(s) of state of the system inside the small cell, expressed with local thermodynamic quantities, is the same as for a macroscopic system in the actual thermodynamic limit of infinitely large volume and particle number.

Consider for instance a non-relativistic classical ideal gas: its (mechanical) equation of state reads $\mathcal{P}\mathcal{V} = Nk_{\mathrm{B}}T$, with N the number of atoms, which occupy a volume $\mathcal{V}$ at uniform pressure $\mathcal{P}$ and temperature T, while k_{B} is the Boltzmann[(a)] constant. This is trivially recast as $\mathcal{P} = nk_{\mathrm{B}}T$, with n the number density of atoms. The local thermodynamic equilibrium assumption then states that under non-uniform conditions of temperature and pressure, the equation of state in a local cell at position $\vec{r}$ is given by

$$\mathcal{P}(t,\vec{r}) = n(t,\vec{r})k_{\mathrm{B}}T(t,\vec{r}) \tag{I.2}$$

at every time t.

The last step towards the continuous-medium model is to promote $\vec{r}$, which till now was simply the discrete label attached to a given cell, to be a continuous variable taking its values in $\mathbb{R}^3$—or rather, in the volume $\mathcal{V}_t$ attached to the system at the corresponding instant t. Accordingly, taking into account the time-dependence of physical quantities, the local variables, in particular the thermodynamic parameters, become *fields* on $\mathbb{R}\times\mathbb{R}^3$.

The replacement of the fine-resolution description, in which atoms are the relevant degrees of freedom, by the lower-resolution model which assimilates small finite volumes of the former to structureless points is called *coarse graining*[(vi)].

> This is a quite generic procedure in theoretical physics, whereby the finer degrees of freedom of a more fundamental description are smoothed away—technically, this is often done by performing averages or integrals, so that these degrees of freedom are "integrated out"—and replaced by novel, effective variables in a theory with a more limited range of applicability, but which is more tractable for "long-range" phenomena.

Coming back to condition (ii), we already stated that it implicitly involves the existence of at least one large length scale L, over which the macroscopic physical properties of the system may vary. This scale can be a characteristic dimension of the system under consideration, as e.g. the diameter of the tube in which a liquid is flowing. In the case of periodic waves propagating in the continuous medium, L also corresponds to their wavelength. More generally, if $\mathcal{G}$ denotes a macroscopic physical quantity, one may consider

$$L \cong \left[\frac{\left|\vec{\nabla}\mathcal{G}(t,\vec{r})\right|}{\left|\mathcal{G}(t,\vec{r})\right|}\right]^{-1}, \tag{I.3}$$

where $\vec{\nabla}$ denotes the (spatial) gradient.

Condition (i) in particular implies that the typical size of the cells which are later coarse grained should be significantly larger than the *mean free path* ℓ_{mfp} of atoms, so that thermodynamic equilibrium holds in the local cells. Since on the other hand this same typical size should be significantly smaller than the scale L of macroscopic variations, one deduces the condition

$$\boxed{\mathrm{Kn} \equiv \frac{\ell_{\mathrm{mfp}}}{L} \ll 1} \tag{I.4}$$

on the dimensionless *Knudsen number* Kn.[(b)]

[(vi)] *Vergröberung*

[(a)] L. Boltzmann, 1844–1906 [(b)] M. Knudsen, 1871–1949

In air under normal conditions $\mathcal{P} = 10^5$ Pa and $T = 300$ K, the mean free path is $\ell_{\text{mfp}} \approx 0.1$ μm. In the study of phenomena with variations on a characteristic scale $L \approx 10$ cm, one finds $\text{Kn} \approx 10^{-6}$, so that air can be meaningfully treated as a continuous medium.

> The opposite regime $\text{Kn} > 1$ is that of a *rarefied medium*, as for instance of the so-called Knudsen gas, in which the collisions between atoms are negligible—and in particular insufficient to ensure thermal equilibrium as an ideal gas. The flow of such systems is not well described by hydrodynamics, but necessitates alternative descriptions like molecular dynamics, in which the degrees of freedom are explicitly atoms.

I.2 Lagrangian description

The *Lagrangian*[c] perspective, which generalizes the approach usually adopted in the description of the motion of a (few) point particle(s), focuses on the trajectories of the material points, where the latter are labeled by their position in the reference configuration. Accordingly, physical quantities are expressed as functions of time t and initial position vectors $\vec{R}$, and any continuity condition has to be formulated with respect to these variables.

I.2.1 Lagrangian coordinates

Consider a material point M in a continuous medium. Given a reference frame $\mathcal{R}$, which allows the definition of its position vector at any time t, one can follow its *trajectory* $\vec{r}(t)$. With a choice of coordinate system, that trajectory is equivalently characterized by the functions $\{x^i(t)\}$ for $i = 1, 2, 3$.

Let $\vec{R}$ resp. $\{X^i\}$ denote the position resp. coordinates of the material point M at t_0. The trajectory obviously depends on this "initial" position, and $\vec{r}$ can thus be viewed as a function of t and $\vec{R}$, where the latter refers to the reference configuration κ_0:

$$\vec{r} = \vec{r}(t, \vec{R}) \tag{I.5a}$$

with the consistency condition

$$\vec{r}(t{=}t_0, \vec{R}) = \vec{R}. \tag{I.5b}$$

In the *Lagrangian description*, also referred to as *material description* or *particle description*, this point of view is generalized, and the various physical quantities $\mathcal{G}$ characterizing a continuous medium are viewed at *any* time as mathematical functions of the variables t and $\vec{R}$:

$$\mathcal{G} = \mathcal{G}(t, \vec{R}), \tag{I.6}$$

where the mapping $\mathcal{G}$—which as often in physics will be denoted with the same notation as the physical quantity represented by its value—is defined for every t on the initial volume $\mathcal{V}_0$ occupied by the reference configuration κ_0.

Together with the time t, the position vector $\vec{R}$—or equivalently its coordinates X^1, X^2, X^3 in a given system—are called *Lagrangian coordinates*.

I.2.2 Continuity assumptions

An important example of physical quantity, function of t and $\vec{R}$, is simply the (vector) position in the reference frame $\mathcal{R}$ of material points at time t, i.e. $\vec{r}$ or equivalently its coordinates $\{x^i\}$, as given by relation (I.5a), which thus relates the configurations κ_0 and κ_t.

[c] J.-L. Lagrange, 1736–1813

More precisely, $\vec{r}(t,\vec{R})$ maps for every t the initial volume $\mathcal{V}_0$ *onto* $\mathcal{V}_t$. To implement mathematically the physical picture of continuity, it will be assumed that the mapping $\vec{r}(t,\cdot): \mathcal{V}_0 \to \mathcal{V}_t$ is also one-to-one for every t—i.e. all in all bijective—, and that the function $\vec{r}$ and its inverse

$$\vec{R} = \vec{R}(t,\vec{r}) \tag{I.7}$$

are *continuous* with respect to both time and space variables. This requirement in particular ensures that neighboring points remain close to each other as time elapses. It also preserves the connectedness of volumes, (closed) surfaces or curves along the evolution: one may then define *material domains*, i.e. connected sets of material points which are transported together in the evolution of the continuous medium.

For the sake of simplicity, it will be assumed that the mapping $\vec{r}$ and its inverse, and more generally every mathematical function $\mathcal{G}$ representing a physical quantity, is at least twice continuously differentiable (i.e. of class $\mathscr{C}^2$). To be able to accommodate for important phenomena that are better modeled with discontinuities, like shock waves in fluids (Sec. VI.2) or ruptures in solids—for instance, in the Earth's crust—, the $\mathscr{C}^2$-character of functions under consideration may hold only piecewise.

I.2.3 Velocity and acceleration of a material point

As mentioned above, for a fixed reference position $\vec{R}$ the function $t \mapsto \vec{r}(t,\vec{R})$ is the trajectory of the material point which passes through $\vec{R}$ at the reference time t_0. As a consequence, the velocity at time t of this material point, measured in the reference frame $\mathcal{R}$, is simply

$$\vec{v}(t,\vec{R}) = \frac{\partial \vec{r}(t,\vec{R})}{\partial t}. \tag{I.8}$$

Since the variable $\vec{R}$ is independent of t, one could actually also write $\vec{v}(t,\vec{R}) = \mathrm{d}\vec{r}(t,\vec{R})/\mathrm{d}t$.
In turn, the acceleration of the material point in $\mathcal{R}$ is given at time t by

$$\vec{a}(t,\vec{R}) = \frac{\partial \vec{v}(t,\vec{R})}{\partial t}. \tag{I.9}$$

Remark: The trajectory (or *pathline*[vii]) of a material point can be visualized, by tagging the point at its position $\vec{R}$ at time t_0, for instance with a fluorescent or radioactive marker, and then imaging the positions at later times $t > t_0$.

On the other hand, if one regularly—say for every instant $t_0 \le t' \le t$—injects some marker at a fixed geometrical point P, the resulting tagged curve at time t is the locus of the geometrical points occupied by medium particles which passed through P in the past. This locus is referred to as *streakline*.[viii] Denoting by $\vec{r}_P$ the position vector of point P, the streakline is the set of geometrical points with position vectors

$$\vec{r} = \vec{r}\big(t, \vec{R}(t', \vec{r}_P)\big) \quad \text{for } t_0 \le t' \le t. \tag{I.10}$$

I.3 Eulerian description

The Lagrangian approach introduced in the previous Section is actually not commonly used in fluid dynamics, at least not in its original form, except for specific problems.

> One reason is that physical quantities at a given time are expressed in terms of a reference configuration in the (far) past: a small uncertainty on this initial condition may actually yield

[vii] *Bahnlinie* [viii] *Streichlinie*

after a finite duration a large uncertainty on the present state of the system, which is problematic. On the other hand, this line of argument explains why the Lagrangian point of view is adopted to investigate *chaos* in many-body systems!

The more usual description is the so-called *Eulerian*[(d)] perspective, in which the evolution between instants t and $t + \mathrm{d}t$ takes the system configuration at time t as a reference.

I.3.1 Eulerian coordinates. Velocity field

In contrast to the "material" Lagrangian point of view, which identifies the medium particles in a reference configuration and follows them in their motion, in the Eulerian description the emphasis is placed on the *geometrical* points. Thus, the *Eulerian coordinates* are time t and a spatial vector $\vec{r}$, where the latter does not label the position of a material point, but rather that of a geometrical point. Accordingly, the physical quantities in the Eulerian specification are described by *fields* on space-time.

Thus, the fundamental field that entirely characterizes the motion of a continuous medium in a given reference frame $\mathcal{R}$ is the *velocity field* $\vec{\mathsf{v}}_t(t,\vec{r})$. The latter is defined such that it gives the value of the Lagrangian velocity $\vec{v}$ [Eq. (I.8)] of a material point passing through $\vec{r}$ at time t:

$$\vec{v} = \vec{\mathsf{v}}_t(t,\vec{r}) \quad \forall t,\ \forall \vec{r} \in \mathcal{V}_t. \tag{I.11}$$

More generally, the value taken at given time and position by a physical quantity $\mathcal{G}$, whether attached to a material point or not, is expressed as a mathematical function $\mathcal{G}_t$ of the same Eulerian variables:

$$\mathcal{G} = \mathcal{G}_t(t,\vec{r}) \quad \forall t,\ \forall \vec{r} \in \mathcal{V}_t. \tag{I.12}$$

Note that the mappings $(t,\vec{R}) \mapsto \mathcal{G}(t,\vec{R})$ in the Lagrangian approach and $(t,\vec{r}) \mapsto \mathcal{G}_t(t,\vec{r})$ in the Eulerian description are in general different. For instance, the domains in $\mathbb{R}^3$ over which their spatial variables take their values differ: constant ($\mathcal{V}_0$) in the Lagrangian specification, time-dependent ($\mathcal{V}_t$) in the case of the Eulerian quantities. Accordingly the latter will be denoted with a subscript t in the next subsection.

I.3.2 Equivalence between the Eulerian and Lagrangian viewpoints

Despite the different choices of variables, the Lagrangian and Eulerian descriptions are fully equivalent. Accordingly, the prevalence in practice of the one over the other is more a technical issue than a conceptual one.

Thus, it is rather clear that the knowledge of the Lagrangian specification can be used to obtain the Eulerian formulation at once, using the mapping $\vec{r} \mapsto \vec{R}(t,\vec{r})$ between present and reference positions of a material point. For instance, the Eulerian velocity field can be expressed as

$$\vec{\mathsf{v}}_t(t,\vec{r}) = \vec{v}\big(t,\vec{R}(t,\vec{r})\big). \tag{I.13a}$$

This identity in particular shows that $\vec{\mathsf{v}}_t$ automatically inherits the smoothness properties of $\vec{v}$: if the mapping $(t,\vec{R}) \mapsto \vec{r}(t,\vec{R})$ and its inverse are piecewise $\mathscr{C}^2$ (cf. § I.2.2), then $\vec{\mathsf{v}}_t$ is (at least) piecewise $\mathscr{C}^1$ in both its variables.

For a generic physical quantity, the transition from the Lagrangian to the Eulerian point of view similarly reads

$$\mathcal{G}_t(t,\vec{r}) = \mathcal{G}\big(t,\vec{R}(t,\vec{r})\big). \tag{I.13b}$$

Reciprocally, given a (well-enough behaved) Eulerian velocity field $\vec{\mathsf{v}}_t$ on a continuous medium, one can uniquely obtain the Lagrangian description of the medium motion by solving the *initial*

[(d)] L. Euler, 1707–1783

value problem

$$\begin{cases} \dfrac{\partial \vec{r}(t,\vec{R})}{\partial t} = \vec{\mathsf{v}}_t\big(t,\vec{r}(t,\vec{R})\big) \\ \vec{r}(t_0,\vec{R}) = \vec{R}, \end{cases} \tag{I.14a}$$

where the second line represents the initial condition. That is, one actually reconstructs the pathline of every material point of the continuous medium. Introducing differential notations, the above system can also be rewritten as

$$\mathrm{d}\vec{r} = \vec{\mathsf{v}}_t(t,\vec{r})\,\mathrm{d}t \quad \text{with} \quad \vec{r}(t_0,\vec{R}) = \vec{R}. \tag{I.14b}$$

Once the pathlines $\vec{r}(t,\vec{R})$ are known, one obtains the Lagrangian function $\mathcal{G}(t,\vec{R})$ for a given physical quantity $\mathcal{G}$ by writing

$$\mathcal{G}(t,\vec{R}) = \mathcal{G}_t\big(t,\vec{r}(t,\vec{R})\big). \tag{I.14c}$$

Since both Lagrangian and Eulerian descriptions are equivalent, we shall from now on drop the subscript t on the mathematical functions representing physical quantities in the Eulerian point of view.

I.3.3 Streamlines

At a given time t, the *streamlines*[(ix)] of the motion are defined as the field lines of $\vec{\mathsf{v}}$. That is, these are curves whose tangent is everywhere parallel to the instantaneous velocity field at the same geometrical point.

Let $\vec{x}(\lambda)$ denote a streamline, parameterized by λ. The definition can be formulated as

$$\frac{\mathrm{d}\vec{x}(\lambda)}{\mathrm{d}\lambda} = \alpha(\lambda)\vec{\mathsf{v}}\big(t,\vec{x}(\lambda)\big) \tag{I.15a}$$

with $\alpha(\lambda)$ a scalar function. Equivalently, denoting by $\mathrm{d}\vec{x}(\lambda)$ a differential line element tangent to the streamline, one has the condition

$$\mathrm{d}\vec{x} \times \vec{\mathsf{v}}\big(t,\vec{x}(\lambda)\big) = \vec{0}. \tag{I.15b}$$

Introducing a Cartesian system of coordinates, the equation for a streamline is conveniently rewritten as

$$\frac{\mathrm{d}x^1(\lambda)}{\mathsf{v}^1\big(t,\vec{x}(\lambda)\big)} = \frac{\mathrm{d}x^2(\lambda)}{\mathsf{v}^2\big(t,\vec{x}(\lambda)\big)} = \frac{\mathrm{d}x^3(\lambda)}{\mathsf{v}^3\big(t,\vec{x}(\lambda)\big)} \tag{I.15c}$$

at a point where none of the component v^i of the velocity field vanishes—if one of the v^i is zero, then so is the corresponding $\mathrm{d}x^i$, thanks to Eq. (I.15b).

Remark: Since the velocity field $\vec{\mathsf{v}}$ depends on the choice of reference frame, this is also the case of its streamlines at a given instant!

Consider now a closed geometrical curve in the volume $\mathcal{V}_t$ occupied by the continuous medium at time t. The streamlines tangent to this curve form in the generic case a tube-like surface, called *stream tube*.[(x)]

Let us introduce two further definitions related to properties of the velocity field:

- If $\vec{\mathsf{v}}(t,\vec{r})$ has at some t the same value at every geometrical point $\vec{r}$ of a (connected) domain $\mathcal{D} \subset \mathcal{V}_t$, then the velocity field is said to be *uniform* across $\mathcal{D}$.
  In that case, the streamlines are parallel to each other over $\mathcal{D}$.

[(ix)] *Stromlinien* [(x)] *Stromröhre*

- If $\vec{\mathsf{v}}(t,\vec{r})$ only depends on the position, not on time, then the velocity field and the corresponding motion of the continuous medium are said to be *steady* or equivalently *stationary*. In that case, the streamlines coincide with the pathlines and the streaklines.

  Indeed, one checks that Eq. (I.14b) for the pathlines, in which the velocity becomes time-independent, can then be recast (at a point where all v^i are non-zero) as

  $$\frac{\mathrm{d}x^1}{\mathsf{v}^1(t,\vec{r})} = \frac{\mathrm{d}x^2}{\mathsf{v}^2(t,\vec{r})} = \frac{\mathrm{d}x^3}{\mathsf{v}^3(t,\vec{r})},$$

  where the variable t plays no role: this is exactly the system (I.15c) defining the streamlines at time t. The equivalence between pathlines and streaklines is also trivial. □

I.3.4 Material derivative

Consider a material point M in a continuous medium, described in a reference frame $\mathcal{R}$. Let $\vec{r}$ resp. $\vec{r}+\mathrm{d}\vec{r}$ denote its position vectors at successive instants t resp. $t+\mathrm{d}t$. The velocity of M at time t resp. $t+\mathrm{d}t$ is by definition equal to the value of the velocity field at that time and at the respective position, namely $\vec{\mathsf{v}}(t,\vec{r})$ resp. $\vec{\mathsf{v}}(t+\mathrm{d}t,\vec{r}+\mathrm{d}\vec{r})$. For small enough $\mathrm{d}t$, the displacement $\mathrm{d}\vec{r}$ of the material point between t and $t+\mathrm{d}t$ is related to its velocity at time t by $\mathrm{d}\vec{r} = \vec{\mathsf{v}}(t,\vec{r})\,\mathrm{d}t$.

Let $\mathrm{d}\vec{\mathsf{v}} \equiv \vec{\mathsf{v}}(t+\mathrm{d}t,\vec{r}+\mathrm{d}\vec{r}) - \vec{\mathsf{v}}(t,\vec{r})$ denote the change in the material point velocity between t and $t+\mathrm{d}t$. Assuming that $\vec{\mathsf{v}}(t,\vec{r})$ is differentiable (cf. § I.3.2) and introducing for simplicity a system of Cartesian coordinates, a Taylor expansion to lowest order yields

$$\mathrm{d}\vec{\mathsf{v}} \simeq \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t}\,\mathrm{d}t + \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial x^1}\,\mathrm{d}x^1 + \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial x^2}\,\mathrm{d}x^2 + \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial x^3}\,\mathrm{d}x^3,$$

up to terms of higher order in $\mathrm{d}t$ or $\mathrm{d}\vec{r}$. Introducing the differential operator

$$\mathrm{d}\vec{r}\cdot\vec{\nabla} \equiv \mathrm{d}x^1\,\frac{\partial}{\partial x^1} + \mathrm{d}x^2\,\frac{\partial}{\partial x^2} + \mathrm{d}x^3\,\frac{\partial}{\partial x^3},$$

this can be recast in the more compact form

$$\mathrm{d}\vec{\mathsf{v}} \simeq \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t}\,\mathrm{d}t + \big(\mathrm{d}\vec{r}\cdot\vec{\nabla}\big)\vec{\mathsf{v}}(t,\vec{r}). \tag{I.16}$$

In the second term on the right-hand side, $\mathrm{d}\vec{r}$ can be replaced by $\vec{\mathsf{v}}(t,\vec{r})\,\mathrm{d}t$. On the other hand, the change in velocity of the material point between t and $t+\mathrm{d}t$ is simply the product of its acceleration $\vec{a}(t)$ at time t by the length $\mathrm{d}t$ of the time interval, at least to lowest order in $\mathrm{d}t$. Dividing both sides of Eq. (I.16) by $\mathrm{d}t$ and taking the limit $\mathrm{d}t \to 0$, in particular in the ratio $\mathrm{d}\vec{\mathsf{v}}/\mathrm{d}t$, yield

$$\vec{a}(t) = \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}). \tag{I.17}$$

That is, the acceleration of the material point consists of two terms:

- the *local acceleration* $\dfrac{\partial\vec{\mathsf{v}}}{\partial t}$, which follows from the non-stationarity of the velocity field;
- the *convective acceleration* $\big(\vec{\mathsf{v}}\cdot\vec{\nabla}\big)\vec{\mathsf{v}}$, due to the non-uniformity of the motion.

More generally, one finds by repeating the same derivation as above that the time derivative of a physical quantity $\mathcal{G}$ attached to a material point or domain, expressed in terms of Eulerian fields, is the sum of a local $(\partial\mathcal{G}/\partial t)$ and a convective $[(\vec{\mathsf{v}}\cdot\vec{\nabla})\mathcal{G}]$ part, irrespective of the tensorial nature of $\mathcal{G}$. Accordingly, one introduces the operator

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}} \tag{I.18}$$

called *material derivative*[(xi)] or (between others) *substantial derivative*,[(xii)] *derivative following the motion*, *hydrodynamic derivative*. Relation (I.17) can thus be recast as

$$\vec{a}(t) = \frac{\mathrm{D}\vec{\mathsf{v}}(t,\vec{r})}{\mathrm{D}t}. \tag{I.19}$$

Remarks:

∗ Equation (I.17) shows that even in the case of a steady motion, the acceleration of a material point may be non-vanishing, thanks to the convective part.

∗ The material derivative (I.18) is also often denoted (and referred to) as total derivative $\mathrm{d}/\mathrm{d}t$.

∗ One also finds in the literature the denomination *convective derivative*.[(xiii)] To the eyes and ears of the author of these lines, that name has the drawback that it does not naturally evoke the local part, but only... the convective one, which comes from the fact that matter is being transported, "conveyed", with a non-vanishing velocity field $\vec{\mathsf{v}}(t,\vec{r})$.

∗ The two terms in Eq. (I.18) actually "merge" together when considering the motion of a material point in Galilean space-time $\mathbb{R}\times\mathbb{R}^3$. As a matter of fact, one easily shows that $\mathrm{D}/\mathrm{D}t$ is the (Lie[(e)]) derivative along the world-line of the material point

> The world-line element corresponding to the motion between t and $t+\mathrm{d}t$ goes from (t,x^1,x^2,x^3) to $(t+\mathrm{d}t, x^1+\mathsf{v}^1\,\mathrm{d}t, x^2+\mathsf{v}^2\,\mathrm{d}t, x^3+\mathsf{v}^3\,\mathrm{d}t)$. The tangent vector to this world-line thus has components $(1,\mathsf{v}^1,\mathsf{v}^2,\mathsf{v}^3)$, i.e. the derivative along the direction of this vector is $\partial_t+\mathsf{v}^1\partial_1+\mathsf{v}^2\partial_2+\mathsf{v}^3\partial_3$, with the usual shorthand notations $\partial_t\equiv\partial/\partial t$ and $\partial_i\equiv\partial/\partial x^i$. □

I.4 Mechanical stress

I.4.1 Forces in a continuous medium

Consider a closed material domain $\mathcal{V}$ inside the volume $\mathcal{V}_t$ occupied by a continuous medium, and let $\mathcal{S}$ denote the (geometric) surface enclosing $\mathcal{V}$. One distinguishes between two classes of forces acting on this domain:

- *Volume* or *body forces*,[(xiv)] which act at each point of the bulk volume of $\mathcal{V}$.
  Examples are weight, long-range electromagnetic forces or, in non-inertial reference frames, fictitious forces (Coriolis, centrifugal).
  For such forces, which tend to be proportional to the volume they act on, it will later be more convenient to introduce the corresponding volumic force density.
- *Surface* or *contact forces*,[(xv)] which act on the surface $\mathcal{S}$, like friction. These will be now discussed in further detail.

Consider an infinitesimally small geometrical surface element $\mathrm{d}^2\mathcal{S}$ at point P. Let $\mathrm{d}^2\vec{F}_s$ denote the surface force through $\mathrm{d}^2\mathcal{S}$. That is, $\mathrm{d}^2\vec{F}_s$ is the contact force, due to the medium exterior to $\mathcal{V}$, that a "test" material surface coinciding with $\mathrm{d}^2\mathcal{S}$ would experience. The vector

$$\vec{T}_{\mathrm{s}} \equiv \frac{\mathrm{d}^2\vec{F}_s}{\mathrm{d}^2\mathcal{S}}, \tag{I.20}$$

representing the surface density of contact forces, is called *(mechanical) stress vector*[(xvi)] on $\mathrm{d}^2\mathcal{S}$.

[(xi)] *Materielle Ableitung* [(xii)] *Substantielle Ableitung* [(xiii)] *Konvektive Ableitung* [(xiv)] *Volumenkräfte* [(xv)] *Oberflächenkräfte*

[(e)] S. Lie, 1842–1899

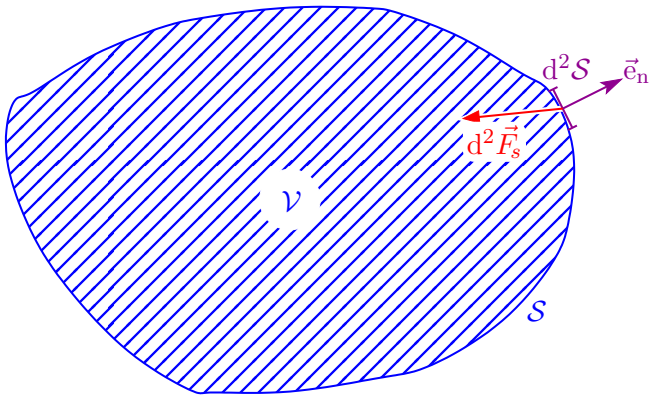

Figure I.2

The corresponding unit in the SI system is the Pascal, with $1\ \mathrm{Pa} = 1\ \mathrm{N}\cdot\mathrm{m}^{-2}$.

Purely geometrically, the stress vector $\vec{T}_s$ on a given surface element $\mathrm{d}^2\mathcal{S}$ at a given point can be decomposed into two components, namely

- a vector orthogonal to the plane tangent at P to $\mathrm{d}^2\mathcal{S}$, the so-called *normal stress*[(xvii)]; when it is directed towards the interior resp. exterior of the medium domain being acted on, it is also referred to as *compression*[(xviii)] resp. *tension*[(xix)];
- a vector in the tangent plane at P, called *shear stress*[(xx)] and often denoted as $\vec{\tau}$.

Despite the short notation adopted in Eq. (I.20), the stress vector depends not only on the position of the geometrical point P where the infinitesimal surface element $\mathrm{d}^2\mathcal{S}$ lies, but also on the orientation of the surface. Let $\vec{\mathrm{e}}_\mathrm{n}$ denote the normal unit vector to the surface element, directed towards the exterior of the volume $\mathcal{V}$ (cf. Fig. I.2), and let $\vec{r}$ denote the position vector of P in a given reference frame. The relation between $\vec{\mathrm{e}}_\mathrm{n}$ and the stress vector $\vec{T}_s$ on $\mathrm{d}^2\mathcal{S}$ is then linear:

$$\vec{T}_s = \boldsymbol{\sigma}(\vec{r})\cdot\vec{\mathrm{e}}_\mathrm{n}, \tag{I.21a}$$

with $\boldsymbol{\sigma}(\vec{r})$ a symmetric tensor of rank 2, the so-called (*Cauchy*[(f)]) *stress tensor*.[(xxi)]

In a given coordinate system, relation (I.21a) yields

$$T_s^i = \sum_{j=1}^{3} \boldsymbol{\sigma}^i{}_j\, \mathrm{e}_\mathrm{n}^j \tag{I.21b}$$

with T_s^i resp. e_n^j the coordinates of the vectors $\vec{T}_s$ resp. $\vec{\mathrm{e}}_\mathrm{n}$, and $\boldsymbol{\sigma}^i{}_j$ the $\binom{1}{1}$-components of the stress tensor.

> While valid in the case of a three-dimensional position space, Eq. (I.21a) should actually be better formulated to become valid in arbitrary dimension. Thus, the unit-length "normal vector" to a surface element at point P is rather a *1-form* acting on the vectors of the tangent space to the surface at P. As such, it should be represented as the transposed of a vector $[(\vec{\mathrm{e}}_\mathrm{n})^\mathsf{T}]$, which multiplies the stress tensor from the left:
>
> $$\vec{T}_s = (\vec{\mathrm{e}}_\mathrm{n})^\mathsf{T}\cdot\boldsymbol{\sigma}(\vec{r}). \tag{I.21c}$$

[(xvi)] *Mechanischer Spannungsvektor* [(xvii)] *Normalspannung* [(xviii)] *Druckspannung* [(xix)] *Zugsspannung* [(xx)] *Scher-, Tangential-* oder *Schubspannung* [(xxi)] (*Cauchy'scher*) *Spannungstensor*

[(f)] A.L. Cauchy, 1789–1857

This shows that the Cauchy stress tensor is a $\binom{2}{0}$-tensor (a "bivector"), which maps 1-forms onto vectors. In terms of coordinates, this gives, using Einstein's summation convention

$$T_s^j = \mathrm{e}_{\mathrm{n},i}\, \boldsymbol{\sigma}^{ij}\,, \tag{I.21d}$$

which thanks to the symmetry of $\boldsymbol{\sigma}$ is equivalent to the relation given above.

Remark: The symmetry property of the Cauchy stress tensor is intimately linked to the assumption that the material points constituting the continuous medium have no intrinsic angular momentum.

I.4.2 Fluids

With the help of the notion of mechanical stress, we may now introduce the definition of a *fluid*, which is the class of continuous media whose motion is described by hydrodynamics:

A fluid is a continuous medium that deforms itself as long as it is submitted to shear stresses. (I.22)

Turning this definition around, one sees that in a fluid *at rest*—or, to be more accurate, studied in a reference frame with respect to which it is at rest—the mechanical stresses are necessarily normal. That is, the stress tensor is in each point diagonal.

More precisely, for a locally isotropic fluid—which means that the material points are isotropic, which is the case throughout these notes—the stress $\binom{2}{0}$-tensor is everywhere proportional to the inverse metric tensor:

$$\boldsymbol{\sigma}(t,\vec{r}) = -\mathcal{P}(t,\vec{r})\, \mathbf{g}^{-1}(t,\vec{r}) \tag{I.23}$$

with $\mathcal{P}(t,\vec{r})$ the *hydrostatic pressure* at position $\vec{r}$ at time t.

Going back to relation (I.21b), the stress vector will be parallel to the "unit normal vector" in any coordinate system if the square matrix of the $\binom{1}{1}$-components $\boldsymbol{\sigma}^i{}_j$ is proportional to the identity matrix, i.e. $\boldsymbol{\sigma}^i{}_j \propto \delta^i{}_j$, where we have introduced the Kronecker symbol. To obtain the $\binom{2}{0}$-components $\boldsymbol{\sigma}^{ik}$, one has to multiply $\boldsymbol{\sigma}^i{}_j$ by the component g^{jk} of the inverse metric tensor, summing over k, which precisely gives Eq. (I.23).

Remarks:

∗ Definition (I.22), as well as the two remarks hereafter, rely on an intuitive picture of "deformations" in a continuous medium. To support this picture with some mathematical background, we shall introduce in Sec. II.A an appropriate strain tensor, which quantifies these deformations, at least as long as they remain small.

∗ A deformable solid will also deform itself when submitted to shear stress! However, for a given fixed amount of tangential stress, the solid will after some time reach a new, deformed equilibrium position—otherwise, it is not a solid, but a fluid.

∗ The previous remark is actually a simplification, valid on the typical time scale of human beings. Thus, materials which in our everyday experience are solids—as for instance those forming the mantle of the Earth—will behave on a longer time scale as fluids—in the previous example, on geological time scales. Whether a given substance behaves as a fluid or a deformable solid is sometimes characterized by the dimensionless *Deborah number* [9], which compares the typical time scale for the response of the substance to a mechanical stress and the observation time.

∗ Even nicer, the fluid vs. deformable solid behavior may actually depend on the intensity of the applied shear stress: ketchup!

Bibliography for Chapter I

- National Committee for Fluid Mechanics films & film notes on *Eulerian Lagrangian description* and on *Flow visualization*;[(1)]
- Faber [1] Chapter 1.1–1.3;
- Feynman [10, 11] Chapter 31–6;
- Guyon *et al.* [2] Chapter 1.1;
- Sedov [12] Chapters 1 & 2.1–2.2;
- Sommerfeld [7, 8] beginning of Chapter II.5.

[(1)] The visualization techniques have evolved since the 1960s, yet pathlines, streaklines or streamlines are still defined in the same way.

CHAPTER II

Kinematics of a continuous medium

The goal of fluid dynamics is to investigate the motion of fluids under consideration of the forces at play, as well as to study the mechanical stresses exerted by moving fluids on bodies with which they are in contact. The description of the motion itself, irrespective of the forces, is the object of *kinematics*.

The possibilities for the motion of a deformable continuous medium, in particular of a fluid, are richer than for a mere point particle or a rigid body: besides translations and global rotations, a deformable medium may also rotate locally and undergo… deformations! The latter term actually encompasses two different yet non-exclusive possibilities, namely either a change of shape or a variation of the volume. All these various types of motion are encoded in the local properties of the velocity field at each instant (Sec. II.1). Generic fluid motions are then classified according to several criteria, especially taking into account kinematics (Sec. II.2).

For the sake of reference, the characterization of deformations themselves, complementing that of their rate of change, is briefly presented in Sec. II.A. That formalism is not needed within fluid dynamics, but rather for the study of deformable solids, like elastic ones.

II.1 Generic motion of a continuous medium

Let $\vec{\mathrm{v}}(t,\vec{r})$ denote the velocity field in a continuous medium, measured with respect to some reference frame $\mathcal{R}$. To illustrate (some of) the possible motions that occur in a deformable body, Fig. II.1 shows the positions at successive instants t and $t+\delta t$ of a small "material vector" $\delta\vec{\ell}(t)$, that is, a continuous set of material points distributed along the straight line element stretching between two neighboring geometrical points. Let $\vec{r}$ and $\vec{r}+\delta\vec{\ell}(t)$ denote the geometrical endpoints of this material vector at time t.

Thanks to the continuity of the mappings $\vec{R}\mapsto\vec{r}(t,\vec{r})$ and its inverse $\vec{r}\mapsto\vec{R}(t,\vec{r})$, the material vector defined at instant t remains a connected set of material points as time evolves, in particular at $t+\delta t$. Assuming that both the initial length $|\delta\vec{\ell}(t)|$ as well as δt are small enough, the evolved set at $t+\delta t$ remains approximately along a straight line, and constitutes a new material vector, denoted $\delta\vec{\ell}(t+\mathrm{d}t)$. The position vectors of its endpoints simply follow from the initial positions of the corresponding material points: $\vec{r}$ resp. $\vec{r}+\delta\vec{\ell}(t)$, to which should be added the respective

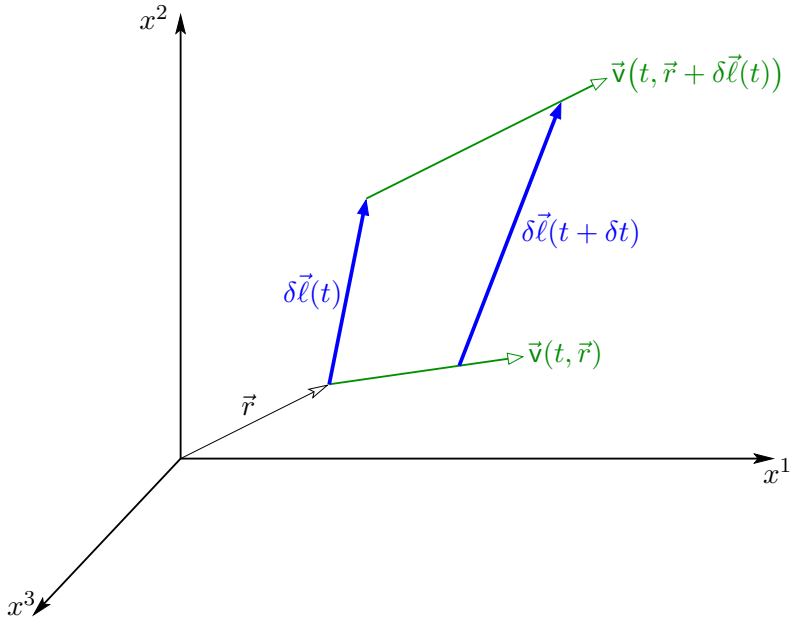

Figure II.1 – Positions of a material line element $\delta\vec{\ell}$ at successive times t and $t+\delta t$.

displacement vectors between t and t+δt, namely the product by δt of the initial velocity $\vec{\mathrm{v}}\big(t,\vec{r}\big)$ resp. $\vec{\mathrm{v}}\big(t,\vec{r}+\delta\vec{\ell}(t)\big)$. That is, one finds

$$\delta\,\vec{\ell}(t+\delta t) = \delta\vec{\ell}(t) + \big[\vec{\mathrm{v}}\big(t,\vec{r}+\delta\vec{\ell}(t)\big) - \vec{\mathrm{v}}\big(t,\vec{r}\big)\big]\delta t + \mathcal{O}\big(\delta t^2\big). \tag{II.1}$$

Figure II.1 already suggests that the motion of the material vector consists not only of a translation, but also of a rotation, as well as an "expansion"—the change in length of the vector.

II.1.1 Local distribution of velocities in a continuous medium

Considering first a fixed time t, let $\vec{\mathrm{v}}(t,\vec{r})$ resp. $\vec{\mathrm{v}}(t,\vec{r})+\delta\vec{\mathrm{v}}$ be the velocity at the geometric point situated at position $\vec{r}$ resp. at $\vec{r}+\delta\vec{r}$ in $\mathcal{R}$.

Introducing for simplicity a system of Cartesian coordinates (x^1,x^2,x^3) in $\mathcal{R}$, the Taylor expansion of the i-th component of the velocity field—which is at least piecewise $\mathscr{C}^1$ in its variables, see § I.3.2—gives to first order

$$\delta\mathrm{v}^i \simeq \sum_{j=1}^{3} \frac{\partial\mathrm{v}^i(t,\vec{r})}{\partial x^j}\,\delta x^j, \tag{II.2a}$$

where $\{\delta x^j\}$ denote the components of $\delta\vec{r}$. Introducing the $\binom{1}{1}$-tensor $\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})$ whose components in the coordinate system used here are the partial derivatives $\partial\mathrm{v}^i(t,\vec{r})/\partial x^j$, the above relation can be recast in the coordinate-independent form

$$\delta\vec{\mathrm{v}} \simeq \vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\cdot\delta\vec{r}. \tag{II.2b}$$

Like every rank 2 tensor, the *velocity gradient tensor* $\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})$ at time t and position $\vec{r}$ can be decomposed into the sum of the symmetric and an antisymmetric part:

$$\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r}) = \mathbf{D}(t,\vec{r}) + \mathbf{R}(t,\vec{r}), \tag{II.3a}$$

where one conventionally writes

$$\mathbf{D}(t,\vec{r}) \equiv \frac{1}{2}\Big(\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r}) + \big[\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}}\Big), \qquad \mathbf{R}(t,\vec{r}) \equiv \frac{1}{2}\Big(\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r}) - \big[\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}}\Big) \tag{II.3b}$$

with $\big[\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}}$ the transposed tensor to $\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})$. These definitions are to be understood as follows:

Using the same Cartesian coordinate system as above, the components of the two tensors $\mathbf{D}$, $\mathbf{R}$, viewed for simplicity as $\binom{0}{2}$-tensors, respectively read

$$\mathbf{D}_{ij}(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial \mathsf{v}_i(t,\vec{r})}{\partial x^j} + \frac{\partial \mathsf{v}_j(t,\vec{r})}{\partial x^i}\right], \qquad \mathbf{R}_{ij}(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial \mathsf{v}_i(t,\vec{r})}{\partial x^j} - \frac{\partial \mathsf{v}_j(t,\vec{r})}{\partial x^i}\right]. \tag{II.3c}$$

Note that here we have silently used the fact that for Cartesian coordinates, the position—subscript or superscript—of the index does not change the value of the component, i.e. numerically $\mathsf{v}_i = \mathsf{v}^i$ for every $i \in \{1,2,3\}$.

> Relations (II.3c) clearly represent the desired symmetric and antisymmetric parts. However, one sees that the definitions would not appear to fulfill their task if the indices were not both either up or down, as e.g.
>
> $$\mathbf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial \mathsf{v}^i(t,\vec{r})}{\partial x^j} + \frac{\partial \mathsf{v}_j(t,\vec{r})}{\partial x_i}\right],$$
>
> in which the symmetry is no longer obvious. The trick is to rewrite the previous identity as
>
> $$\mathbf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}\delta^{ik}\delta^l{}_j\left[\frac{\partial \mathsf{v}_k(t,\vec{r})}{\partial x^l} + \frac{\partial \mathsf{v}_l(t,\vec{r})}{\partial x^k}\right] = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\partial \mathsf{v}_k(t,\vec{r})}{\partial x^l} + \frac{\partial \mathsf{v}_l(t,\vec{r})}{\partial x^k}\right],$$
>
> where we have used the fact that the metric tensor of Cartesian coordinates coincides with the Kronecker symbol. To fully generalize to curvilinear coordinates, the partial derivatives in the rightmost term should be replaced by the covariant derivatives discussed in Appendix **??**, leading eventually to
>
> $$\mathbf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}_k(t,\vec{r})}{\mathrm{d}x^l} + \frac{\mathrm{d}\mathsf{v}_l(t,\vec{r})}{\mathrm{d}x^k}\right] \tag{II.4a}$$
>
> $$\mathbf{R}^i{}_j(t,\vec{r}) = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}_k(t,\vec{r})}{\mathrm{d}x^l} - \frac{\mathrm{d}\mathsf{v}_l(t,\vec{r})}{\mathrm{d}x^k}\right] \tag{II.4b}$$
>
> With these new forms, which are valid in any coordinate system, the raising or lowering of indices does not affect the visual symmetric or antisymmetric aspect of the tensor.

Using the tensors $\mathbf{D}$ and $\mathbf{R}$ we just introduced, whose physical meaning will be discussed at length in § II.1.2–II.1.3, relation (II.2b) can be recast as

$$\vec{\mathsf{v}}\big(t,\vec{r}+\delta\vec{r}\big) = \vec{\mathsf{v}}\big(t,\vec{r}\big) + \mathbf{D}(t,\vec{r})\cdot\delta\vec{r} + \mathbf{R}(t,\vec{r})\cdot\delta\vec{r} + \mathcal{O}\big(|\delta\vec{r}|^2\big) \tag{II.5}$$

where as stated at the beginning every field is considered at the same time.

Under consideration of relation (II.5) with $\delta\vec{r} = \delta\vec{\ell}(t)$, Eq. (II.1) for the time evolution of the material line element becomes

$$\delta\vec{\ell}(t+\delta t) = \delta\vec{\ell}(t) + \big[\mathbf{D}(t,\vec{r})\cdot\delta\vec{\ell}(t) + \mathbf{R}(t,\vec{r})\cdot\delta\vec{\ell}(t)\big]\delta t + \mathcal{O}\big(\delta t^2\big).$$

Subtracting $\delta\vec{\ell}(t)$ from both sides, dividing by δt and taking the limit $\delta t \to 0$, one finds for the rate of change of the material vector:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{\ell}(t) = \mathbf{D}(t,\vec{r})\cdot\delta\vec{\ell}(t) + \mathbf{R}(t,\vec{r})\cdot\delta\vec{\ell}(t) \tag{II.6}$$

In the following two subsections, we shall investigate the physical content of each of the tensors $\mathbf{R}(t,\vec{r})$ and $\mathbf{D}(t,\vec{r})$.

II.1.2 Rotation rate tensor and vorticity vector

The tensor $\mathbf{R}(t,\vec{r})$ defined by Eq. (II.3b) is called, for reasons that will become clearer below, *rotation rate tensor*.[(xxii)]

By construction, this tensor is antisymmetric. Accordingly, one can naturally associate with it a dual (pseudo)-vector $\vec{\Omega}(t,\vec{r})$, such that for any vector $\vec{V}$

$$\mathbf{R}\big(t,\vec{r}\big)\cdot\vec{V}=\vec{\Omega}\big(t,\vec{r}\big)\times\vec{V}\quad\forall\vec{V}\in\mathbb{R}^3. \tag{II.7}$$

In Cartesian coordinates, the components of $\vec{\Omega}(t,\vec{r})$ are related to those of the rotation rate tensor by

$$\Omega^i(t,\vec{r})\equiv-\frac{1}{2}\sum_{j,k=1}^{3}\epsilon^{ijk}\mathbf{R}_{jk}(t,\vec{r}) \tag{II.8a}$$

with ϵ^{ijk} the totally antisymmetric Levi-Civita symbol. Using the antisymmetry of $\mathbf{R}\big(t,\vec{r}\big)$, this equivalently reads

$$\Omega^1(t,\vec{r})\equiv-\mathbf{R}_{23}(t,\vec{r}),\quad\Omega^2(t,\vec{r})\equiv-\mathbf{R}_{31}(t,\vec{r}),\quad\Omega^3(t,\vec{r})\equiv-\mathbf{R}_{12}(t,\vec{r}). \tag{II.8b}$$

Comparing with Eq. (II.3c), one finds

$$\vec{\Omega}(t,\vec{r})=\frac{1}{2}\vec{\nabla}\times\vec{\mathsf{v}}(t,\vec{r}). \tag{II.9}$$

Proof of Eqs. (II.7), (II.9): introducing the Cartesian components $\{V^j\}$ of $\vec{V}$ and dropping for brevity the $(t,\vec{r})$-dependence of fields, the i-th component of $\mathbf{R}\cdot\vec{V}$ reads

$$\mathbf{R}_{ij}V^j=\frac{1}{2}\big(\partial_j\mathsf{v}_i-\partial_i\mathsf{v}_j\big)V^j,$$

where we used the summation convention over the repeated index j and the shorthand notation ∂_i for the partial derivative with respect to x^i. This may further be rewritten as

$$\mathbf{R}_{ij}V^j=-\frac{1}{2}\big(\delta_i^k\delta_j^l-\delta_j^k\delta_i^l\big)(\partial_k\mathsf{v}_l)V^j,$$

which now involves three sums. The term with the four Kronecker symbols is in fact the sum (over a fifth index m) of the product $\epsilon_{ijm}\epsilon^{mkl}$ of Levi-Civita symbols:

$$\mathbf{R}_{ij}V^j=-\frac{1}{2}\epsilon_{ijm}\epsilon^{mkl}(\partial_k\mathsf{v}_l)V^j.$$

On the right hand side of this identity, $\epsilon^{mkl}\partial_k\mathsf{v}_l$ is the m-th component of the curl $\vec{\nabla}\times\vec{\mathsf{v}}$, i.e. using definition (II.9):

$$\mathbf{R}_{ij}V^j=-\epsilon_{ijm}\Omega^mV^j=\epsilon_{imj}\Omega_mV^j,$$

which is precisely the i-th component of $\Omega\times\vec{\mathsf{v}}$. □

Let us now rewrite relation (II.6) with the help of the vector $\vec{\Omega}(t,\vec{r})$, assuming that $\mathbf{D}(t,\vec{r})$ vanishes so as to isolate the effect of the remaining term. Under this assumption, the rate of change of the material vector between two neighboring points reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{\ell}(t)=\mathbf{R}(t,\vec{r})\cdot\delta\vec{\ell}(t)=\vec{\Omega}(t,\vec{r})\times\delta\vec{\ell}(t). \tag{II.10}$$

The term on the right hand side is then exactly the rate of rotation of a vector $\delta\vec{\ell}(t)$ in the motion of a rigid body with instantaneous angular velocity $\vec{\Omega}(t,\vec{r})$. Accordingly, the pseudovector $\vec{\Omega}(t,\vec{r})$ is referred to as *local angular velocity*.[(xxiii)] This a posteriori justifies the denomination *rotation rate tensor* for the antisymmetric tensor $\mathbf{R}(t,\vec{r})$.

[(xxii)] *Wirbeltensor* [(xxiii)] *Wirbelvektor*

Remarks:

∗ Besides the local angular velocity $\vec{\Omega}(t,\vec{r})$, one also defines the *vorticity vector*[(xxiv)] as the curl of the velocity field

$$\boxed{\vec{\omega}(t,\vec{r}) \equiv \vec{\nabla}\times\vec{\mathsf{v}}(t,\vec{r})} = 2\vec{\Omega}(t,\vec{r}). \tag{II.11}$$

In fluid mechanics, the vorticity is actually more often used than the local angular velocity.

∗ The local angular velocity $\vec{\Omega}(t,\vec{r})$ or equivalently the vorticity vector $\vec{\omega}(t,\vec{r})$ define, at fixed t, divergence-free (pseudo)vector fields, since obviously $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\mathsf{v}})=0$. The corresponding field lines are called *vorticity lines*[(xxv)] and are given by [cf. Eq. (I.15)]

$$\mathrm{d}\vec{x}\times\vec{\omega}(t,\vec{r}) = \vec{0} \tag{II.12a}$$

or equivalently, at a point where none of the components of the vorticity vector vanishes,

$$\frac{\mathrm{d}x_1}{\omega^1(t,\vec{r})} = \frac{\mathrm{d}x_2}{\omega^2(t,\vec{r})} = \frac{\mathrm{d}x_3}{\omega^3(t,\vec{r})}. \tag{II.12b}$$

II.1.3 Strain rate tensor

According to the previous subsection, the local rotational motion of a material vector is governed by the (local and instantaneous) rotation rate tensor $\mathbf{R}(t,\vec{r})$. In turn, the translational motion is simply the displacement—which must be described in an affine space, not a vector one—of one of the endpoints of $\delta\vec{\ell}$ by an amount given by the product of velocity and length of time interval. That is, both components of the motion of a rigid body are already accounted for without invoking the symmetric tensor $\mathbf{D}(t,\vec{r})$.

In other words, the tensor $\mathbf{D}(t,\vec{r})$ characterizes the local deviation between the velocity fields in a deformable body, in particular a fluid, and in a rigid body rotating with angular velocity $\vec{\Omega}(t,\vec{r})$. Accordingly, it is called *strain rate tensor* or *deformation rate tensor*.[(xxvi)]

As we shall now see, the diagonal and off-diagonal components of $\mathbf{D}(t,\vec{r})$ actually describe the rates of change of different kinds of deformation. For simplicity, we assume throughout this subsection that $\vec{\Omega}(t,\vec{r})=\vec{0}$.

II.1.3 a Diagonal components

We first assume that all off-diagonal terms in the strain rate tensor vanish: $\mathbf{D}_{ij}(t,\vec{r})=0$ for $i\neq j$, so as to isolate the meaning of the diagonal components.

Going back to Eq. (II.1), let us simply project it along one of the axes of the coordinate system, say along direction i. Denoting $\delta\ell^i$ the i-th component of $\delta\vec{\ell}$, one thus finds

$$\delta\ell^i(t+\delta t) = \delta\ell^i(t) + \big[\mathsf{v}^i\big(t,\vec{r}+\delta\vec{\ell}(t)\big) - \mathsf{v}^i\big(t,\vec{r}\big)\big]\delta t + \mathcal{O}\big(\delta t^2\big).$$

Taylor-expanding the term between square brackets to first order then yields

$$\delta\ell^i(t+\delta t) - \delta\ell^i(t) \simeq \sum_{j=1}^{3}\frac{\partial\mathsf{v}^i(t,\vec{r})}{\partial x^j}\,\delta\ell^j(t)\,\delta t,$$

up to terms of higher order in $|\delta\vec{\ell}(t)|$ or δt. Since we have assumed that both $\vec{\Omega}(t,\vec{r})$—or equivalently the components $\mathbf{R}_{ij}(t,\vec{r})$ of the rotation rate tensor—and the off-diagonal $\mathbf{D}_{ij}(t,\vec{r})$ with $i\neq j$ vanish, one checks that the partial derivative $\partial\mathsf{v}^i(t,\vec{r})/\partial x^j$ vanishes for $i\neq j$. That is, the only non-zero term in the sum is that with $j=i$, so that the equation simplifies to

$$\delta\ell^i(t+\delta t) - \delta\ell^i(t) \simeq \frac{\partial\mathsf{v}^i(t,\vec{r})}{\partial x^i}\,\delta\ell^i(t)\,\delta t = \mathbf{D}^i{}_i(t,\vec{r})\,\delta\ell^i(t)\,\delta t.$$

[(xxiv)] *Wirbligkeit* [(xxv)] *Wirbellinien* [(xxvi)] *Verzerrungsgeschwindigkeitstensor, Deformationsgeschwindigkeitstensor*

Thus, the relative elongation in δt of the i-th component—remember that there is no local rotation, so that the change in $\delta\ell^i$ is entirely due to a variation of the length of the material vector—is given by

$$\frac{\delta\ell^i(t+\delta t)-\delta\ell^i(t)}{\delta\ell^i(t)} = \mathbf{D}^i{}_i(t,\vec{r})\,\delta t \tag{II.13}$$

or alternatively, taking the limit $\delta t \to 0$

$$\frac{1}{\delta\ell^i(t)}\frac{\mathrm{d}}{\mathrm{d}t}\delta\ell^i(t) = \mathbf{D}^i{}_i(t,\vec{r}). \tag{II.14}$$

This equation means that the diagonal component $\mathbf{D}^i{}_i(t,\vec{r})$ represents the local rate of linear elongation in direction i.

Volume expansion rate

Instead of considering a one-dimensional material vector, one can study the evolution of a small "material rectangular parallelepiped" of continuous medium, situated at time t at position $\vec{r}$ with instantaneous edge lengths $\delta L^1(t)$, $\delta L^2(t)$, $\delta L^3(t)$—where for simplicity the coordinate axes are taken along the parallelepiped edges—, so that its volume at time t is simply $\delta\mathcal{V}(t) = \delta L^1(t)\,\delta L^2(t)\,\delta L^3(t)$.

Taking into account Eq. (II.13) for the relative elongation of each edge length, one finds that the relative change in volume between t and $t+\delta t$ is

$$\begin{aligned}\frac{\delta\mathcal{V}(t+\delta t)-\delta\mathcal{V}(t)}{\delta\mathcal{V}(t)} &= \frac{\delta L^1(t+\delta t)-\delta L^1(t)}{\delta L^1(t)} + \frac{\delta L^2(t+\delta t)-\delta L^2(t)}{\delta L^2(t)} + \frac{\delta L^3(t+\delta t)-\delta L^3(t)}{\delta L^3(t)} \\ &= \left[\mathbf{D}^1{}_1(t,\vec{r}) + \mathbf{D}^2{}_2(t,\vec{r}) + \mathbf{D}^3{}_3(t,\vec{r})\right]\delta t.\end{aligned}$$

In the second line, one recognizes the *trace* of the tensor $\mathbf{D}(t,\vec{r})$, which going back to the definition of the latter is equal to the divergence of the velocity fluid:

$$\mathbf{D}^1{}_1(t,\vec{r}) + \mathbf{D}^2{}_2(t,\vec{r}) + \mathbf{D}^3{}_3(t,\vec{r}) = \frac{\partial \mathsf{v}^1(t,\vec{r})}{\partial x^1} + \frac{\partial \mathsf{v}^2(t,\vec{r})}{\partial x^2} + \frac{\partial \mathsf{v}^3(t,\vec{r})}{\partial x^3} = \vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}).$$

That is, this divergence represents the local and instantaneous *volume expansion rate* of the continuous medium. Accordingly, the flow of a fluid is referred to as *incompressible* in some region when the velocity field in that region is divergence-free:

$$\boxed{\text{incompressible flow} \quad\Leftrightarrow\quad \vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}) = 0.} \tag{II.15}$$

We shall comment on this definition in § II.2.3.

II.1.3 b Off-diagonal components

Let us now assume that $\mathbf{D}_{12}(t,\vec{r})$, and thereby automatically $\mathbf{D}_{21}(t,\vec{r})$, is the only non-vanishing component of the strain rate tensor. To see the influence of that component, we need to consider the time evolution of a different object than a material vector, since anything that can affect the latter—translation, rotation, dilatation—has already been described above.

Accordingly, we now look at the change between successive instants t and $t+\delta t$ of an elementary "material rectangle", as pictured in Fig. II.2. We denote by $\vec{\mathsf{v}}$ resp. $\vec{\mathsf{v}}+\delta\vec{\mathsf{v}}$ the velocity at time t at the lower left resp. upper right corner of the rectangle. Taylor expansions give for the Cartesian components of the shift $\delta\vec{\mathsf{v}}$

$$\delta\mathsf{v}_1 = \frac{\partial \mathsf{v}_1(t,\vec{r})}{\partial x^2}\,\delta\ell^2, \qquad \delta\mathsf{v}_2 = \frac{\partial \mathsf{v}_2(t,\vec{r})}{\partial x^1}\,\mathrm{d}\ell^1.$$

Figure II.2 shows that what is a right angle at time t becomes an angle $\pi/2-\delta\alpha$ at $t+\mathrm{d}t$, where $\delta\alpha = \delta\alpha_1 - \delta\alpha_2$. In the limit of small δt, both $\delta\alpha_1$ and $\delta\alpha_2$ will be small and thus approximately

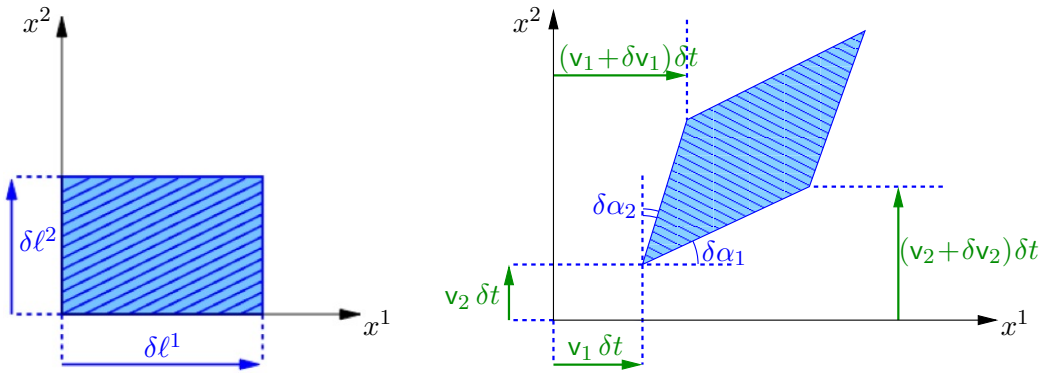

Figure II.2 – Evolution of a material rectangle caught in the motion of a continuous medium between times t (left) and $t+\delta t$ (right).

equal to their respective tangents. Using the fact that the parallelogram still has the same area—since the diagonal components of $\mathbf{D}$ vanish—the projection of any side of the deformed rectangle at time $t+\delta t$ on its original direction at time t keeps approximately the same length, up to corrections of order δt. One thus finds for the oriented angles

$$\delta\alpha_1 \simeq \frac{\delta\mathsf{v}_2\,\delta t}{\delta\ell^1} \quad \text{and} \quad \delta\alpha_2 \simeq -\frac{\delta\mathsf{v}_1\,\delta t}{\delta\ell^2}.$$

With the Taylor expansions given above, this leads to

$$\delta\alpha_1 \simeq \frac{\partial\mathsf{v}_2(t,\vec{r})}{\partial x^1}\,\delta t, \qquad \delta\alpha_2 \simeq -\frac{\partial\mathsf{v}_1(t,\vec{r})}{\partial x^2}\,\delta t.$$

Gathering all pieces, one finds

$$\frac{\delta\alpha}{\delta t} \simeq \frac{\partial\mathsf{v}_2(t,\vec{r})}{\partial x^1} + \frac{\partial\mathsf{v}_1(t,\vec{r})}{\partial x^2} = 2\,\mathbf{D}_{12}(t,\vec{r}). \tag{II.16}$$

In the limit $\delta t \to 0$, one sees that the off-diagonal component $\mathbf{D}_{12}(t,\vec{r})$ represents half the local velocity of the "angular deformation"—the *shear*—around direction x^3.

Remark: To separate the two physical effects present in the strain rate tensor, the latter is often written as the sum of a diagonal *rate-of-expansion tensor* proportional to the identity $\mathbb{1}$—which is in fact the $\binom{1}{1}$-form of the metric tensor $\mathbf{g}$ of Cartesian coordinates—and a traceless *rate-of-shear tensor* $\mathbf{S}$:

$$\mathbf{D}(t,\vec{r}) = \frac{1}{3}\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\mathbb{1} + \mathbf{S}(t,\vec{r}) \tag{II.17a}$$

with

$$\mathbf{S}(t,\vec{r}) \equiv \frac{1}{2}\left(\vec{\nabla}\vec{\mathbf{v}}(t,\vec{r}) + \big[\vec{\nabla}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}} - \frac{2}{3}\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\mathbb{1}\right). \tag{II.17b}$$

Component-wise, and generalizing to curvilinear coordinates, this reads

$$\mathbf{D}_{ij}(t,\vec{r}) = \frac{1}{3}\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]g_{ij}(t,\vec{r}) + \mathbf{S}_{ij}(t,\vec{r}) \tag{II.17c}$$

with [cf. Eq. (II.4a)]

$$\mathbf{S}_{ij}(t,\vec{r}) \equiv \frac{1}{2}\left[g^k_{\,i}(t,\vec{r})\,g^l_{\,j}(t,\vec{r})\left(\frac{\mathrm{d}\mathsf{v}_k(t,\vec{r})}{\mathrm{d}x^l} + \frac{\mathrm{d}\mathsf{v}_l(t,\vec{r})}{\mathrm{d}x^k}\right) - \frac{2}{3}\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]g_{ij}(t,\vec{r})\right]. \tag{II.17d}$$

Summary

Gathering the findings of this Section, the most general motion of a material volume element inside a continuous medium, in particular in a fluid, can be decomposed in four elements:

- a translation;
- a rotation, with a local angular velocity $\vec{\Omega}(t,\vec{r})$ given by Eq. (II.9)—i.e. related to the anti-symmetric part $\mathbf{R}(t,\vec{r})$ of the velocity gradient—and equal to twice the (local) vorticity vector $\vec{\omega}(t,\vec{r})$;
- a local dilatation or contraction, in which the geometric form of the material volume element remains unchanged, whose rate is given by the divergence of the velocity field $\vec{\nabla}\cdot\vec{v}(t,\vec{r})$, i.e. encoded in the diagonal elements of the strain rate tensor $\mathbf{D}(t,\vec{r})$;
- a change of shape ("deformation") of the material volume element at constant volume, controlled by the rate-of-shear tensor $\mathbf{S}(t,\vec{r})$ [Eqs. (II.17b), (II.17d)], obtained by taking the traceless symmetric part of the velocity gradient.

Remark: In the case of a uniform motion, all spatial derivatives are by definition zero, so that the vorticity $\vec{\omega}(t,\vec{r})$, the expansion rate $\vec{\nabla}\cdot\vec{v}(t,\vec{r})$ and the rate-of-shear tensor $\mathbf{S}(t,\vec{r})$ actually vanish everywhere in the flow. Accordingly, the motion of a material element in that case is simply a pure translation, without deformation or rotation.

II.2 Classification of fluid flows

The motion, or *flow*[(xxvii)], of a fluid can be characterized according to several criteria, either purely geometrical (§ II.2.1), kinematic (§ II.2.2), or of a more physical nature (§ II.2.3), that takes into account the physical behavior of the flowing fluid in its evolution.

II.2.1 Geometrical criteria

In the general case, the quantities characterizing the properties of a fluid flow will depend on time as well as on three spatial coordinates.

For some more or less idealized models of actual flows, it may turn out that only two spatial coordinates play a role, in which case one talks of a *two-dimensional flow*. An example is the flow of air around the wing of an airplane, which in first approximation is "infinitely" long compared to its transverse profile: the (important!) effects at the ends of the wing, which introduce the dependence on the spatial dimension along the wing, may be left aside in a first approach, then considered in a second, more detailed step.

In some cases, e.g. for fluid flows in pipes, one may even assume that the properties only depend on a single spatial coordinate, so that the flow is *one-dimensional*. In that approximation, the physical local quantities are actually often replaced by their average value over the cross section of the pipe.

On a different level, one also distinguishes between *internal* und *external* fluid flows, according to whether the fluid is enclosed inside solid walls—e.g. in a pipe—or flowing around a body—e.g. around an airplane wing.

II.2.2 Kinematic criteria

The notions of *uniform*—that is independent of position—and *steady*—independent of time—motions were already introduced at the end of § I.3.3. Accordingly, there are *non-uniform* and *unsteady* fluids flows.

If the vorticity vector $\vec{\omega}(t,\vec{r})$ vanishes at every point $\vec{r}$ of a flowing fluid, then the corresponding motion is referred to as an *irrotational flow*[(xxviii)] or, for reasons that will be clarified in Sec. IV.4, *potential flow*. The opposite case is that of a *vortical* or *rotational flow*.[(xxix)]

[(xxvii)] *Strömung* [(xxviii)] *wirbelfreie Strömung* [(xxix)] *Wirbelströmung*

According to whether the flow velocity v is smaller or larger than the (local) speed of sound c_s in the fluid, one talks of *subsonic* or *supersonic* motion[(xxx)], corresponding respectively to a dimensionless *Mach number*[(g)]

$$\boxed{\mathrm{Ma} \equiv \frac{\mathsf{v}}{c_s}} \tag{II.18}$$

smaller or larger than 1. Note that the Mach number can a priori be defined, and take different values $\mathrm{Ma}(t, \vec{r})$, at every point in a flow.

When the fluid flows in layers that do not mix with each other, so that the streamlines remain parallel, the flow is referred to as *laminar*. In the opposite case the flow is *turbulent*.

II.2.3 Physical criteria

All fluids are compressible, more or less according to the substance and its thermodynamic state. Nevertheless, this compressibility is sometimes irrelevant for a given motion, in which case it may be fruitful to consider that the fluid flow is *incompressible*, which, as seen in § II.1.3 a, technically means that its volume expansion rate vanishes, $\vec{\nabla} \cdot \vec{\mathsf{v}} = 0$. In the opposite case ($\vec{\nabla} \cdot \vec{\mathsf{v}} \neq 0$), the flow is said to be *compressible*. It is however important to realize that the statement is more a kinematic one, than really reflecting the thermodynamic compressibility of the fluid.

> In practice, flows are compressible in regions where the fluid velocity is "large", namely where the Mach number (II.18) is not much smaller than 1, i.e. roughly speaking $\mathrm{Ma} \gtrsim 0.2$.

In an analogous manner, one speaks of viscous resp. non-viscous flows to express the fact that the fluid under consideration is modeled as viscous resp. inviscid—which leads to different equations of motion—, irrespective of the fact that every real fluid has a non-zero viscosity.

Other thermodynamic criteria are also used to characterize possible fluid motions: isothermal flows—i.e. in which the temperature is uniform and remains constant—, isentropic flows—i.e. without production of entropy—, and so on.

Bibliography for Chapter II

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- Faber [1] Chapter 2.4;
- Feynman [10, 11] Chapter 39–1;
- Guyon *et al.* [2] Chapters 3.1, 3.2;
- Sommerfeld [7, 8] Chapter I.

(xxx) *Unterschall-* bzw. *Überschallströmung*

(g) E. Mach, 1838–1916

Appendix to Chapter II

II.A Deformations in a continuous medium

Strain tensor...

Chapter III

Fundamental equations of non-relativistic fluid dynamics

Some of the most fundamental laws of physics are conservation equations for various quantities: energy, linear and angular momentum, electric charge, and so on. When applying these laws to many-body systems, in particular to continuous media like moving fluids, care must be taken to consider isolated and closed systems, to ensure their validity. At the very least, the amount of quantity exchanged with the exterior of the system—for example the change in linear momentum per unit time due to external forces, as given by Newton's second law, or the change in energy due to the mechanical work of these forces—must be quantifiable.

When this is the case, it is possible to re-express global conservation laws or more generally balance equations—given in terms of macroscopic quantities like total mass, total energy, total momentum, etc.—in a local form involving densities, using the generic recipe provided by Reynolds'[(h)] transport theorem (Sec. III.1). In the framework of a non-relativistic theory, in which the mass or equivalently the particle number of a closed system is conserved, one may thus derive a general continuity equation, holding at every point of the fluid (Sec. III.2).

The same approach may be followed to derive equations expressing the time evolution of momentum or energy under the influence of external forces acting at every point of the fluid. In either case, it is however necessary to account for the possibility that several physical phenomena may contribute to the transport of momentum and energy. One defines various fluid-dynamical models according to the level of approximation at which the possible forms of transport are incorporated in the description. To those different models correspond distinct sets of equations for the local expression of Newton's second law (Sec. III.3) or of energy balance (Sec. III.4).

[(h)] O. Reynolds, 1842–1912

III.1 Reynolds transport theorem

The material derivative of a quantity was already introduced in § I.3.4, where its action on a local function of both time t and position vector $\vec{r}$ was defined. In this Section, we shall derive a formula for the substantial derivative of an extensive physical quantity carried by a "macroscopic" material system. This formula will in the remainder of the Chapter represent the key relation that will allow us to express the usual (conservation) laws of Newtonian mechanics, which hold for closed systems, in terms of Eulerian variables.

III.1.1 Closed system, open system

Consider the motion of a continuous medium, in particular a flowing fluid, described in a reference frame $\mathcal{R}$. Let $\mathcal{S}$ be an arbitrary closed geometrical surface that remains fixed in $\mathcal{R}$. This surface will hereafter be referred to as *control surface*, and the geometrical volume $\mathcal{V}$ it encloses as *control volume*. Due to the macroscopic transport of matter in the flowing medium, the fluid contained inside a given control surface represents an *open system*, which can exchange matter and energy with its exterior as time elapses.

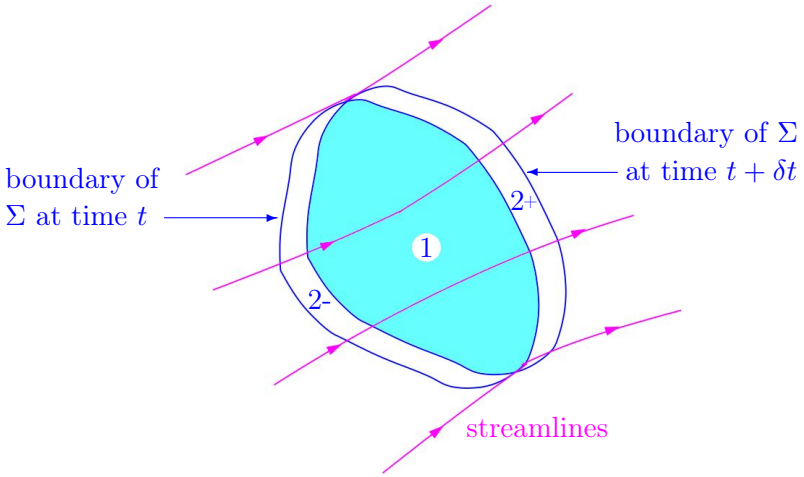


Figure III.1 – Time evolution of a closed material system transported in the motion of a continuous medium.

In contrast, let Σ be a *closed* system, whose constituent material points occupy at some given time t the volume $\mathcal{V}_\Sigma(t)$. At a shortly later time $t+\delta t$, the material system Σ has moved and now occupies a new volume $\mathcal{V}_\Sigma(t+\delta t)$. On Fig. III.1, one can distinguish between three regions in position space:

- (1), which is common to the successive positions of Σ at t and $t+\delta t$; the corresponding volume will be denoted $\mathcal{V}$ and its surface $\partial\mathcal{V}$;
- (2−), which is left behind by Σ between t and $t+\delta t$;
- (2+), into which Σ penetrates between t and $t+\delta t$.

Note already that in the limit δt going to zero, the volumes $\mathcal{V}_\Sigma(t)$, $\mathcal{V}_\Sigma(t+\delta t)$ and $\mathcal{V}$ coincide; more precisely, their difference is of order δt.

III.1.2 Material derivative of an extensive quantity

Let $\mathcal{G}(t)$ be one of the extensive quantities that characterize a macroscopic physical property of the closed material system Σ. To this extensive quantity, one can associate at every point $\vec{r}$ the corresponding intensive *specific density* $g_{\rm m}(t,\vec{r})$, defined as the local amount of $\mathcal{G}$ per unit mass of

matter. Denoting by $\mathrm{d}\mathcal{G}(t,\vec{r})$ resp. $\mathrm{d}M(t,\vec{r})$ the amount of $\mathcal{G}$ resp. the mass inside a small material volume at position $\vec{r}$ at time t, one can write symbolically

$$\mathcal{g}_{\mathrm{m}}(t,\vec{r}) = \frac{\mathrm{d}\mathcal{G}(t,\vec{r})}{\mathrm{d}M(t,\vec{r})}, \tag{III.1}$$

where the notation with differentials is used to suggest that the identity holds in the limit of a small material volume.

For instance, the linear momentum resp. the kinetic energy of a mass $\mathrm{d}M$ of fluid moving with velocity $\vec{\mathrm{v}}$ is $\mathrm{d}\vec{P} = (\mathrm{d}M)\vec{\mathrm{v}}$ resp. $\mathrm{d}K = \frac{1}{2}(\mathrm{d}M)\vec{\mathrm{v}}^2$, so that the associated specific density is $\mathrm{d}\vec{P}/\mathrm{d}M = \vec{\mathrm{v}}$ resp. $\mathrm{d}K/\mathrm{d}M = \frac{1}{2}\vec{\mathrm{v}}^2$.

Remark: These examples illustrate the fact that the tensorial nature—scalar, vector, tensor of higher rank—of the function associated with quantity $\mathcal{G}$ can be arbitrary.

For a material system Σ occupying at time t a volume $\mathcal{V}_\Sigma(t)$, Eq. (III.1) leads to

$$\mathcal{G}(t) = \int_{\mathcal{V}_\Sigma(t)} \mathcal{g}_{\mathrm{m}}(t,\vec{r})\,\mathrm{d}M(t,\vec{r}) = \int_{\mathcal{V}_\Sigma(t)} \mathcal{g}_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\,\mathrm{d}^3\vec{r} \tag{III.2}$$

for the value of $\mathcal{G}$ of the system, where in the second identity $\rho(t,\vec{r}) = \mathrm{d}M/\mathrm{d}^3\vec{r}$ is the local mass density.

Let us now assume that the material system Σ is moving as part of a larger, flowing continuous medium. To find the substantial derivative $\mathrm{D}\mathcal{G}(t)/\mathrm{D}t$ of $\mathcal{G}(t)$, we shall first compute the variation $\delta\mathcal{G}$ for the material system Σ between times t and $t+\delta t$, where δt is assumed to be small. At the end of the calculation, we shall take the limit $\delta t \to 0$.

Going back to the regions (1), (2−), (2+) defined in Fig. III.1, one can write

$$\delta\mathcal{G} = \big(\mathcal{G}_1 + \mathcal{G}_{2+}\big)_{t+\delta t} - \big(\mathcal{G}_1 + \mathcal{G}_{2-}\big)_t = \delta\mathcal{G}_1 + \delta\mathcal{G}_2,$$

where the various indices denote the respective spatial domains and instants, and

$$\delta\mathcal{G}_1 \equiv \big(\mathcal{G}_1\big)_{t+\delta t} - \big(\mathcal{G}_1\big)_t, \qquad \delta\mathcal{G}_2 \equiv \big(\mathcal{G}_{2+}\big)_{t+\delta t} - \big(\mathcal{G}_{2-}\big)_t.$$

- $\delta\mathcal{G}_1$ represents the variation of $\mathcal{G}$ inside region (1) due to the non-stationarity of the fluid flow. Using a Taylor expansion to linear order in δt, one may write

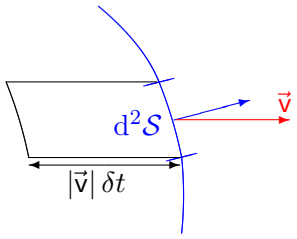
$$\delta\mathcal{G}_1 \simeq \frac{\mathrm{d}\mathcal{G}_1(t)}{\mathrm{d}t}\,\delta t = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{V}} \mathcal{g}_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\,\mathrm{d}^3\vec{r}\right]\delta t.$$

  Since the integration volume $\mathcal{V}$ is time-independent, one can pull the time derivative inside the integral, where it becomes a partial derivative, leading to

$$\delta\mathcal{G}_1 \simeq \int_{\mathcal{V}_\Sigma(t)} \frac{\partial}{\partial t}\big[\mathcal{g}_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\big]\,\mathrm{d}^3\vec{r}\;\delta t. \tag{III.3}$$

- $\delta\mathcal{G}_2$ represents the net amount of $\mathcal{G}$ traversing between t and $t+\delta t$ the control surface $\partial\mathcal{V}$, either leaving (region 2+) or entering (region 2−) the volume $\mathcal{V}$, where in the latter case the amount is counted negatively. This is precisely the *flux*—in the mathematical acceptation of the term—through the surface $\partial\mathcal{V}$, oriented towards the exterior, of an appropriate flux density for quantity $\mathcal{G}$.[(2)]

[(2)] This flux density can be read off Eq. (III.4), namely $\mathcal{g}_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})$.



Let $\vec{\mathrm{v}}$ denote the velocity of the continuous medium at position $\vec{r}$ at time t. The amount of quantity $\mathcal{G}$ that traverses in δt a surface element $\mathrm{d}^2\mathcal{S}$ situated in $\vec{r}$ equals the amount inside an elementary cylinder with base $\mathrm{d}^2\mathcal{S}$ and height $|\vec{\mathrm{v}}|\,\delta t$, i.e. $\mathrm{d}^3\mathcal{G} = g_{\mathrm{m}}\rho\,\mathrm{d}^3\mathcal{V}$, with $\mathrm{d}^3\mathcal{V} = |\mathrm{d}^2\vec{\mathcal{S}}\cdot\vec{\mathrm{v}}|\,\delta t$, where the vector $\mathrm{d}^2\vec{\mathcal{S}}$ is normal to the surface element.

Integrating over all surface elements all over the control surface, the net amount of quantity $\mathcal{G}$ flowing through $\partial\mathcal{V}$ thus reads

$$\delta\mathcal{G}_2 = \oint_{\partial\mathcal{V}} \mathrm{d}^3\mathcal{G} = \oint_{\partial\mathcal{V}} \left[g_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right]\cdot\mathrm{d}^2\vec{\mathcal{S}}\,\delta t. \tag{III.4}$$

All in all, Eqs. (III.3)–(III.4) yield after dividing by δt and taking the limit $\delta t \to 0$ the so-called *Reynolds transport theorem*:[(xxxi)]

$$\boxed{\frac{\mathrm{D}\mathcal{G}(t)}{\mathrm{D}t} = \int_{\mathcal{V}} \frac{\partial}{\partial t}\left[g_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\right]\mathrm{d}^3\vec{r} + \oint_{\partial\mathcal{V}} \left[g_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right]\cdot\mathrm{d}^2\vec{\mathcal{S}}.} \tag{III.5}$$

The first term on the right hand side of this relation represents a *local* time derivative, similar to the first term in Eq. (I.18), for which reason it is often denoted $\partial\mathcal{G}/\partial t$. In contrast, the second term is of *convective* type, i.e. directly caused by the motion of matter, and represents the transport of $\mathcal{G}$.

Anticipating on the remainder of the Chapter, this theorem will help us as follows. The "usual" laws of dynamics are valid for closed, material systems Σ, rather than for open ones. Accordingly, these laws involve time derivatives "following the system in its motion", which is precisely what the material derivative $\mathrm{D}/\mathrm{D}t$ describes. Reynolds' transport theorem (III.5) expresses the latter, for extensive quantities $\mathcal{G}(t)$, in terms of local densities attached to fixed spatial positions, i.e. in Eulerian variables.

Remarks:

∗ When computing the flux through a surface element, we assumed that the latter is at rest, which defines the reference frame in which velocity $\vec{\mathrm{v}}(t,\vec{r})$ of the flowing medium is to be measured. Similarly, the flow velocity $\vec{\mathrm{v}}(t,\vec{r})$ in Reynolds' transport theorem is measured with respect to a reference frame in which the control surface $\partial\mathcal{V}$ is motionless.

∗ Since relation (III.5) is traditionally referred to as a theorem, one may wonder what are its assumptions. Obviously, the derivation of the result relies on the assumption that the specific density $g_{\mathrm{m}}(t,\vec{r})$ and the velocity field $\vec{\mathrm{v}}(t,\vec{r})$ are both continuous and differentiable, in agreement with the generic hypotheses in § I.2.2. Figure III.1 actually also embodies the hidden, but necessary assumption that the motion is continuous, which leads to the smooth evolution of the connected system of material points which are together inside the control surface $\partial\mathcal{V}$ at time t. Again, this follows from suitable properties of $\vec{\mathrm{v}}$.

∗ Accordingly, the Reynolds transport theorem (III.5) does not hold if the velocity field, or the specific density g_{m}, is discontinuous. As was already mentioned in § I.2.2, such discontinuities are however necessary to account for some phenomena (shock waves, boundary between two immiscible fluids...). In such cases, it will be necessary to reformulate the transport theorem to take into account the discontinuities.

(xxxi) *Reynolds'scher Transportsatz*

III.2 Mass and particle number conservation: continuity equation

The mass M and the particle number N of a closed non-relativistic system Σ remain constant in its motion:

$$\frac{\mathrm{D}M(t)}{\mathrm{D}t} = 0, \qquad \frac{\mathrm{D}N(t)}{\mathrm{D}t} = 0. \tag{III.6}$$

These conservation laws lead with the help of Reynolds' transport theorem to partial differential equations for some of the local fields that characterize a fluid flow.

III.2.1 Integral formulation

For an arbitrary control volume $\mathcal{V}$ delimited by surface $\partial\mathcal{V}$, the Reynolds transport theorem (III.5) with $\mathcal{G}(t) = M$, to which corresponds the specific density $g_{\mathrm{m}}(t,\vec{r}) = 1$, reads

$$\frac{\mathrm{D}M(t)}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{V}} \rho(t,\vec{r})\,\mathrm{d}^3\vec{r}\right] + \oint_{\partial\mathcal{V}} \left[\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right]\cdot \mathrm{d}^2\vec{\mathcal{S}} = 0. \tag{III.7}$$

That is, the time derivative of the mass contained in $\mathcal{V}$ equals the negative of the mass flow rate through $\partial\mathcal{V}$. In agreement with footnote (2), $\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})$ is the *mass flux density*,[(xxxii)] while its integral is the *mass flow rate*.[(xxxiii)]

Taking now $\mathcal{G}(t) = N$, the associated specific density is $g_{\mathrm{m}}(t,\vec{r}) = N/M$. Since the product of N/M with the mass density $\rho(t,\vec{r})$ is precisely the particle number density $n(t,\vec{r})$, Reynolds' theorem (III.5) leads to

$$\frac{\mathrm{D}N(t)}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{V}} n(t,\vec{r})\,\mathrm{d}^3\vec{r}\right] + \oint_{\partial\mathcal{V}} \left[n(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right]\cdot \mathrm{d}^2\vec{\mathcal{S}} = 0, \tag{III.8}$$

where $n(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})$ represents the *particle number flux density*.[(xxxiv)]

Equation (III.7) resp. (III.8) constitutes the *integral formulation* of mass resp. particle number conservation.

Remarks:

∗ In the case of a steady motion, Eq. (III.7) shows that the net mass flow rate through an arbitrary closed geometrical surface $\partial\mathcal{V}$ vanishes. That is, the entrance of some amount of fluid into a (control) volume $\mathcal{V}$ must be compensated by the simultaneous departure of an equal mass from the volume.

∗ Considering "particle number" (and its conservation) in a fluid-based description is somewhat paradoxical. This number is to be understood as representative for any quantum number, carried at the microscopic level by particles, which is conserved in the fluid motion, like electric charge.

III.2.2 Local formulation

Since the control volume $\mathcal{V}$ in Eq. (III.7) resp. (III.8) is time-independent, the time derivative can be exchanged with the integration over volume. Besides, the surface integral can be transformed with the help of Gauss's theorem into a volume integral. All in all, this yields

$$\int_{\mathcal{V}} \left\{\frac{\partial\rho(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\left[\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right]\right\}\mathrm{d}^3\vec{r} = 0,$$

resp.

$$\int_{\mathcal{V}} \left\{\frac{\partial n(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\left[n(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right]\right\}\mathrm{d}^3\vec{r} = 0.$$

(xxxii) *Massenstromdichte* (xxxiii) *Massenstrom* (xxxiv) *Teilchenstromdichte*

These identities hold for an arbitrary integration volume $\mathcal{V}$. Using the continuity of the respective integrands, one deduces the following so-called *continuity equations*:

$$\boxed{\frac{\partial \rho(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\left[\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right] = 0} \tag{III.9}$$

resp.

$$\frac{\partial n(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\left[n(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right] = 0. \tag{III.10}$$

Equation (III.9) represents the first of five dynamical (partial differential) equations which govern the evolution of a non-relativistic fluid flow.

Remarks:

∗ The form of the continuity equation (III.9) does not depend on the properties of the flowing medium, as for instance whether dissipative effects play a significant role or not. This should be contrasted with the findings of the next two Sections.

∗ In the case of a steady fluid flow, Eq. (III.9) gives $\vec{\nabla}\cdot\left[\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right] = 0$, i.e.

$$\rho(t,\vec{r})\,\vec{\nabla}\cdot\,\vec{\mathrm{v}}(t,\vec{r}) + \vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\rho(t,\vec{r}) = 0.$$

Thus, the stationary flow of a *homogeneous* fluid, i.e. for which $\rho(t,\vec{r})$ is position independent, will be incompressible [$\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r}) = 0$, cf. Eq. (II.15)].

III.3 Momentum balance: Euler and Navier–Stokes equations

For a closed system Σ with total linear momentum $\vec{P}$ with respect to a given reference frame $\mathcal{R}$, Newton's second law reads

$$\frac{\mathrm{D}\vec{P}(t)}{\mathrm{D}t} = \vec{F}(t), \tag{III.11}$$

with $\vec{F}$ the sum of the "external" forces acting on Σ.

The left hand side of this equation can be transformed with the help of Reynolds' transport theorem (III.5), irrespective of any assumption on the fluid under consideration (§ III.3.1). In contrast, the forces acting on a fluid element, more precisely the forces exerted by the neighboring elements, do depend on the (assumed) properties of the fluid. The two most widespread models used for fluids are that of a perfect fluid, which leads to the Euler equation (§ III.3.2), and of a Newtonian fluid, for which Newton's second law (III.11) translates into the Navier–Stokes equation (§ III.3.3).

Throughout this Section, we use the shorter designation "momentum" instead of the more accurate "linear momentum".

III.3.1 Material derivative of momentum

As already noted shortly below Eq. (III.1), the specific density associated with the momentum $\vec{P}(t)$ is simply the flow velocity $\vec{\mathrm{v}}(t,\vec{r})$. Applying Reynolds' theorem (III.5) to the momentum of the material system contained at time t inside a control volume $\mathcal{V}$, the material derivative on the left hand side of Newton's law (III.11) can be recast as

$$\frac{\mathrm{D}\vec{P}(t)}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{V}} \vec{\mathrm{v}}(t,\vec{r})\,\rho(t,\vec{r})\,\mathrm{d}^3\vec{r}\right] + \oint_{\partial\mathcal{V}} \vec{\mathrm{v}}(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\cdot\mathrm{d}^2\vec{\mathcal{S}}. \tag{III.12}$$

Both terms on the right hand side can be transformed to yield more tractable expressions. On the one hand, since the volume $\mathcal{V}$ is motionless in the reference frame $\mathcal{R}$, the time derivative can

be taken inside the integral. Its action on $\rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})$ is then given by the usual product rule. On the other hand, one can show the identity

$$\oint_{\partial\mathcal{V}} \vec{\mathsf{v}}(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})\cdot \mathrm{d}^2\vec{\mathcal{S}} = \int_{\mathcal{V}} \left\{ -\vec{\mathsf{v}}(t,\vec{r})\,\frac{\partial\rho(t,\vec{r})}{\partial t} + \rho(t,\vec{r})\big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}) \right\} \mathrm{d}^3\vec{r}. \tag{III.13}$$

All in all, one thus obtains

$$\frac{\mathrm{D}\vec{P}(t)}{\mathrm{D}t} = \int_{\mathcal{V}} \rho(t,\vec{r}) \left\{ \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}) \right\} \mathrm{d}^3\vec{r} = \int_{\mathcal{V}} \rho(t,\vec{r})\,\frac{\mathrm{D}\vec{\mathsf{v}}(t,\vec{r})}{\mathrm{D}t}\,\mathrm{d}^3\vec{r}. \tag{III.14}$$

Proof of relation (III.13): consider the i-th component of the vector defined by the surface integral on the left hand side of that identity. Gauss's divergence theorem gives

$$\oint_{\partial\mathcal{V}} \big[\mathsf{v}^i(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})\big]\cdot \mathrm{d}^2\vec{\mathcal{S}} = \int_{\mathcal{V}} \vec{\nabla}\cdot\big[\mathsf{v}^i(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})\big]\,\mathrm{d}^3\vec{r}.$$

The action of the differential operator yields $\mathsf{v}^i(t,\vec{r})\,\vec{\nabla}\cdot\big[\rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})\big] + \rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\mathsf{v}^i(t,\vec{r})$: the divergence in the first term can be expressed according to the continuity equation (III.9) as the negative of the time derivative of the mass density, leading to

$$\vec{\nabla}\cdot\big[\mathsf{v}^i(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})\big] = -\mathsf{v}^i(t,\vec{r})\,\frac{\partial\rho(t,\vec{r})}{\partial t} + \rho(t,\vec{r})\big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\mathsf{v}^i(t,\vec{r}).$$

This relation holds for all three components, from where Eq. (III.13) follows. □

Remark: The derivation of Eq. (III.14) relies on purely algebraic transformations, either as encoded in Reynolds' transport theorem, or when going from relation (III.12) to (III.14). That is, it does not imply any model—apart from that of a continuous medium—for the fluid properties. In particular, whether or not dissipative effects are important in the fluid did not play any role here.

III.3.2 Perfect fluid: Euler equation

In this section, we first introduce the notion, or rather the model, of a *perfect fluid*, which is defined by the choice of a specific ansatz for the stress tensor that encodes the contact forces between neighboring fluid elements (§ III.3.2 a). Using that ansatz and the results of the previous paragraph, Newton's second law (III.11) is shown in § III.3.2 b to be equivalent to a local formulation, the so-called *Euler equation*. Eventually, the latter is recast in the generic form for a local conservation or balance equation, involving the time derivative of a local density and the divergence of the corresponding flux density (§ III.3.2 e).

III.3.2 a Forces in a perfect fluid

The forces in a fluid were already discussed on a general level in § I.4.1. Thus, the total force on the right hand side of Eq. (III.11) consists of volume and surface forces, which can respectively be expressed as a volume or a surface integral

$$\vec{F}(t) = \int_{\mathcal{V}} \vec{f}_V(t,\vec{r})\,\mathrm{d}^3\vec{r} + \oint_{\mathcal{S}} \vec{T}_s(t,\vec{r})\,\mathrm{d}^2\mathcal{S}, \tag{III.15}$$

where $\vec{f}_V$ denotes the local density of body forces, while $\vec{T}_s$ is the mechanical stress vector introduced in Eq. (I.20). The latter will now allow us to introduce various models of fluids.

The first, simplest model is that of a *perfect fluid*, or *ideal fluid*:

A perfect fluid *is a fluid in which there are no shear stresses nor heat conduction.* (III.16a)

Stated differently, at every point of a perfect fluid the stress vector $\vec{T}_s$ on a (test) surface element $\mathrm{d}^2\mathcal{S}$ moving with the fluid is normal to $\mathrm{d}^2\mathcal{S}$, irrespective of whether the fluid is at rest or in motion.

That is, introducing the normal unit vector $\vec{\mathrm{e}}_n(\vec{r})$ to $\mathrm{d}^2\mathcal{S}$ oriented towards the exterior of the material region acted upon,[3] one may write

$$\vec{T}_s(t,\vec{r}) = -\mathcal{P}(t,\vec{r})\,\vec{\mathrm{e}}_n(\vec{r}), \tag{III.16b}$$

with $\mathcal{P}(t,\vec{r})$ the pressure at position $\vec{r}$. Accordingly, the mechanical stress tensor in a perfect fluid in a reference frame $\mathcal{R}$ which is moving with the fluid is given by

$$\boldsymbol{\sigma}(t,\vec{r}) = -\mathcal{P}(t,\vec{r})\,\mathbf{g}^{-1}(t,\vec{r}), \tag{III.16c}$$

with $\mathbf{g}^{-1}$ the inverse metric tensor, just like in a fluid at rest [Eq. (I.23)]. In a given coordinate system in $\mathcal{R}$, the $\binom{2}{0}$-components of $\boldsymbol{\sigma}$ thus simply read

$$\boldsymbol{\sigma}^{ij}(t,\vec{r}) = -\mathcal{P}(t,\vec{r})\,g^{ij}(t,\vec{r}) \tag{III.16d}$$

i.e. the $\binom{1}{1}$-components are $\boldsymbol{\sigma}^i{}_j(t,\vec{r}) = -\mathcal{P}(t,\vec{r})\,\delta^i{}_j$.

Using relation (III.16b), the total surface forces in Eq. (III.15) can be transformed into a volume integral:

$$\oint_{\mathcal{S}} \vec{T}_s(t,\vec{r})\,\mathrm{d}^2\mathcal{S} = -\oint_{\mathcal{S}} \mathcal{P}(t,\vec{r})\,\vec{\mathrm{e}}_n(\vec{r})\,\mathrm{d}^2\mathcal{S} = -\oint_{\mathcal{S}} \mathcal{P}(t,\vec{r})\,\mathrm{d}^2\vec{\mathcal{S}} = -\int_{\mathcal{V}} \vec{\nabla}\mathcal{P}(t,\vec{r})\,\mathrm{d}^3\vec{r}, \tag{III.17}$$

where the last identity follows from a corollary of the usual divergence theorem.

Remark: Although this might not be intuitive at first, the pressure $\mathcal{P}(t,\vec{r})$ entering Eqs. (III.16b)–(III.16d) is actually the hydrostatic pressure already introduced in the definition of the mechanical stress in a fluid at rest, see Eq. (I.23). One heuristic justification is that the stresses are defined as the forces per unit area exerted by a piece of fluid situated on one side of a surface on the fluid situated on the other side. Even if the fluid is moving, the two fluid elements on both sides of the surface—as well as the comoving test surface—have the same velocity,[4] i.e. their relative velocity vanishes, just like in a fluid at rest.

III.3.2 b Euler equation

Gathering Eqs. (III.11), (III.14), (III.15) and (III.17) yields

$$\int_{\mathcal{V}} \rho(t,\vec{r}) \left\{ \frac{\partial \vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathrm{v}}(t,\vec{r}) \right\} \mathrm{d}^3\vec{r} = \int_{\mathcal{V}} \left[-\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_V(t,\vec{r}) \right] \mathrm{d}^3\vec{r}.$$

Since this identity must hold irrespective of the control volume $\mathcal{V}$, the integrands on both sides must be equal. That is, the various fields they involve obey the *Euler equation*

$$\boxed{\rho(t,\vec{r}) \left\{ \frac{\partial \vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathrm{v}}(t,\vec{r}) \right\} = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_V(t,\vec{r}).} \tag{III.18}$$

Remarks:

∗ The term in curly brackets on the left hand side is exactly the acceleration (I.17) of a material point, as in Newton's second law.

∗ Due to the convective term $(\vec{\mathrm{v}}\cdot\vec{\nabla})\vec{\mathrm{v}}$, the Euler equation is a *nonlinear* partial differential equation.

[3] Cf. the discussion between Eqs. (I.21a)–(I.21c).

[4] ... thanks to the usual continuity assumption: this no longer holds at a discontinuity!

∗ Besides Newton's second law for linear momentum, one could also think of investigating the consequence of its analogue for angular momentum. Since we have assumed that the material points do not have any intrinsic spin, the conservation of angular momentum, apart from leading to the necessary symmetry of the stress tensor—which is realized in a perfect fluid, see Eq. (III.16c) or (III.16d), and will also hold in a Newtonian fluid, see Eq. (III.27)—does not bring any new dynamical equation.

III.3.2 c Boundary conditions

To fully formulate the mathematical problem representing a given fluid flow, one must also specify *boundary conditions* for the various partial differential equations. These conditions reflect the geometry of the problem under consideration.

- Far from an obstacle or from walls, one may specify a given pattern for the flow velocity field. For instance, one may require that the flow be uniform "at infinity", as e.g. for the motion far from the rotating cylinder in Fig. IV.5 illustrating the geometry of the Magnus effect.

- At an obstacle, in particular at a wall, the component of velocity *perpendicular* to the obstacle should vanish: that is, the fluid cannot penetrate the obstacle or wall, which makes sense and will be hereafter often referred to as *impermeability* condition. In case the obstacle is itself in motion, one should consider the (normal component of the) *relative* velocity of the fluid with respect to the obstacle.

  On the other hand, the model of a perfect fluid, in which there is by definition no friction, does not specify the value of the tangential component of the fluid relative velocity at an obstacle.

III.3.2 d Alternative forms of the Euler equation

In practice, the volume forces acting on a fluid element are often proportional to its mass, as are e.g. the gravitational, Coriolis or centrifugal forces. Therefore, it is rather natural to introduce the corresponding force density per unit mass, instead of per unit volume:

$$\vec{a}_V(t,\vec{r}) \equiv \frac{\mathrm{d}\vec{F}_V(t,\vec{r})}{\mathrm{d}M(t,\vec{r})} = \frac{\vec{f}_V(t,\vec{r})}{\rho(t,\vec{r})}.$$

With the help of this "specific density of body forces", which has the dimension of an acceleration, the Euler equation (III.18) can be recast as

$$\boxed{\frac{\mathrm{D}\vec{\mathrm{v}}(t,\vec{r})}{\mathrm{D}t} = -\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{a}_V(t,\vec{r}).} \tag{III.19}$$

The interpretation of this form is quite straightforward: the acceleration of a material point (left hand side) is the sum of the acceleration due to the pressure forces and the acceleration due to volume forces (right hand side).

Alternatively, one may use the identity (in which the time and position variables have been omitted for the sake of brevity)

$$\vec{\mathrm{v}} \times \left(\vec{\nabla} \times \vec{\mathrm{v}}\right) = \vec{\nabla}\left(\frac{\vec{\mathrm{v}}^2}{2}\right) - \left(\vec{\mathrm{v}} \cdot \vec{\nabla}\right)\vec{\mathrm{v}},$$

which can be proved either starting from the usual formula for the double cross product—with a small twist when applying the differential operator to a vector squared—or by working component by component. Recognizing in the rightmost term the convective part of the Euler equation, one can rewrite the latter, or equivalently Eq. (III.19), as

$$\frac{\partial \vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \vec{\nabla}\left[\frac{\vec{\mathrm{v}}(t,\vec{r})^2}{2}\right] - \vec{\mathrm{v}}(t,\vec{r}) \times \vec{\omega}(t,\vec{r}) = -\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{a}_V(t,\vec{r}), \tag{III.20}$$

where we have made use of the vorticity vector (II.11). Note that the second term on the left hand side of this equation involves the (gradient of the) kinetic energy per unit mass $\mathrm{d}K/\mathrm{d}M$.

In § IV.2.1, we shall see yet another form of the Euler equation [Eq. (IV.8)], involving thermodynamic functions other than the pressure.

III.3.2 e The Euler equation as a balance equation

The Euler equation can be rewritten in the generic form for a balance equation, namely as the identity of the sum of the time derivative of a density and the divergence of a flux density with a source term—which vanishes if the quantity under consideration is conserved. Accordingly, we first introduce two

Definitions: One associates with the i-th component in a given coordinate system of the momentum of a material system its

- *density*[(xxxv)] $\rho(t,\vec{r})\,\mathrm{v}^i(t,\vec{r})$ and (III.21a)
- *flux density*[(xxxvi)](in direction j) $\mathbf{T}^{ij}(t,\vec{r}) \equiv \rho(t,\vec{r})\,\mathrm{v}^i(t,\vec{r})\,\mathrm{v}^j(t,\vec{r}) + \mathcal{P}(t,\vec{r})\,g^{ij}(t,\vec{r})$, (III.21b)

with g^{ij} the components of the inverse metric tensor $\mathbf{g}^{-1}$.

Physically, $\mathbf{T}^{ij}$ represents the amount of momentum along $\vec{\mathrm{e}}_i$ transported per unit time through a unit surface[(5)] perpendicular to the direction of $\vec{\mathrm{e}}_j$—i.e. transported in direction j. That is, it is the i-th component of the force upon a test unit surface with normal unit vector $\vec{\mathrm{e}}_j$.

The first contribution to $\mathbf{T}^{ij}$—namely the transported momentum density multiplied by the velocity—arises from the convective transport represented by the macroscopic motion. On the other hand, the second term, involving pressure, represents the momentum transport due to the thermal, random motion of the atoms of the fluid.

Remarks:

∗ As thermal motion is random and (statistically) isotropic, it does not contribute to the momentum density $\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})$, only to the momentum flux density.

∗ In tensor notation, the momentum flux density (III.21b), viewed as a $\binom{2}{0}$-tensor, is given by

$$\mathbf{T}(t,\vec{r}) = \rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r}) \otimes \vec{\mathrm{v}}(t,\vec{r}) + \mathcal{P}(t,\vec{r})\mathbf{g}^{-1}(t,\vec{r}) \quad \text{for a perfect fluid.} \tag{III.22}$$

∗ Given its physical meaning, the momentum flux (density) tensor $\mathbf{T}$ is obviously related to the Cauchy stress tensor $\boldsymbol{\sigma}$. More precisely, $\mathbf{T}$ represents the forces exerted by a material point on its neighbors, as measured in an arbitrary reference frame, in which the fluid velocity is $\vec{\mathrm{v}}(t,\vec{r})$. In turn $\boldsymbol{\sigma}$ stands for the stresses acting upon the material point due to its neighbors, as measured in the reference frame in which the material point is at rest, such that $\vec{\mathrm{v}}(t,\vec{r}) = \vec{0}$. In the latter reference frame, invoking Newton's third law—which in continuum mechanics is referred to as *Cauchy's fundamental lemma*—, the two tensors are simply opposite to each other. Coming back to an arbitrary reference frame, one may write

$$\mathbf{T}(t,\vec{r}) = \rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r}) \otimes \vec{\mathrm{v}}(t,\vec{r}) - \boldsymbol{\sigma}(t,\vec{r}) \tag{III.23}$$

where $\boldsymbol{\sigma}(t,\vec{r})$ is given by Eq. (III.16c) for a perfect fluid.

[(5)] ... which must be immobile in the reference frame in which the fluid has the velocity $\vec{\mathrm{v}}$ entering definition (III.21b).

[(xxxv)] *Impulsdichte* [(xxxvi)] *Impulsstromdichte*

∗ Building on the previous remark, the absence of shear stress defining a perfect fluid can be reformulated as a condition of the momentum flux tensor:

> *A perfect fluid is a fluid at each point of which one can find a local velocity, such that for an observer moving with that velocity the fluid is locally isotropic. The momentum flux tensor is thus diagonal in the observer's reference frame.* (III.24)

The observer traveling with the same velocity as the local fluid velocity will be hereafter referred to as *comoving observer*, and the reference frame in which she is sitting as *local rest frame*.

Using definitions (III.21), one easily checks that the Euler equation (III.18) is equivalent to the balance equations (for $j = 1, 2, 3$)

$$\frac{\partial}{\partial t}\big[\rho(t,\vec{r})\,\mathsf{v}^j(t,\vec{r})\big] + \sum_{i=1}^{3}\frac{\mathrm{d}\mathbf{T}^{ij}(t,\vec{r})}{\mathrm{d}x^i} = f_V^j(t,\vec{r}). \tag{III.25a}$$

with f_V^j the j-th component of the volume force density and $\mathrm{d}/\mathrm{d}x^i$ the covariant derivatives (see Appendix **??**), that coincide with the partial derivatives in Cartesian coordinates.

Proof: For the sake of brevity, the $(t,\vec{r})$-dependence of the various fields will not be specified. With the product rule and the definition of $\mathbf{T}^{ij}$, one finds

$$\begin{aligned}\frac{\partial(\rho\mathsf{v}^j)}{\partial t} + \sum_{i=1}^{3}\frac{\mathrm{d}\mathbf{T}^{ij}}{\mathrm{d}x^i} &= \frac{\partial\rho}{\partial t}\,\mathsf{v}^j + \rho\frac{\partial\mathsf{v}^j}{\partial t} + \sum_{i=1}^{3}\mathsf{v}^j\frac{\mathrm{d}(\rho\mathsf{v}^i)}{\mathrm{d}x^i} + \sum_{i=1}^{3}\rho\mathsf{v}^i\frac{\mathrm{d}\mathsf{v}^j}{\mathrm{d}x^i} + \sum_{i=1}^{3}g^{ij}\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}x^i}\\ &= \mathsf{v}^j\left[\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{\mathsf{v}})\right] + \rho\left[\frac{\partial\mathsf{v}^j}{\partial t} + (\vec{\mathsf{v}}\cdot\vec{\nabla})\mathsf{v}^j\right] + \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}x_j},\end{aligned}$$

where we have used $\sum_i g^{ij}\mathrm{d}/\mathrm{d}x^j = \mathrm{d}/\mathrm{d}x_j$. The first term between square brackets vanishes thanks to the continuity equation (III.9). In turn, the second term is precisely the j-th component of the left member of the Euler equation (III.18), i.e. it equals the j-th component of $\vec{f}_V$ minus the third term, which represents the j-th component of $\vec{\nabla}\mathcal{P}$. □

In tensor notation, Eq. (III.25a) reads

$$\frac{\partial}{\partial t}\big[\rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})\big] + \vec{\nabla}\cdot\mathbf{T}(t,\vec{r}) = \vec{f}_V(t,\vec{r}), \tag{III.25b}$$

where the action of the divergence on a $\binom{2}{0}$-tensor is defined through its components, which is to be read off Eq. (III.25a).

III.3.3 Newtonian fluid: Navier–Stokes equation

In a real moving fluid, there are *friction forces* that contribute to the transport of momentum between neighboring fluid layers when the latter are in relative motion. Accordingly, the momentum flux-density tensor is no longer given by Eq. (III.21b) or (III.22), but now contains extra terms, involving derivatives of the flow velocity. The Euler equation must therefore be replaced by an alternative dynamical equation, including the friction forces.

III.3.3 a Momentum flux density in a Newtonian fluid

The momentum flux density (III.21b) in a perfect fluid only contains two terms—one proportional to $\mathsf{v}^i(t,\vec{r})\,\mathsf{v}^j(t,\vec{r})$, the other proportional to the components g^{ij} of the inverse metric tensor. Since the coefficients in front of these two terms could a priori depend on $\vec{\mathsf{v}}^2$, this represents the most general symmetric tensor of degree 2 which can be constructed with the help of the flow velocity only.

If the use of terms that depend on the spatial derivatives of the velocity field is also allowed, the components of the momentum flux-density tensor can be of the following form, where for the sake of brevity the variables t and $\vec{r}$ are omitted

$$\mathbf{T}^{ij} = \rho \mathsf{v}^i \mathsf{v}^j + \mathcal{P} g^{ij} + A \frac{\mathrm{d}\mathsf{v}^i}{\mathrm{d}x_j} + B \frac{\mathrm{d}\mathsf{v}^j}{\mathrm{d}x_i} + \mathcal{O}\left(\frac{\mathrm{d}^2 \mathsf{v}_i}{\mathrm{d}x_j\, \mathrm{d}x_k}\right) + \cdots , \tag{III.26}$$

with coefficients A, B that depend on i, j and on the fluid under consideration.

> This ansatz for $\mathbf{T}^{ij}$, as well as the form of the energy flux density involved in Eq. (III.37) below, can be "justified" by starting from a microscopic *kinetic theory* of the fluid and writing the solutions of the corresponding equation of motion as a specific expansion—which turns out to be in powers of the Knudsen number (I.4). This also explains why terms of the type $\mathsf{v}^i \partial \mathcal{P}/\partial x_j$ or $\mathsf{v}^i \partial T/\partial x_j$, with T the temperature, were not considered in Eq. (III.26).
> Despite these theoretical considerations, in the end the actual justification for the choices of momentum or energy flux density is the agreement with the measured behavior of fluids.

As discussed in § I.1.3, the description of a system of particles as a continuous medium, and in particular as a fluid, in local thermodynamic equilibrium, rests on the assumption that the macroscopic quantities of relevance for the medium vary slowly both in space and time. Accordingly, (spatial) gradients should be small: the third and fourth terms in Eq. (III.26) should thus be on the one hand much smaller than the first two ones, on the other hand much larger than the rightmost term as well as those involving higher-order derivatives or of powers of the first derivatives. Neglecting these smaller terms, one obtains "first-order dissipative fluid dynamics", which describes the motion of *Newtonian fluids*—this actually defines the latter.

Using the necessary symmetry of $\mathbf{T}^{ij}$, the third and fourth terms in Eq. (III.26) can be rewritten as the sum of a traceless symmetric contribution and a tensor proportional to the inverse metric tensor. This leads to the momentum flux-density tensor

$$\begin{aligned} \mathbf{T}^{ij}(t,\vec{r}) = {} & \rho(t,\vec{r}) \mathsf{v}^i(t,\vec{r}) \mathsf{v}^j(t,\vec{r}) + \mathcal{P}(t,\vec{r})\, g^{ij}(t,\vec{r}) \\ & - \eta(t,\vec{r}) \left[\frac{\mathrm{d}\mathsf{v}^i(t,\vec{r})}{\mathrm{d}x_j} + \frac{\mathrm{d}\mathsf{v}^j(t,\vec{r})}{\mathrm{d}x_i} - \frac{2}{3} g^{ij}(t,\vec{r}) \vec{\nabla} \cdot \vec{\mathsf{v}}(t,\vec{r}) \right] \\ & - \zeta(t,\vec{r}) g^{ij}(t,\vec{r}) \vec{\nabla} \cdot \vec{\mathsf{v}}(t,\vec{r}). \end{aligned} \tag{III.27a}$$

In geometric formulation, this reads

$$\boxed{\mathbf{T}(t,\vec{r}) = \rho(t,\vec{r})\, \vec{\mathsf{v}}(t,\vec{r}) \otimes \vec{\mathsf{v}}(t,\vec{r}) + \mathcal{P}(t,\vec{r})\, \mathbf{g}^{-1}(t,\vec{r}) + \boldsymbol{\pi}(t,\vec{r})} \tag{III.27b}$$

where dissipative effects are encoded in the *viscous stress tensor*[(xxxvii)]

$$\boxed{\begin{array}{l} \text{for a Newtonian fluid:} \\ \boldsymbol{\pi}(t,\vec{r}) \equiv -2\eta(t,\vec{r}) \left[\mathbf{D}(t,\vec{r}) - \frac{1}{3} \big[\vec{\nabla} \cdot \vec{\mathsf{v}}(t,\vec{r}) \big] \mathbf{g}^{-1}(t,\vec{r}) \right] - \zeta(t,\vec{r}) \big[\vec{\nabla} \cdot \vec{\mathsf{v}}(t,\vec{r}) \big] \mathbf{g}^{-1}(t,\vec{r}) \end{array}} \tag{III.27c}$$

with $\mathbf{D}(t,\vec{r})$ the strain rate tensor discussed in § II.1.3. Component-wise, this reads

$$\pi^{ij}(t,\vec{r}) \equiv -2\eta(t,\vec{r}) \left[\mathbf{D}^{ij}(t,\vec{r}) - \frac{1}{3} \big[\vec{\nabla} \cdot \vec{\mathsf{v}}(t,\vec{r}) \big] g^{ij}(t,\vec{r}) \right] - \zeta(t,\vec{r}) \big[\vec{\nabla} \cdot \vec{\mathsf{v}}(t,\vec{r}) \big] g^{ij}(t,\vec{r}). \tag{III.27d}$$

In terms of the traceless rate-of-shear tensor (II.17b) or of its components (II.17d), one may alternatively write

$$\boldsymbol{\pi}(t,\vec{r}) \equiv -2\eta(t,\vec{r})\, \mathbf{S}(t,\vec{r}) - \zeta(t,\vec{r}) \big[\vec{\nabla} \cdot \vec{\mathsf{v}}(t,\vec{r}) \big] \mathbf{g}^{-1}(t,\vec{r}) \tag{III.27e}$$

$$\pi^{ij}(t,\vec{r}) \equiv -2\eta(t,\vec{r})\, \mathbf{S}^{ij}(t,\vec{r}) - \zeta(t,\vec{r}) \big[\vec{\nabla} \cdot \vec{\mathsf{v}}(t,\vec{r}) \big] g^{ij}(t,\vec{r}). \tag{III.27f}$$

[(xxxvii)] *viskoser Spannungstensor*

This viscous stress tensor involves two novel characteristics of the medium, so-called *transport coefficients*, namely:

- the (dynamical) *shear viscosity*[(xxxviii)] η, which multiplies the traceless symmetric part of the velocity gradient tensor, i.e. the conveniently termed rate-of-shear tensor;
- the *bulk viscosity*, also called *second viscosity*,[(xxxix)] ζ, which multiplies the volume-expansion part of the velocity gradient tensor, i.e. the term proportional to $\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})$.

The two corresponding contributions represent a *diffusive* transport of momentum in the fluid—representing a third type of transport besides the convective and thermal ones.

Remarks:

∗ In the case of a Newtonian fluid, the viscosity coefficients η and ζ are independent of the flow velocity. However, they still depend on the temperature and pressure of the fluid, so that they are not necessarily uniform and constant in a real flowing fluid.

∗ In an incompressible flow, $\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r}) = 0$, the last contribution to the momentum flux density (III.27) drops out. Thus, the bulk viscosity ζ only plays a role in compressible fluid motions.[(6)]

∗ Expression (III.27c) or (III.27d) of the viscous stress tensor assumes implicitly that the fluid is (locally) isotropic, since the coefficients η, ζ are independent of the directions i, j.

III.3.3 b Surface forces in a Newtonian fluid

The Cauchy stress tensor corresponding to the momentum flux density (III.27) of a Newtonian fluid is [cf. Eq. (III.23)]

$$\boldsymbol{\sigma}(t,\vec{r}) = -\mathcal{P}(t,\vec{r})\,\mathbf{g}^{-1}(t,\vec{r}) - \boldsymbol{\pi}(t,\vec{r}) \tag{III.28a}$$

that is, using the form (III.27e) of the viscous stress tensor

$$\boldsymbol{\sigma}(t,\vec{r}) = -\mathcal{P}(t,\vec{r})\,\mathbf{g}^{-1}(t,\vec{r}) + 2\eta(t,\vec{r})\,\mathbf{S}(t,\vec{r}) + \zeta(t,\vec{r})\big[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]\mathbf{g}^{-1}(t,\vec{r}). \tag{III.28b}$$

Component-wise, this becomes

$$\sigma^{ij}(t,\vec{r}) = \left\{-\mathcal{P}(t,\vec{r}) + \left[\zeta(t,\vec{r}) - \frac{2}{3}\eta(t,\vec{r})\right]\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\right\} g^{ij}(t,\vec{r}) + \eta(t,\vec{r})\left[\frac{\mathrm{d}\mathrm{v}^i(t,\vec{r})}{\mathrm{d}x_j} + \frac{\mathrm{d}\mathrm{v}^j(t,\vec{r})}{\mathrm{d}x_i}\right]. \tag{III.28c}$$

Accordingly, the mechanical stress vector on an infinitesimally small surface element situated at point $\vec{r}$ with unit normal vector $\vec{\mathrm{e}}_n(\vec{r})$ reads

$$\vec{T}_s(t,\vec{r}) = \boldsymbol{\sigma}(t,\vec{r})\cdot\vec{\mathrm{e}}_n(\vec{r}) = \sum_{i,j=1}^{3}\left\{\left[-\mathcal{P}(t,\vec{r}) + \left(\zeta(t,\vec{r}) - \frac{2}{3}\eta(t,\vec{r})\right)\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\right] g^i{}_j(t,\vec{r}) + \eta(t,\vec{r})\left(\frac{\mathrm{d}\mathrm{v}^i(t,\vec{r})}{\mathrm{d}x^j} + \frac{\mathrm{d}\mathrm{v}_j(t,\vec{r})}{\mathrm{d}x_i}\right)\right\} n^j(\vec{r})\,\vec{\mathrm{e}}_i(t,\vec{r}), \tag{III.29}$$

with $n^j(\vec{r})$ the coordinate of $\vec{\mathrm{e}}_n(\vec{r})$ along direction j. One easily identifies the two components of this stress vector (cf. § I.4.1)

- the term proportional to $\sum g^i{}_j n^j \vec{\mathrm{e}}_i = \vec{\mathrm{e}}_n$ is the *normal stress* on the surface element. It consists of the usual hydrostatic pressure term $-\mathcal{P}\,\vec{\mathrm{e}}_n$, and a second contribution, proportional to the local expansion rate $\vec{\nabla}\cdot\vec{\mathrm{v}}$: in the compressible motion of a Newtonian—and more generally a dissipative—fluid, the normal stress is thus not only given by $-\mathcal{P}\,\vec{\mathrm{e}}_n$, but includes additional contributions that vanish in the static case.

(6) As a consequence, the bulk viscosity is often hard to measure—one has to devise a compressible flow—so that it is actually not so well known for many substances, even well-studied ones [13].

(xxxviii) *Scherviskosität* (xxxix) *Dehnviskosität, Volumenviskosität, zweite Viskosität*

- the remaining term is the *tangential stress*, proportional to the shear viscosity η. Accordingly, the value of the latter can be deduced from a measurement of the tangential force acting on a surface element, see § V.1.2.

As in § III.3.2 a, the external contact forces acting on a fluid element delimited by a surface $\mathcal{S}$ can easily be computed. Invoking the Stokes theorem yields

$$\begin{aligned}\oint_{\mathcal{S}} \vec{T}_s(t,\vec{r})\,\mathrm{d}^2\mathcal{S} &= -\oint_{\mathcal{S}} \mathcal{P}(t,\vec{r})\,\vec{e}_n(\vec{r})\,\mathrm{d}^2\mathcal{S} - \oint_{\mathcal{S}} \boldsymbol{\pi}(t,\vec{r})\cdot\vec{e}_n(\vec{r})\,\mathrm{d}^2\mathcal{S}\\ &= -\int_{\mathcal{V}} \vec{\nabla}\mathcal{P}(t,\vec{r})\,\mathrm{d}^3\mathcal{V} + \int_{\mathcal{V}} \vec{\nabla}\cdot\boldsymbol{\pi}(t,\vec{r})\,\mathrm{d}^3\mathcal{V}\\ &= -\int_{\mathcal{V}} \vec{\nabla}\mathcal{P}(t,\vec{r})\,\mathrm{d}^3\mathcal{V} + \int_{\mathcal{V}} \vec{f}_{\mathrm{visc}}(t,\vec{r})\,\mathrm{d}^3\mathcal{V}, \end{aligned} \tag{III.30a}$$

with the local viscous friction force density

$$\begin{aligned}\vec{f}_{\mathrm{visc}}(t,\vec{r}) = \sum_{i,j=1}^{3} \frac{\mathrm{d}}{\mathrm{d}x^i}\left\{\eta(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}^i(t,\vec{r})}{\mathrm{d}x_j} + \frac{\mathrm{d}\mathsf{v}^j(t,\vec{r})}{\mathrm{d}x_i}\right]\right\}\vec{e}_j(t,\vec{r})\\ + \vec{\nabla}\left\{\left[\zeta(t,\vec{r}) - \frac{2}{3}\eta(t,\vec{r})\right]\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\right\}. \end{aligned} \tag{III.30b}$$

Remark: Equations (III.16c) or equivalently (III.16d)—in the case of a perfect fluid— and Eqs. (III.28b)–(III.28c) for a Newtonian fluid, relating the Cauchy stress tensor to the flow velocity (and the pressure), are so-called *constitutive relations*. These actually define—together with expressions for the heat currents—the respective models.

III.3.3 c Navier–Stokes equation

Combining the viscous force (III.30b) with the generic equations (III.12), (III.14) and (III.15), the application of Newton's second law to a volume $\mathcal{V}$ of fluid leads to an identity between sums of volume integrals. Since this relation holds for any volume $\mathcal{V}$, it translates into an identity between the integrands, namely

$$\rho(t,\vec{r})\left\{\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathsf{v}}(t,\vec{r})\right\} = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_{\mathrm{visc}}(t,\vec{r}) + \vec{f}_V(t,\vec{r}) \tag{III.31a}$$

or component-wise

$$\begin{aligned}\rho(t,\vec{r})\left\{\frac{\partial\mathsf{v}^i(t,\vec{r})}{\partial t} + \left[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\mathsf{v}^i(t,\vec{r})\right\} = -\frac{\mathrm{d}\mathcal{P}(t,\vec{r})}{\mathrm{d}x_i} + \frac{\mathrm{d}}{\mathrm{d}x_i}\left\{\left[\zeta(t,\vec{r}) - \frac{2}{3}\eta(t,\vec{r})\right]\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\right\}\\ + \sum_{j=1}^{3}\frac{\mathrm{d}}{\mathrm{d}x^j}\left\{\eta(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}^i(t,\vec{r})}{\mathrm{d}x_j} + \frac{\mathrm{d}\mathsf{v}^j(t,\vec{r})}{\mathrm{d}x_i}\right]\right\} + \left[\vec{f}_V(t,\vec{r})\right]^i \end{aligned} \tag{III.31b}$$

for $i = 1, 2, 3$.

If the implicit dependence of the viscosity coefficients η and ζ on time and position is negligible, one may take them outside of the spatial derivatives. As a result, one obtains the (compressible) *Navier*[(i)]*–Stokes*[(j)] *equation*

$$\rho(t,\vec{r})\left\{\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathsf{v}}(t,\vec{r})\right\} = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \eta\triangle\vec{\mathsf{v}}(t,\vec{r}) + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}\left[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\right] + \vec{f}_V(t,\vec{r}) \tag{III.32}$$

with $\triangle = \vec{\nabla}^2$ the Laplacian. This is a non-linear partial differential equation of second order, while the Euler equation (III.18) is of first order.

[(i)] C.-L. Navier, 1785–1836 [(j)] G. G. Stokes, 1819–1903

The difference between the order of the equations is not a mere detail: while the Euler equation looks like the limit $\eta, \zeta \to 0$ of the Navier–Stokes equation, this may not necessarily hold true for their solutions. This is for instance due to the fact that their respective boundary conditions differ.

In the case of an incompressible flow, the local expansion rate in the Navier–Stokes equation (III.32) vanishes, leading to the *incompressible Navier–Stokes equation*

$$\frac{\partial \vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathrm{v}}(t,\vec{r}) = -\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}) + \nu\triangle\vec{\mathrm{v}}(t,\vec{r}), \tag{III.33}$$

with

$$\nu \equiv \eta/\rho \tag{III.34}$$

the *kinematic shear viscosity*.

Remark: The dimension of the dynamic viscosity coefficients η, ζ is $\mathsf{M}\mathsf{L}^{-1}\mathsf{T}^{-1}$ and the corresponding unit in the SI system is the Poiseuille[(k)], abbreviated $\mathrm{Pa}\cdot\mathrm{s}$. In contrast, the kinematic viscosity has dimension $\mathsf{L}^2\mathsf{T}^{-1}$, i.e. depends only on space and time, hence its denomination.

III.3.3 d Boundary conditions

At the interface between a viscous fluid, in particular a Newtonian one, and another body—be it an obstacle in the flow, a wall containing the fluid, or even a second viscous fluid which is immiscible with the first one—the relative velocity between the fluid and the body must vanish. This holds not only for the normal component of the velocity ("impermeability" condition), as in perfect fluids, but also for the tangential one, to account for the friction forces. The latter requirement is often referred to as *no-slip condition*.

III.3.4 Higher-order dissipative fluid dynamics

Instead of considering only the first spatial derivatives of the velocity field in the momentum flux-density tensor (III.26), one may wish to also include the second derivatives, or even higher ones. Such assumptions lead to partial differential equations of motion, replacing the Navier–Stokes equation, of increasing order: *Burnett equation*, *super Burnett equation* [14, 15].

The domain of validity of such higher-order dissipative fluid models is a priori larger than that of first-order fluid dynamics, since it becomes possible to account for stronger gradients. On the other hand, this is at the cost of introducing a large number of new parameters besides the transport coefficients already present in Newtonian fluids. In parallel, the numerical implementation of the model becomes more involved, so that a macroscopic description does not necessarily represent the best approach.

III.4 Energy conservation, entropy balance

The conservation of mass and Newton's second law for linear momentum lead to four partial differential equations, one scalar—continuity equation (III.9)—and one vectorial—Euler (III.18) or Navier–Stokes (III.32)—, describing the coupled evolutions of five fields: $\rho(t,\vec{r})$, the three components of $\vec{\mathrm{v}}(t,\vec{r})$ and $\mathcal{P}(t,\vec{r})$.[(7)] To fully determine the latter, a fifth equation is needed. For this last constraint, there are several possibilities.

A first alternative is if some of the *kinematic* properties of the fluid flow are imposed a priori. Thus, requiring that the motion should be steady or irrotational or incompressible… might suffice

[(7)] The density of volume forces $\vec{f}_V$ or equivalently the corresponding potential energy per unit mass Φ, which stand for gravity or inertial forces, are given "from the outside" and not counted as a degree of freedom.

[(k)] J.-L.-M. Poiseuille, 1797–1869

to fully constrain the fluid flow for the geometry under consideration: we shall see several examples in the next three Chapters.

A second possibility, which will also be illustrated in Chap. IV–V, is that of a *thermodynamic* constraint: isothermal flow, isentropic flow... For instance, one sees in thermodynamics that in an adiabatic process for an ideal gas, the pressure and volume of the latter obey the relation $\mathcal{P}\mathcal{V}^\gamma = \text{constant}$, where γ denotes the ratio of the heat capacities at constant pressure ($C_\mathcal{P}$) and constant volume ($C_\mathcal{V}$). Since $\mathcal{V}$ is proportional to $1/\rho$, this so-called "adiabatic equation of state" provides the needed constraint relating pressure and mass density.

Eventually, one may argue that non-relativistic physics automatically implies a further conservation law besides those for mass and linear momentum, namely energy conservation. Thus, using the reasoning adopted in Secs. (III.2) and (III.3), the rate of change of the total energy—internal, kinetic and potential—of the matter inside a given volume equals the negative of the flow of energy through the surface delimiting this volume. In agreement with the first law of thermodynamics, one must take into account in the energy exchanged with the exterior of the volume not only the convective transport of internal, kinetic and potential energies, but also the exchange of the mechanical work of contact forces and—for dissipative fluids—of heat.

III.4.1 Energy and entropy conservation in perfect fluids

In non-dissipative non-relativistic fluids, energy is either transported convectively—as it accompanies some flowing mass of fluid—or exchanged in the form of the mechanical work of the pressure forces between neighboring regions. Mathematically, this is expressed at the local level by the equation

$$\begin{aligned}\frac{\partial}{\partial t}\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})^2 + e(t,\vec{r}) + \rho(t,\vec{r})\,\Phi(t,\vec{r})\right] \\ + \vec{\nabla}\cdot\left\{\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})^2 + e(t,\vec{r}) + \mathcal{P}(t,\vec{r}) + \rho(t,\vec{r})\,\Phi(t,\vec{r})\right]\vec{\mathrm{v}}(t,\vec{r})\right\} = 0,\end{aligned} \tag{III.35}$$

where e denotes the local density of internal energy and Φ the potential energy per unit mass of volume forces—assumed to be conservative—such that the acceleration $\vec{a}_V$ present in Eq. (III.19) equals $-\vec{\nabla}\Phi$.

Equation (III.35) will not be proven here—we shall see later in § X.3.3 that it emerges as low-velocity limit of one of the equations of non-dissipative relativistic fluid dynamics. It is however clearly of the usual form for a conservation equation, involving

- the total energy density, consisting of the kinetic ($\frac{1}{2}\rho\vec{\mathrm{v}}^2$), internal ($e$) and potential ($\rho\Phi$) energy densities; and
- the total energy flux density, which involves the previous three forms of energy, as well as that exchanged as mechanical work of the pressure forces.[8]

Remarks:

∗ The presence of pressure in the flux density, however not in the density, is reminiscent of the same property in definitions (III.21).

∗ The assumption that the volume forces are conservative is of course not innocuous. For instance, it does not hold for Coriolis forces, which means that one must be careful when working in a rotating reference frame.

[8] Remember that when a system with pressure $\mathcal{P}$ increases its volume by an amount $\mathrm{d}\mathcal{V}$, it exerts a mechanical work $\mathcal{P}\,\mathrm{d}\mathcal{V}$, "provided" to its exterior.

∗ The careful reader will have noticed that energy conservation (III.35) constitutes a fifth equation complementing the continuity and Euler equations (III.9) and (III.18), yet at the cost of introducing a new scalar field, the energy density, so that now a sixth equation is needed. The latter is provided by the thermal equation of state of the fluid, which relates its energy density, mass density and pressure.[(9)] In contrast to the other equations, this equation of state is not "dynamical", i.e. for instance it does not involve time or spatial derivatives, but is purely algebraic.

One can show—again, this will be done in the relativistic case (§ X.3.2), and can also be seen as special case of the result obtained for Newtonian fluids in § III.4.3—that in a perfect, non-dissipative fluid, the relation (III.35) expressing energy conservation locally, together with thermodynamic relations, lead to the local conservation of entropy, expressed as

$$\frac{\partial s(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\big[s(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})\big] = 0, \tag{III.36}$$

where $s(t,\vec{r})$ is the entropy density, while $s(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})$ represents the entropy flux density. The motion of a perfect fluid is thus automatically *isentropic*.

This equation, together with a thermodynamic relation, is sometimes more practical than the energy conservation equation (III.35), to which it is however totally equivalent.

III.4.2 Energy conservation in Newtonian fluids

In a real fluid, there exist further mechanisms for transporting energy besides the convective transport due to the fluid motion, namely *diffusion*, either of momentum or of energy:

- On the one hand, the viscous friction forces in the fluid, which already lead to the transport of momentum between neighboring fluid layers moving with different velocities, exert some work in the motion, which induces a diffusive transport of energy. This is accounted for by a contribution $\boldsymbol{\pi}\cdot\vec{\mathsf{v}}$ to the energy flux density—component-wise, a contribution $\sum_j \pi^i{}_j\,\mathsf{v}^j$ to the i-th component of the flux density—, with $\boldsymbol{\pi}$ the viscous stress tensor, given in the case of a Newtonian fluid by Eq. (III.27c).

- On the other hand, there is also *heat conduction* from the regions with higher temperatures towards those with lower temperatures. This transport is described by the introduction in the energy flux density of a heat current[(xl)] $\vec{\jmath}_Q(t,\vec{r}) = -\kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r})$—in accordance with the local formulation of *Fourier's law*,[(1)] see e.g. Sec. 1.2.1 in Ref. [2]—, with κ the *heat conductivity*[(xli)] of the fluid.

Taking into account these additional contributions, the local formulation of energy conservation in a Newtonian fluid reads

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})^2 + e(t,\vec{r}) + \rho(t,\vec{r})\Phi(t,\vec{r})\right] \\
&\quad + \vec{\nabla}\cdot\left\{\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})^2 + e(t,\vec{r}) + \mathcal{P}(t,\vec{r}) + \rho(t,\vec{r})\Phi(t,\vec{r})\right]\vec{\mathsf{v}}(t,\vec{r})\right. \\
&\qquad\qquad - \eta(t,\vec{r})\left[\Big(\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\Big)\vec{\mathsf{v}}(t,\vec{r}) + \vec{\nabla}\left(\frac{\vec{\mathsf{v}}(t,\vec{r})^2}{2}\right)\right] \\
&\qquad\qquad \left. - \left[\zeta(t,\vec{r}) - \frac{2\eta(t,\vec{r})}{3}\right]\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\,\vec{\mathsf{v}}(t,\vec{r}) - \kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r})\right\} = 0.
\end{aligned} \tag{III.37}$$

[(9)] This is where the assumption of local thermodynamic equilibrium (§ I.1.3) plays a crucial role.

[(xl)] *Wärmestromvektor* [(xli)] *Wärmeleitfähigkeit*

[(1)] J. B. Fourier, 1768–1830

with Φ the potential energy per unit mass due to conservative external volume forces. If the three transport coefficients η, ζ and κ vanish, this equation simplifies to that for perfect fluids, Eq. (III.35).

Remark: The energy flux density can be read off Eq. (III.37), since it represents the term between curly brackets. Dropping the external potential Φ, one can check that it can also be written as

$$\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})^2 + e(t,\vec{r}) + \mathcal{P}(t,\vec{r})\right]\vec{\mathrm{v}}(t,\vec{r}) - 2\eta(t,\vec{r})\,\mathbf{S}(t,\vec{r})\cdot\vec{\mathrm{v}}(t,\vec{r}) \\ - \zeta(t,\vec{r})\big[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]\vec{\mathrm{v}}(t,\vec{r}) - \kappa(t,\vec{r})\,\vec{\nabla}T(t,\vec{r}), \quad \text{(III.38)}$$

with $\mathbf{S}(t,\vec{r})$ the traceless symmetric rate-of-shear tensor. One recognizes the various physical sources of energy transport.

III.4.3 Entropy balance in Newtonian fluids

In a real fluid, with viscous friction forces and heat conductivity, one can expect a priori that the transformation of mechanical energy into heat will lead in general to an increase in entropy, at least as long as one considers a closed and isolated system.

Consider a volume $\mathcal{V}$ of flowing Newtonian fluid, delimited by a surface $\mathcal{S}$ at each point $\vec{r}$ of which the boundary conditions

$$\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\mathrm{e}}_n(\vec{r}) = 0 \quad \text{and} \quad \vec{\jmath}_Q(t,\vec{r})\cdot\vec{\mathrm{e}}_n(\vec{r}) = 0$$

hold, where $\vec{\mathrm{e}}_n(\vec{r})$ denotes the unit normal vector to $\mathcal{S}$ at $\vec{r}$. Physically, these boundary conditions mean than neither matter nor heat flows across the surface $\mathcal{S}$, so that the system inside $\mathcal{S}$ is closed and isolated. To completely exclude energy exchanges with the exterior of $\mathcal{S}$, it is also assumed that there are no external volume forces acting on the fluid inside volume $\mathcal{V}$. We shall investigate the implications of the continuity equation (III.10), the Navier–Stokes equation (III.32), and the energy conservation equation (III.37) for the total entropy S of the fluid inside $\mathcal{V}$. For the sake of brevity, the variables t, $\vec{r}$ will be omitted in the remainder of this Section.

Starting with the energy conservation equation (III.37), the contribution

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\,\vec{\mathrm{v}}^2\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho\,\vec{\mathrm{v}}^2\right)\vec{\mathrm{v}}\right]$$

in its first two lines can be replaced by

$$\rho\,\vec{\mathrm{v}}\cdot\frac{\partial\vec{\mathrm{v}}}{\partial t} + \frac{1}{2}\frac{\partial\rho}{\partial t}\,\vec{\mathrm{v}}^2 + \frac{1}{2}\Big[\vec{\nabla}\cdot\big(\rho\,\vec{\mathrm{v}}\big)\Big]\,\vec{\mathrm{v}}^2 + \sum_{i=1}^{3}\rho\mathrm{v}_i\big(\vec{\mathrm{v}}\cdot\vec{\nabla}\big)\mathrm{v}^i = \sum_{i=1}^{3}\rho\mathrm{v}_i\left[\frac{\partial\mathrm{v}^i}{\partial t} + \big(\vec{\mathrm{v}}\cdot\vec{\nabla}\big)\mathrm{v}^i\right], \quad \text{(III.39a)}$$

where the continuity equation (III.9) was used.
As recalled in Appendix A, the fundamental thermodynamic relation $U = TS - \mathcal{P}\mathcal{V} + \mu N$ gives on the one hand $e + \mathcal{P} = Ts + \mu n$, which leads to

$$\vec{\nabla}\cdot\big[(e+\mathcal{P})\vec{\mathrm{v}}\big] = T\vec{\nabla}\cdot\big(s\vec{\mathrm{v}}\big) + \mu\vec{\nabla}\cdot\big(n\vec{\mathrm{v}}\big) + \vec{\mathrm{v}}\cdot\big(s\vec{\nabla}T + n\vec{\nabla}\mu\big) = T\vec{\nabla}\cdot\big(s\vec{\mathrm{v}}\big) + \mu\vec{\nabla}\cdot\big(n\vec{\mathrm{v}}\big) + \vec{\mathrm{v}}\cdot\vec{\nabla}\mathcal{P}, \quad \text{(III.39b)}$$

where the second identity follows from the Gibbs–Duhem relation $\mathrm{d}\mathcal{P} = s\,\mathrm{d}T + n\,\mathrm{d}\mu$. On the other hand, it leads to $\mathrm{d}e = T\,\mathrm{d}s + \mu\,\mathrm{d}n$, which under consideration of the continuity equation for particle number yields

$$\frac{\partial e}{\partial t} = T\frac{\partial s}{\partial t} + \mu\frac{\partial n}{\partial t} = T\frac{\partial s}{\partial t} - \mu\vec{\nabla}\cdot\big(n\vec{\mathrm{v}}\big). \quad \text{(III.39c)}$$

With the help of relations (III.39a)–(III.39c), the energy conservation equation (III.37) can be rewritten as

$$\sum_{i=1}^{3}\rho \mathsf{v}_i\left[\frac{\partial \mathsf{v}^i}{\partial t}+\big(\vec{\mathsf{v}}\cdot\vec{\nabla}\big)\mathsf{v}^i\right]+T\frac{\partial s}{\partial t}+T\vec{\nabla}\cdot\big(s\vec{\mathsf{v}}\big)+\vec{\mathsf{v}}\cdot\vec{\nabla}\mathcal{P}=$$
$$\sum_{i,j=1}^{3}\frac{\partial}{\partial x^j}\left[\eta\left(\frac{\partial \mathsf{v}^i}{\partial x_j}+\frac{\partial \mathsf{v}^j}{\partial x_i}-\frac{2}{3}g^{ij}\vec{\nabla}\cdot\vec{\mathsf{v}}\right)\mathsf{v}_i\right]+\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\big[\zeta\big(\vec{\nabla}\cdot\vec{\mathsf{v}}\big)\mathsf{v}_i\big]+\vec{\nabla}\cdot\big(\kappa\vec{\nabla}T\big). \tag{III.39d}$$

Multiplying the i-th component of Eq. (III.31b) by v_i gives

$$\rho\mathsf{v}_i\left[\frac{\partial \mathsf{v}^i}{\partial t}+\big(\vec{\mathsf{v}}\cdot\vec{\nabla}\big)\mathsf{v}^i\right]+\mathsf{v}_i\frac{\partial\mathcal{P}}{\partial x_i}=\sum_{j=1}^{3}\mathsf{v}_i\frac{\partial}{\partial x^j}\left[\eta\left(\frac{\partial \mathsf{v}^i}{\partial x_j}+\frac{\partial \mathsf{v}^j}{\partial x_i}-\frac{2}{3}g^{ij}\vec{\nabla}\cdot\vec{\mathsf{v}}\right)\right]+\mathsf{v}_i\frac{\partial}{\partial x^i}\big(\zeta\vec{\nabla}\cdot\vec{\mathsf{v}}\big).$$

Subtracting this identity, summed over $i=1,2,3$, from Eq. (III.39d), yields

$$T\frac{\partial s}{\partial t}+T\vec{\nabla}\cdot\big(s\vec{\mathsf{v}}\big)=\eta\sum_{i,j=1}^{3}\frac{\partial \mathsf{v}_i}{\partial x^j}\left(\frac{\partial \mathsf{v}^i}{\partial x_j}+\frac{\partial \mathsf{v}^j}{\partial x_i}-\frac{2}{3}g^{ij}\vec{\nabla}\cdot\vec{v}\right)+\zeta\big(\vec{\nabla}\cdot\vec{\mathsf{v}}\big)^2+\vec{\nabla}\cdot\big(\kappa\vec{\nabla}T\big). \tag{III.40}$$

On the right hand side of this equation, one may use the identity

$$\frac{1}{2}\sum_{i,j=1}^{3}\left(\frac{\partial \mathsf{v}^i}{\partial x_j}+\frac{\partial \mathsf{v}^j}{\partial x_i}-\frac{2}{3}g^{ij}\vec{\nabla}\cdot\vec{v}\right)\left(\frac{\partial \mathsf{v}_i}{\partial x^j}+\frac{\partial \mathsf{v}_j}{\partial x^i}-\frac{2}{3}g_{ij}\vec{\nabla}\cdot\vec{v}\right)=\sum_{i,j=1}^{3}\left(\frac{\partial \mathsf{v}^i}{\partial x_j}+\frac{\partial \mathsf{v}^j}{\partial x_i}-\frac{2}{3}g^{ij}\vec{\nabla}\cdot\vec{v}\right)\frac{\partial \mathsf{v}_j}{\partial x^i}, \tag{III.41a}$$

which follows from the fact that both symmetric terms $\partial\mathsf{v}_i/\partial x^j$ and $\partial\mathsf{v}_j/\partial x^i$ from the left member give the same contribution, while the term in g_{ij} yields a zero contribution, since it multiplies a traceless term.
Additionally, one has

$$\vec{\nabla}\cdot\big(\kappa\vec{\nabla}T\big)=T\vec{\nabla}\cdot\left(\frac{\kappa\vec{\nabla}T}{T}\right)+\frac{\kappa}{T}\big(\vec{\nabla}T\big)^2. \tag{III.41b}$$

All in all, Eqs. (III.40) and (III.41) lead to

$$\frac{\partial s}{\partial t}+\vec{\nabla}\cdot\big(s\vec{\mathsf{v}}\big)-\vec{\nabla}\cdot\left(\frac{\kappa\vec{\nabla}T}{T}\right)=\frac{\eta}{2T}\sum_{i,j=1}^{3}\left(\frac{\partial \mathsf{v}^i}{\partial x_j}+\frac{\partial \mathsf{v}^j}{\partial x_i}-\frac{2}{3}g^{ij}\vec{\nabla}\cdot\vec{v}\right)\left(\frac{\partial \mathsf{v}_i}{\partial x^j}+\frac{\partial \mathsf{v}_j}{\partial x^i}-\frac{2}{3}g_{ij}\vec{\nabla}\cdot\vec{\mathsf{v}}\right)$$
$$+\frac{\zeta}{T}\big(\vec{\nabla}\cdot\vec{\mathsf{v}}\big)^2+\kappa\frac{\big(\vec{\nabla}T\big)^2}{T^2}. \tag{III.42a}$$

This may still be recast in the slightly more compact form

$$\frac{\partial s(t,\vec{r})}{\partial t}+\vec{\nabla}\cdot\left[s(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})-\kappa(t,\vec{r})\frac{\vec{\nabla}T(t,\vec{r})}{T(t,\vec{r})}\right]=$$
$$\frac{1}{T(t,\vec{r})}\left\{2\eta(t,\vec{r})\,\mathbf{S}(t,\vec{r}):\mathbf{S}(t,\vec{r})+\zeta(t,\vec{r})\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]^2+\kappa(t,\vec{r})\frac{\big[\vec{\nabla}T(t,\vec{r})\big]^2}{T(t,\vec{r})}\right\} \tag{III.42b}$$

with $\mathbf{S}:\mathbf{S}\equiv\mathbf{S}_{ij}\mathbf{S}^{ij}$ the scalar obtained by doubly contracting the rate-of-shear tensor with itself.
This equation can then be integrated over the $\mathcal{V}$ occupied by the fluid:

- When computing the integral of the divergence term on the left hand side with the Stokes theorem, it vanishes thanks to the boundary conditions imposed at the surface $\mathcal{S}$;
- the remaining term in the left member is simply the time derivative $\mathrm{d}S/\mathrm{d}t$ of the total entropy of the closed system;

- if all three transport coefficients η, ζ, κ are *positive*, then it is also the case of the three terms on the right hand side.

One thus finds $\dfrac{\mathrm{d}S}{\mathrm{d}t} \geq 0$, in agreement with the second law of thermodynamics.

Remarks:

$*$ The previous derivation may be seen as a proof that the transport coefficients must be positive to ensure that the second law of thermodynamics holds.

$*$ If all three transport coefficients η, ζ, κ vanish, i.e. in the case of a non-dissipative fluid, Eq. (III.42) simply reduces to the entropy conservation equation in perfect fluids (III.36).

Bibliography for Chapter III

- Feynman [10, 11] Chapter 40–2 & 41–1, 41–2.
- Guyon *et al.* [2] Chapters 3.3, 4.1–4.3, 5.1, 5.2.
- Landau–Lifshitz [4, 5] Chapter I § 1,2 & § 6,7 (perfect fluids) and Chapters II § 15,16 & V § 49 (Newtonian fluids).
- Fließbach [16] Chapter 32.

CHAPTER IV

Non-relativistic flows of perfect fluids

In the previous Chapter, we have introduced the coupled dynamical equations that govern the flows of perfect fluids in the non-relativistic regime, namely the continuity (III.9), Euler (III.18), and energy conservation (III.35) equations for the mass density $\rho(t,\vec{r})$, fluid velocity $\vec{\mathrm{v}}(t,\vec{r})$ and pressure $\mathcal{P}(t,\vec{r})$. The present Chapter discusses solutions of that system of equations, i.e. possible motions of perfect fluids,[(10)] obtained when using various assumptions to simplify the problem so as to render the equations tractable analytically.

In the simplest possible case, there is simply no motion at all in the fluid; yet the volume forces acting at each point still drive the behavior of the pressure and local mass density throughout the medium (Sec. IV.1). Steady flows, in which there is by definition no real dynamics, are also easily dealt with: both the Euler and energy conservation equations yield the Bernoulli equation, which can be further simplified by kinematic assumptions on the flow (Sec. IV.2).

Section IV.3 deals with the dynamics of vortices, i.e. of the vorticity vector field, in the motion of a perfect fluid. In such fluids, in case the pressure only depends on the mass density, there exists a quantity, related to vorticity, that remains conserved if the volume forces at play are conservative.

The latter assumption is also necessary to define potential flows (Sec. IV.4), in which the further hypothesis of an incompressible motion leads to simplified equations of motion, for which a number of exact mathematical results are known, especially in the case of two-dimensional flows.

Throughout the Chapter, it is assumed that the body forces in the fluid, whose volume density was denoted by $\vec{f}_V$ in Chapter III, are conservative, so that they derive from a potential. More

(10) . . . at least in an idealized world. Yet the reader is encouraged to relate the results to observations of her everyday life—beyond the few illustrative examples provided by the author—, and to wonder how a small set of seemingly "simple" mathematical equations can describe a wide variety of physical phenomena.

specifically, anticipating the fact that these volume forces are proportional to the amount of mass they act upon, we introduce the potential energy per unit mass Φ, such that

$$\vec{f}_V(t,\vec{r}) = -\rho(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}). \tag{IV.1}$$

IV.1 Hydrostatics of a perfect fluid

The simplest possibility is that of *static* solutions of the system of equations governing the dynamics of perfect fluids, namely those with $\vec{\mathrm{v}} = \vec{0}$ everywhere—in an appropriate global reference frame—and additionally $\partial/\partial t = 0$. Accordingly, there is strictly speaking no fluid flow: this is the regime of *hydrostatics*, for which the only[(11)] non-trivial equation—following from the Euler equation (III.18)—reads

$$\boxed{\frac{1}{\rho(\vec{r})}\vec{\nabla}\mathcal{P}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}).} \tag{IV.2}$$

Throughout this Section, we adopt a fixed system of Cartesian coordinates $(x^1, x^2, x^3) = (x, y, z)$, with the basis vector $\vec{\mathrm{e}}_3$ oriented vertically and pointing upwards. In the following examples, we shall consider the case of fluids in a homogeneous gravity field, leading to $\Phi(\vec{r}) = gz$, with $g = 9.8\ \mathrm{m\cdot s^{-2}}$.

Remark: If the stationarity condition is relaxed, the continuity equation still leads to $\partial\rho/\partial t = 0$, i.e. to a time-independent mass density. Whether time derivatives vanish or not makes no change in the Euler equation when $\vec{\mathrm{v}} = \vec{0}$. Eventually, energy conservation requires that the internal energy density e—and thereby the pressure—follow the same time evolution as the "external" potential energy Φ. Thus, there is a non-stationary hydrostatics, but in which the time evolution decouples from the spatial problem.

IV.1.1 Archimedes' principle

Consider first a fluid, or a system of several fluids, at rest, occupying some region of space. Let $\mathcal{S}$ be a closed control surface inside that region, as depicted in Fig. IV.1 (left), and $\mathcal{V}$ be the volume delimited by $\mathcal{S}$. The fluid inside $\mathcal{S}$ will be denoted by Σ, and that outside by Σ'.

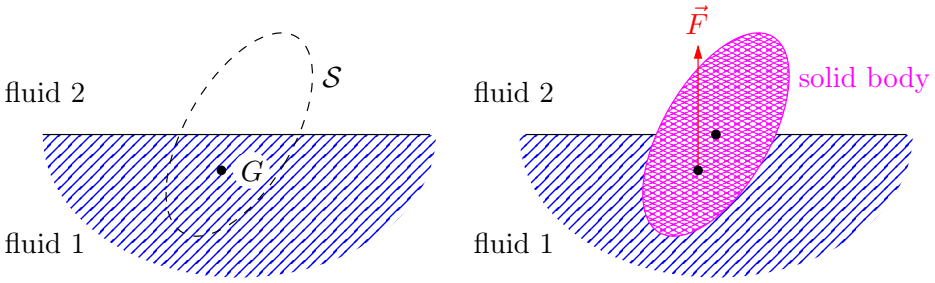


Figure IV.1 – Gedankenexperiment to illustrate Archimedes' principle.

The system Σ is in mechanical equilibrium, i.e. the sum of the gravity forces acting at each point of the volume $\mathcal{V}$ and the pressure forces exerted at each point of $\mathcal{S}$ by the fluid Σ' must vanish:

- The gravity forces at each point result in a single force $\vec{F}_G$, applied at the center of mass G of Σ, whose direction and magnitude are those of the weight of the system Σ.
- According to the equilibrium condition, the resultant of the pressure forces must equal $-\vec{F}_G$:

$$\oint_{\mathcal{S}} \mathcal{P}(\vec{r})\,\mathrm{d}^2\vec{\mathcal{S}} = -\vec{F}_G.$$

[(11)] This is true only in the case of perfect fluids; for dissipative ones, there emerge new possibilities, see § V.1.1.

If one now replaces the fluid system Σ by a rigid body $\mathcal{B}$, while keeping the fluids Σ' outside $\mathcal{S}$ in the same equilibrium state, the mechanical stresses inside Σ' remain unchanged. Thus, the resultant of the contact forces exerted by Σ' on $\mathcal{B}$ is still given by $\vec{F} = -\vec{F}_G$, which still applies at the center of mass G of the fluid system Σ. This constitutes the celebrated *Archimedes*[m] *principle*:

> *Any object, wholly or partially immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object.* (IV.3)

In addition, we have obtained the point of application of the resultant force ("buoyancy"[xlii]) from the fluid.

Remark: If the center of mass G of the "displaced" fluid system does not coincide with the center of mass of the body $\mathcal{B}$, the latter will be submitted to a torque, since $\vec{F}$ and its weight are applied at two different points. This is e.g. the case in Fig. IV.1, which describes a mechanically unstable situation.

IV.1.2 Incompressible fluid

Consider first an incompressible fluid—or, more correctly, a fluid whose compressibility can as a first approximation be neglected—with constant, uniform mass density ρ.
The fundamental equation of hydrostatics (IV.2) in the uniform gravitational field $-g\,\vec{e}_z$ then yields

$$\frac{\partial \mathcal{P}(\vec{r})}{\partial x} = \frac{\partial \mathcal{P}(\vec{r})}{\partial y} = 0, \qquad \frac{\partial \mathcal{P}(\vec{r})}{\partial z} = -\rho g.$$

That is, one recovers Pascal's law[n]

$$\mathcal{P}(\vec{r}) = \mathcal{P}(z) = \mathcal{P}_0 - \rho g z, \tag{IV.4}$$

with $\mathcal{P}_0$ the pressure at the reference point with altitude $z = 0$.

For instance, the reader is possibly aware that at a depth of 10 meters under (liquid) water ($\rho = 10^3$ kg·m^{-3}), the pressure is

$$\mathcal{P}(-10\,\mathrm{m}) = \mathcal{P}(0) + 10^3 \cdot g \cdot 10 \approx 2 \times 10^5 \text{ Pa},$$

with $\mathcal{P}(0) \approx 10^5$ Pa the typical atmospheric pressure at sea level.

IV.1.3 Fluid at global thermal equilibrium

To depart from the assumption of incompressibility, whose range of validity is quite limited, let us instead consider a fluid at global thermal equilibrium i.e. with a uniform temperature T; for instance, an ideal gas, obeying at each point the mechanical equation of state $\mathcal{P}(\vec{r}) = n(\vec{r})k_\mathrm{B}T$, where n denotes the number density.

Denoting by m the mass of a molecule of that gas, the mass density is related to pressure and temperature by $\rho = m\mathcal{P}/k_\mathrm{B}T$, so that Eq. (IV.2) reads

$$\frac{\partial \mathcal{P}(\vec{r})}{\partial x} = \frac{\partial \mathcal{P}(\vec{r})}{\partial y} = 0, \qquad \frac{\partial \mathcal{P}(\vec{r})}{\partial z} = -\frac{mg}{k_\mathrm{B}T}\,\mathcal{P}(\vec{r}),$$

i.e. one obtains the *barometric formula*[xliii]

$$\mathcal{P}(\vec{r}) = \mathcal{P}(z) = \mathcal{P}_0 \exp\left(-\frac{mgz}{k_\mathrm{B}T}\right).$$

[xlii] *statischer Auftrieb* [xliii] *barometrische Höhenformel*

[m] Archimedes, c.287–c.212 BC [n] B. Pascal, 1623–1662

Invoking the equation of state, one sees that the molecule number density $n(\vec{r})$ is also exponentially distributed, in agreement with the Maxwell distribution of statistical mechanics since mgz is the potential gravitational energy of a molecule at altitude z.

Taking as example air—which is a fictive ideal gas with molar mass[12] $\mathcal{N}_A\, m_{\text{air}} = 29\ \text{g}\cdot\text{mol}^{-1}$—the ratio $k_{\text{B}}T/m_{\text{air}}g$ equals 8.8×10^3 m for $T = 300$ K, i.e. the pressure drops by a factor 2 for every elevation gain of ca. 6 km. Obviously, however, assuming a constant temperature in the Earth atmosphere over such a length scale is unrealistic.

IV.1.4 Isentropic fluid

Let us now assume that the entropy per particle is constant throughout the perfect fluid at rest under study: $s/n = \text{constant}$, with s the entropy density and n the particle number density.

> We shall show in § X.3.2 that the ratio s/n is always conserved in the motion of a relativistic perfect fluid. Taking the low-velocity limit, one deduces the conservation of s/n in a non-relativistic non-dissipative flow: $\mathrm{D}(s/n)/\mathrm{D}t = 0$, implying that s/n is constant along pathlines, i.e. in the stationary regime along streamlines. Here we assume that s/n is constant everywhere.

Consider now the enthalpy $H = U + \mathcal{P}\mathcal{V}$ of the fluid, whose change in an infinitesimal process is the differential $\mathrm{d}H = T\,\mathrm{d}S + \mathcal{V}\,\mathrm{d}\mathcal{P} + \mu\,\mathrm{d}N$.[13] In this relation, μ denotes the chemical potential, which will however play no further role as we assume that the number of molecules in the fluid is constant, leading to $\mathrm{d}N = 0$. Dividing by N gives

$$\mathrm{d}\left(\frac{H}{N}\right) = T\,\mathrm{d}\left(\frac{S}{N}\right) + \frac{\mathcal{V}}{N}\,\mathrm{d}\mathcal{P},$$

or equivalently, in terms of the respective densities

$$\mathrm{d}\left(\frac{w}{n}\right) = T\,\mathrm{d}\left(\frac{s}{n}\right) + \frac{1}{n}\,\mathrm{d}\mathcal{P},$$

where w denotes the enthalpy density. Dropping the first term on the right-hand side, since s/n is assumed to be constant, and dividing by the mass of a molecule of the fluid, one finds

$$\mathrm{d}\left(\frac{w}{\rho}\right) = \frac{1}{\rho}\,\mathrm{d}\mathcal{P}. \tag{IV.5}$$

This identity relates the change in specific enthalpy w/ρ to the pressure variation $\mathcal{P}$ in an elementary isentropic process. If one considers a fluid at local thermodynamic equilibrium, in which w/ρ and $\mathcal{P}$ take different values at different places, the identity relates the difference in w/ρ to that in $\mathcal{P}$ between two (neighboring) points. Dividing by the distance between the two points, and in the limit where this distance vanishes, one derives an identity similar to (IV.5) with gradients instead of differentials:

$$\vec{\nabla}\left[\frac{w(\vec{r})}{\rho(\vec{r})}\right] = \frac{1}{\rho(\vec{r})}\,\vec{\nabla}\mathcal{P}(\vec{r})$$

Together with Eq. (IV.2), one thus obtains

$$\vec{\nabla}\left[\frac{w(\vec{r})}{\rho(\vec{r})} + \Phi(\vec{r})\right] = \vec{0} \tag{IV.6}$$

that is $\dfrac{w(z)}{\rho(z)} + gz = \text{constant}$ in the case of a homogeneous gravity field.

[12] $\mathcal{N}_A$ denotes the Avogadro number.

[13] The reader in need of a short reminder on thermodynamics is referred to Appendix A.

Taking as example an ideal diatomic gas, its internal energy is $U = \frac{5}{2}Nk_{\mathrm{B}}T$, resulting in the enthalpy density

$$w = e + \mathcal{P} = \frac{5}{2}nk_{\mathrm{B}}T + nk_{\mathrm{B}}T = \frac{7}{2}nk_{\mathrm{B}}T.$$

That is, $\dfrac{w}{\rho} = \dfrac{7}{2}\dfrac{k_{\mathrm{B}}T}{m}$, with m the mass of a molecule of gas. Equation (IV.6) then gives

$$\frac{\mathrm{d}T(z)}{\mathrm{d}z} = -\frac{mg}{\frac{7}{2}k_{\mathrm{B}}}.$$

In the case of air, the term on the right hand side equals $9.77 \times 10^{-3}\ \mathrm{K\cdot m^{-1}} = 9.77\ \mathrm{K\cdot km^{-1}}$, i.e. the temperature drops by ca. 10 degrees for an elevation gain of 1 km. This represents a much better modeling of the (lower) Earth atmosphere than the isothermal assumption of § IV.1.3.

Remarks:

∗ The *International Standard Atmosphere* (ISA)[(14)] model of the Earth atmosphere assumes a (piecewise) linear dependence of the temperature on the altitude. The adopted value of the temperature gradient in the troposphere is smaller than the above, namely $6.5\ \mathrm{K\cdot km^{-1}}$, to take into account the possible condensation of water vapor into droplets or even ice.

∗ Coming back to the derivation of relation (IV.6), if we had not assumed s/n constant, we would have found the relation

$$\frac{1}{\rho(\vec{r})}\,\vec{\nabla}\mathcal{P}(\vec{r}) = \vec{\nabla}\left[\frac{w(\vec{r})}{\rho(\vec{r})}\right] - T(\vec{r})\,\vec{\nabla}\left[\frac{s(\vec{r})}{\rho(\vec{r})}\right], \tag{IV.7}$$

which we shall use in § IV.2.1.

IV.2 Steady inviscid flows

We now turn to stationary solutions of the equations of motion for perfect fluids: all partial time derivatives vanish—and accordingly we shall drop the t variable—, yet the flow velocity $\vec{\mathrm{v}}(\vec{r})$ may now be non-zero. Under those conditions, the equations (III.18) and (III.35) expressing the conservations of momentum and energy collapse onto a single equation (§ IV.2.1). Some applications of the latter in the particular case of an incompressible flow are then presented (§ IV.2.2).

IV.2.1 Bernoulli equation

Replacing in the Euler equation (III.20) the pressure term with the help of relation (IV.7) and the acceleration due to volume forces by its expression in terms of the potential energy per unit mass, one finds

$$\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \vec{\nabla}\left[\frac{\vec{\mathrm{v}}(t,\vec{r})^2}{2}\right] - \vec{\mathrm{v}}(t,\vec{r})\times\vec{\omega}(t,\vec{r}) = T(t,\vec{r})\,\vec{\nabla}\left[\frac{s(t,\vec{r})}{\rho(t,\vec{r})}\right] - \vec{\nabla}\left[\frac{w(t,\vec{r})}{\rho(t,\vec{r})}\right] - \vec{\nabla}\Phi(t,\vec{r}), \tag{IV.8}$$

which is rather more clumsy than the starting point (III.20), due to the many thermodynamic quantities it involves on its right hand side. Gathering all gradient terms together, one obtains

$$\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \vec{\nabla}\left[\frac{\vec{\mathrm{v}}(t,\vec{r})^2}{2} + \frac{w(t,\vec{r})}{\rho(t,\vec{r})} + \Phi(t,\vec{r})\right] = \vec{\mathrm{v}}(t,\vec{r})\times\vec{\omega}(t,\vec{r}) + T(t,\vec{r})\,\vec{\nabla}\left[\frac{s(t,\vec{r})}{\rho(t,\vec{r})}\right]. \tag{IV.9}$$

In the stationary regime, the first term on the left-hand side disappears[(15)]—and we now omit the time variable from the equations.

[(14)] See e.g. `https://en.wikipedia.org/wiki/International_Standard_Atmosphere`.

[(15)] This yields a relation known as *Crocco's*[(o)] *theorem*[(xliv)]

[(xliv)] *Croccos Wirbelsatz*

[(o)] L. Crocco, 1909–1986

Let $\mathrm{d}\vec{\ell}(\vec{r})$ denote a vector tangential to the streamline at position $\vec{r}$, i.e. parallel to $\vec{\mathrm{v}}(\vec{r})$. In the scalar product of $\mathrm{d}\vec{\ell}(\vec{r})$ with Eq. (IV.9), both terms on the right hand side yield zero:

- the mixed product $\mathrm{d}\vec{\ell}(\vec{r}) \cdot [\vec{\mathrm{v}}(\vec{r}) \times \vec{\omega}(\vec{r})]$ is zero for it involves two collinear vectors;
- $\mathrm{d}\vec{\ell}(\vec{r}) \cdot \vec{\nabla}[s(\vec{r})/\rho(\vec{r})]$ vanishes due to the conservation of s/n in flows of perfect fluids, which together with the stationarity reads $\vec{\mathrm{v}}(\vec{r}) \cdot \vec{\nabla}[s(\vec{r})/n(\vec{r})] = 0$, where n is proportional to ρ.

In turn, $\mathrm{d}\vec{\ell}(\vec{r}) \cdot \vec{\nabla}$ represents the derivative along the direction of $\mathrm{d}\vec{\ell}(\vec{r})$, i.e. along the streamline at $\vec{r}$. Thus, the derivative of the term in squared brackets on the left hand side of Eq. (IV.9) vanishes along a streamline, i.e. the term remains constant on a streamline:

$$\boxed{\frac{\vec{\mathrm{v}}(\vec{r})^2}{2} + \frac{w(\vec{r})}{\rho(\vec{r})} + \Phi(\vec{r}) = \text{constant along a streamline}} \tag{IV.10}$$

where the value of the constant depends on the streamline. Relation (IV.10) is referred to as the *Bernoulli equation*.[(p)]

In the stationary regime, the energy conservation equation (III.35), in which one recognizes the enthalpy density $w(\vec{r}) = e(\vec{r}) + \mathcal{P}(\vec{r})$ in the flux term, leads to the same relation (IV.10).

> The first term in Eq. (III.35) vanishes due to the stationarity condition, leaving (we drop the variables)
>
> $$\vec{\nabla} \cdot \left[\left(\frac{\vec{\mathrm{v}}^2}{2} + \frac{w}{\rho} + \Phi \right) \rho \vec{\mathrm{v}} \right] = 0.$$
>
> Applying the product rule to the left member, one finds a first term proportional to $\vec{\nabla} \cdot (\rho \vec{\mathrm{v}})$—which vanishes thanks to the continuity equation (III.9)—, leaving only the other term, which is precisely ρ times the derivative along $\vec{\mathrm{v}}$ of the left hand side of the Bernoulli equation. □

Bernoulli equation in particular cases

Coming back to Eq. (IV.9), if the steady flow is irrotational, i.e. $\vec{\omega}(\vec{r}) = \vec{0}$ everywhere, and isentropic, i.e. $s(\vec{r})/n(\vec{r})$ is uniform, then the gradient on the left hand side vanishes. That is, the constant in the Bernoulli equation (IV.10) is independent of the streamline, i.e. it is the same everywhere.

In case the flow is incompressible, i.e. $\vec{\nabla} \cdot \vec{\mathrm{v}}(\vec{r}) = 0$, then the continuity equation shows that the mass density ρ becomes uniform throughout the fluid. One may then pull the factor $1/\rho$ inside the pressure gradient in the Euler equation (III.20). Repeating then the same steps as below Eq. (IV.9), one finds that the Bernoulli equation now reads

$$\boxed{\text{In incompressible flows } \quad \frac{\vec{\mathrm{v}}(\vec{r})^2}{2} + \frac{\mathcal{P}(\vec{r})}{\rho} + \Phi(\vec{r}) \quad \text{is constant along a streamline.}} \tag{IV.11}$$

This is the form which we shall use in the applications hereafter.

> Can this form be reconciled with the other one (IV.10), which is still what follows from the energy conservation equation? Subtracting one from the other, one finds that the ratio $e(\vec{r})/\rho$ is constant along streamlines. That is, since ρ is uniform, the internal energy density is constant along pathlines—which coincide with streamlines in a steady flow. Now, thermodynamics expresses the differential $\mathrm{d}e$ through $\mathrm{d}s$ and $\mathrm{d}n$: since both entropy and particle number are conserved along a pathline, so is internal energy, i.e. Eq. (IV.10) is compatible with Eq. (IV.11).

[(p)] D. Bernoulli, 1700–1782

IV.2.2 Applications of the Bernoulli equation

Throughout this Section, we assume that the flow is incompressible, i.e. the mass density is uniform, and rely on Eq. (IV.11). Of course, one may release this assumption, in which case one should replace pressure by enthalpy density everywhere below.[16]

IV.2.2 a Drainage of a vessel. Torricelli's law

Consider a liquid contained in a vessel with a small hole at its bottom, through which the liquid can flow (Fig. IV.2).

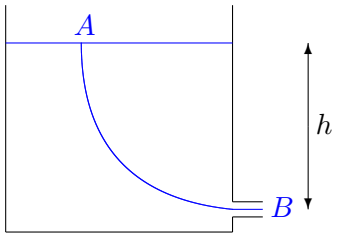


Figure IV.2

At points A and B that lie on the same streamline, the pressure in the liquid equals the atmospheric pressure[17] $\mathcal{P}_A = \mathcal{P}_B = \mathcal{P}_0$. The Bernoulli equation (at constant pressure) then yields

$$\frac{\mathsf{v}_A^2}{2} + g z_A = \frac{\mathsf{v}_B^2}{2} + g z_B,$$

with z_A resp. z_B the height of point A resp. B, i.e.

$$\mathsf{v}_B^2 = \mathsf{v}_A^2 + 2gh.$$

If the velocity at point A vanishes, one finds *Torricelli's law*[(xlv)(q)]

$$\mathsf{v}_B = \sqrt{2gh}.$$

That is, the speed of efflux is the same as that acquired by a body *in free fall* from the same height h in the same gravity field.

Remark: To be allowed to apply the Bernoulli equation, one should first show that the liquid flows steadily. If the horizontal cross section of the vessel is much larger than the aperture of the hole and h large enough, this holds to a good approximation.

IV.2.2 b Venturi effect

Consider now the incompressible flow of a fluid inside the geometry illustrated in Fig. IV.3. As we shall only be interested in the average velocity or pressure of the fluid across a cross section of the tube, the flow is effectively one-dimensional.

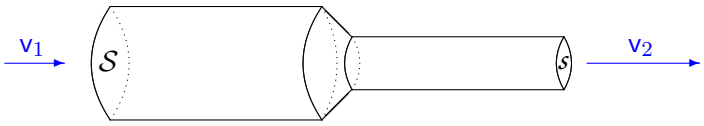


Figure IV.3

The conservation of the mass flow rate in the tube, which represents the integral formulation of the continuity equation (III.9), leads to $\rho \mathcal{S} \mathsf{v}_1 = \rho s \mathsf{v}_2$, i.e. $\mathsf{v}_2 = (\mathcal{S}/s)\mathsf{v}_1 > \mathsf{v}_1$, with $\mathcal{S}$ resp. s the area of the tube cross section in its broad resp. narrow section.

On the other hand, the Bernoulli equation at constant height, and thus potential energy, gives

$$\frac{\mathsf{v}_1^2}{2} + \frac{\mathcal{P}_1}{\rho} = \frac{\mathsf{v}_2^2}{2} + \frac{\mathcal{P}_2}{\rho}. \tag{IV.12}$$

All in all, the pressure in the narrow section is thus smaller than in the broad section, $\mathcal{P}_2 < \mathcal{P}_1$, which constitutes the *Venturi effect*.[(r)]

[16] The author confesses that he has a better physical intuition of pressure than of enthalpy, hence his parti pris.

[17] One can show that the pressure in the liquid at point B equals the atmospheric pressure provided the local streamlines are parallel to each other—that is, if the flow is laminar.

[(xlv)] *Torricellis Theorem*

[(q)] E. Torricelli, 1608–1647 [(r)] G. B. Venturi, 1746–1822

Remarks:

∗ Using mass conservation and the Bernoulli equation, one can express v_1 or v_2 in terms of the tube cross section areas and the pressure difference. For instance, the mass flow rate reads

$$\rho\mathcal{S}\left[2\frac{\mathcal{P}_1-\mathcal{P}_2}{\rho}\Big/\left(\frac{\mathcal{S}^2}{s^2}-1\right)\right]^{1/2}.$$

∗ From Eq. (IV.12) and the relation between the velocities, one sees that is the ratio $\mathcal{S}/s$, and thus v_2, is large enough, then $\mathcal{P}_2$ may be negative—which seems somewhat unsettling. Physically, the pressure does not truly become negative, but instead the assumed model breaks down. More precisely, when $\mathcal{P}_2$ reaches the saturated vapor pressure of the liquid, then the thermodynamically stable state is locally the gas phase, so that vapor bubbles will appear. This phenomenon is referred to as (hydrodynamic) *cavitation*.

IV.2.2 c Pitot tube

Figure IV.4 represents schematically the flow of a fluid around a (static) *Pitot tube*,[(s)] which is a device used to estimate a flow velocity through the measurement of a pressure difference. Three streamlines are shown, starting far away from the Pitot tube, where the flow is (approximately) uniform and has velocity $\vec{\mathsf{v}}$, which one wants to measure. The flow is assumed to be incompressible.

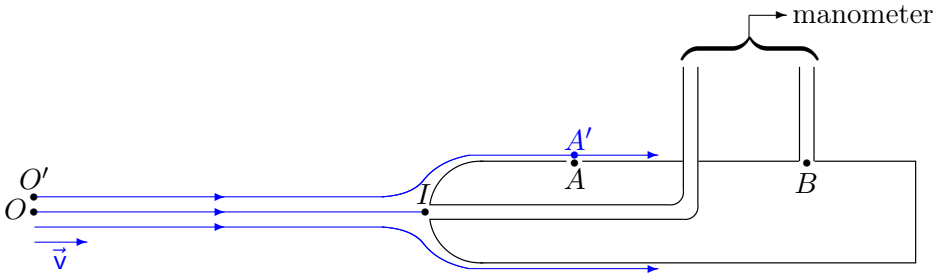


Figure IV.4 – Flow around a Pitot tube.

The Pitot tube consists of two long thin concentric tubes:

- Despite the presence of the hole at the end point I, the flow does not penetrate in the inner tube, so that $\vec{\mathsf{v}}_I = \vec{0}$ to a good approximation.
- In the broader tube, there is a hole at a point A, which is far enough from I to ensure that the fluid flow in the vicinity of A is no longer perturbed by the extremity of the tube: $\vec{\mathsf{v}}_A = \vec{\mathsf{v}}_{A'} \simeq \vec{\mathsf{v}}$, where the second identity holds thanks to the thinness of the tube, which thus perturbs the flow properties minimally. In addition, the pressure in the broader tube is uniform, so that $\mathcal{P}_A = \mathcal{P}_B$.

If one neglects the height differences—which is a posteriori justified by the numerical values we shall find—the (incompressible) Bernoulli equation gives first

$$\mathcal{P}_O + \rho\frac{\vec{\mathsf{v}}^2}{2} = \mathcal{P}_I$$

along the streamline OI, and

$$\mathcal{P}_{O'} + \rho\frac{\vec{\mathsf{v}}^2}{2} = \mathcal{P}_{A'} + \rho\frac{\vec{\mathsf{v}}_{A'}^2}{2}$$

along the streamline $O'A'$. Using $\mathcal{P}_{O'} \simeq \mathcal{P}_O$, $\mathcal{P}_{A'} \simeq \mathcal{P}_A$ and $\vec{\mathsf{v}}_{A'} \simeq \vec{\mathsf{v}}$, the latter identity leads to

[(s)] H. Pitot, 1695–1771

$\mathcal{P}_O \simeq \mathcal{P}_A = \mathcal{P}_B$. One thus finds

$$\mathcal{P}_I - \mathcal{P}_B = \rho \frac{\vec{\mathrm{v}}^2}{2},$$

so that a measurement of $\mathcal{P}_I - \mathcal{P}_B$ gives an estimate of $|\vec{\mathrm{v}}|$.

For instance, in air ($\rho \sim 1.3\ \mathrm{kg \cdot m^{-3}}$) a velocity of $100\ \mathrm{m \cdot s^{-1}}$ results in a pressure difference of 6.5×10^3 Pa $= 6.5 \times 10^{-2}$ atm. With a height difference h of a few centimeters between O and A', the neglected term $\rho g h$ is of order 0.1–1 Pa.

Remarks:

∗ The flow of a fluid with velocity $\vec{\mathrm{v}}$ around a motionless Pitot tube is equivalent to the motion of a Pitot tube with velocity $-\vec{\mathrm{v}}$ in a fluid at rest. Thus Pitot tubes are used to measure the speed of airplanes.[18]

∗ Is the flow of air really incompressible at velocities of $100\ \mathrm{m \cdot s^{-1}}$ or higher? Not really, since the Mach number (II.18) becomes larger than 0.3. In practice, one thus rather uses the "compressible" Bernoulli equation (IV.10), yet the basic principles presented above remain valid.

IV.2.2 d Magnus effect

Consider an initially uniform and steady flow with velocity $\vec{\mathrm{v}}_0$. One introduces in it a cylinder that rotates about its axis with angular velocity $\vec{\omega}_{\mathcal{C}}$ perpendicular to the flow velocity (Fig. IV.5).

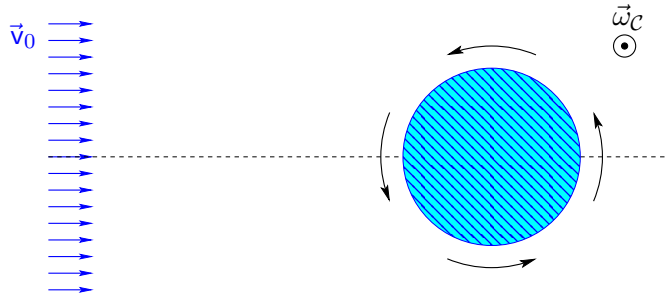


Figure IV.5 – Fluid flow around a rotating cylinder.

Intuitively, one can expect that the cylinder will drag the neighboring fluid layers along in its rotation.[19] In that case, the fluid velocity due to that rotation will add up to resp. be subtracted from the initial flow velocity in the lower resp. upper region close to the cylinder in Fig. IV.5.

Invoking now the Bernoulli equation—in which the height difference between both sides of the cylinder is neglected—, the pressure will be larger above the cylinder than below it. Accordingly, the cylinder will experience a resulting force directed downwards—more precisely, it is proportional to $\vec{\mathrm{v}}_0 \times \vec{\omega}_{\mathcal{C}}$—, which constitutes the *Magnus effect*.[t]

IV.3 Vortex dynamics in perfect fluids

We now turn back to the case of an arbitrary flow $\vec{\mathrm{v}}(t, \vec{r})$, still in the case of a perfect fluid. The vorticity vector field, defined as the curl of the flow velocity field, was introduced in § II.1.2, together with the vorticity lines. Modulo a few assumptions on the fluid equation of state and the volume forces, one can show that vorticity is "frozen" in the flow of a perfect fluid, in the sense that the

[18] When he introduced the idea in 1732, Pitot rather had the velocity of ships in his mind.

[19] Strictly speaking, this is not true in perfect fluids, only in real fluids with friction! Nevertheless, the tangential forces due to viscosity in the latter may be small enough that the Bernoulli equation remains approximately valid, as is assumed here.

[t] G. Magnus, 1802–1870

flux of vorticity across a material surface remains constant as the latter is being transported. This behavior will be investigated and formulated both at the integral level (§ IV.3.1) and differentially (§ IV.3.2).

IV.3.1 Circulation of the flow velocity. Kelvin's theorem

Definition: Let $\vec{\gamma}(t,\lambda)$ be a closed curve, parametrized by a real number $\lambda \in [0,1]$, which is being swept along by the fluid in its motion. The integral

$$\Gamma_{\vec{\gamma}}(t) \equiv \oint_{\vec{\gamma}} \vec{\mathrm{v}}(t,\vec{\gamma}(t,\lambda)) \cdot \mathrm{d}\vec{\ell} \tag{IV.13}$$

is called the *circulation* around the curve of the velocity field.

Remark: According to Stokes' theorem,[(20)] if the area bounded by the contour $\vec{\gamma}(t,\lambda)$ is simply connected, $\Gamma_{\vec{\gamma}}(t)$ equals the surface integral—the "flux"—of the vorticity field over every surface $\mathcal{S}_{\vec{\gamma}}(t)$ delimited by $\vec{\gamma}$:

$$\Gamma_{\vec{\gamma}}(t) = \int_{\mathcal{S}_{\vec{\gamma}}} \left[\vec{\nabla} \times \vec{\mathrm{v}}(t,\vec{r})\right] \cdot \mathrm{d}^2\vec{\mathcal{S}} = \int_{\mathcal{S}_{\vec{\gamma}}} \vec{\omega}(t,\vec{r}) \cdot \mathrm{d}^2\vec{\mathcal{S}}. \tag{IV.14}$$

Stated differently, the vorticity field is the flux density of the circulation of the velocity.

This relationship between circulation and vorticity will now be further exploited: we shall first establish and formulate results at the integral level, namely for the circulation, which will then be expressed at the differential level, i.e. in terms of the vorticity, in § IV.3.2.

Many results take a simpler form in a so-called *barotropic fluid*,[(xlvi)] in which the pressure can be expressed as function of only the mass density: $\mathcal{P} = \mathcal{P}(\rho)$, irrespective of whether the fluid is otherwise perfect or dissipative. An example of such a result is

Kelvin's circulation theorem:[(u)]

> *In a perfect barotropic fluid with conservative volume forces, the circulation of the flow velocity around a closed curve (enclosing a simply connected region) comoving with the fluid is conserved.* (IV.15a)

Denoting by $\vec{\gamma}(t,\lambda)$ the closed contour in the theorem,

$$\boxed{\frac{\mathrm{D}\Gamma_{\vec{\gamma}}(t)}{\mathrm{D}t} = 0.} \tag{IV.15b}$$

This result is also sometimes called *Thomson's theorem*.

Proof: For the sake of brevity, the arguments of the fields are omitted in case it is not necessary to specify them. Differentiating definition (IV.13) first gives

$$\frac{\mathrm{D}\Gamma_{\vec{\gamma}}}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\int_0^1 \frac{\partial\vec{\gamma}(t,\lambda)}{\partial\lambda} \cdot \vec{\mathrm{v}}(t,\vec{\gamma}(t,\lambda))\,\mathrm{d}\lambda = \int_0^1 \left[\frac{\partial^2\vec{\gamma}}{\partial\lambda\,\partial t} \cdot \vec{\mathrm{v}} + \frac{\partial\vec{\gamma}}{\partial\lambda} \cdot \left(\frac{\partial\vec{\mathrm{v}}}{\partial t} + \sum_i \frac{\partial\vec{\mathrm{v}}}{\partial x^i}\frac{\partial\gamma^i}{\partial t}\right)\right]\mathrm{d}\lambda.$$

Since the contour $\vec{\gamma}(t,\lambda)$ flows with the fluid, $\dfrac{\partial\vec{\gamma}(t,\lambda)}{\partial t} = \vec{\mathrm{v}}(t,\vec{\gamma}(t,\lambda))$, which leads to

$$\frac{\mathrm{D}\Gamma_{\vec{\gamma}}}{\mathrm{D}t} = \int_0^1 \left\{\frac{\partial\vec{\mathrm{v}}}{\partial\lambda} \cdot \vec{\mathrm{v}} + \frac{\partial\vec{\gamma}}{\partial\lambda} \cdot \left[\frac{\partial\vec{\mathrm{v}}}{\partial t} + (\vec{\mathrm{v}} \cdot \vec{\nabla})\vec{\mathrm{v}}\right]\right\}\mathrm{d}\lambda.$$

[(20)] which in its classical form used here is also known as Kelvin–Stokes theorem...

[(xlvi)] *barotropes Fluid*

[(u)] W. Thomson, Baron Kelvin, 1824–1907

The first term in the curly brackets is clearly the derivative with respect to λ of $\vec{\mathrm{v}}^2/2$, so that its integral along a closed curve vanishes. The second term involves the material derivative of $\vec{\mathrm{v}}$, as given by the Euler equation. Using Eq. (III.19) with $\vec{a}_V = -\vec{\nabla}\Phi$ leads to

$$\frac{\mathrm{D}\Gamma_{\vec{\gamma}}}{\mathrm{D}t} = \int_0^1 \left(-\frac{\vec{\nabla}\mathcal{P}}{\rho} - \vec{\nabla}\Phi \right) \cdot \frac{\partial \vec{\gamma}}{\partial \lambda} \, \mathrm{d}\lambda.$$

Again, the circulation of the gradient $\vec{\nabla}\Phi$ around a closed contour vanishes, leaving

$$\frac{\mathrm{D}\Gamma_{\vec{\gamma}}(t)}{\mathrm{D}t} = -\oint_{\vec{\gamma}} \frac{\vec{\nabla}\mathcal{P}(t,\vec{r})}{\rho(t,\vec{r})} \cdot \mathrm{d}\vec{\ell}, \tag{IV.16}$$

which constitutes the general case of Kelvin's circulation theorem for perfect fluids with conservative volume forces.
Transforming the contour integral with Stokes' theorem yields the surface integral of

$$\vec{\nabla} \times \left(\frac{\vec{\nabla}\mathcal{P}}{\rho} \right) = \frac{\vec{\nabla} \times \vec{\nabla}\mathcal{P}}{\rho} + \frac{\vec{\nabla}\mathcal{P} \times \vec{\nabla}\rho}{\rho^2} = \frac{\vec{\nabla}\mathcal{P} \times \vec{\nabla}\rho}{\rho^2}. \tag{IV.17}$$

In a barotropic fluid, the rightmost term of this identity vanishes since $\vec{\nabla}\mathcal{P}$ and $\vec{\nabla}\rho$ are collinear, which yields relation (IV.15). □

Remark: Using relation (IV.14) and the fact that the area $S_{\vec{\gamma}}(t)$ bounded by the curve $\vec{\gamma}$ at time t defines a material surface, which will be transported in the fluid motion, Kelvin's theorem (IV.15) can be restated as

> *In a perfect barotropic fluid with conservative volume forces, the flux of the vorticity across a material surface is conserved.* (IV.18)

Kelvin's theorem leads to two trivial corollaries, namely

Helmholtz's theorem:[(v)]

> *In the flow of a perfect barotropic fluid with conservative volume forces, the vorticity lines and vorticity tubes move with the fluid.* (IV.19)

Similar to the definition of stream tubes in § I.3.3, a vorticity tube is defined as the surface formed by the vorticity lines tangent to a given closed geometrical curve.
In the case of vanishing vorticity $\vec{\omega} = \vec{0}$, one has

Lagrange's theorem:

> *In a perfect barotropic fluid with conservative volume forces, if the flow is irrotational at a given instant t, it remains irrotational at later times.* (IV.20)

Kelvin's circulation theorem (IV.15) and its corollaries imply that vorticity cannot be created nor destroyed in the flow of a perfect barotropic fluid with conservative volume forces. However, the more general form (IV.16) already show that in a non-barotropic fluid, there is a "source" for vorticity, which leads to the non-conservation of the circulation of the flow velocity. Similarly, non-conservative forces—for instance the Coriolis force in a rotating reference frame—may contribute a non-vanishing term in Eq. (IV.16) leading to a change in the circulation. We shall see in Sec. V.5 that viscous stresses also affect the transport of vorticity in a fluid.

[(v)] H. von Helmholtz, 1821–1894

IV.3.2 Vorticity transport equation in perfect fluids

Consider the Euler equation (III.20), in the case of conservative volume forces, $\vec{a}_V = -\vec{\nabla}\Phi$. Taking the rotational curl of both sides yields after some straightforward algebra

$$\frac{\partial\vec{\omega}(t,\vec{r})}{\partial t} - \vec{\nabla}\times\big[\vec{\mathrm{v}}(t,\vec{r})\times\vec{\omega}(t,\vec{r})\big] = -\frac{\vec{\nabla}\mathcal{P}(t,\vec{r})\times\vec{\nabla}\rho(t,\vec{r})}{\rho(t,\vec{r})^2}. \tag{IV.21}$$

This relation can be further transformed using the identity (we omit the variables)

$$\vec{\nabla}\times\big(\vec{\mathrm{v}}\times\vec{\omega}\big) = \big(\vec{\omega}\cdot\vec{\nabla}\big)\vec{\mathrm{v}} + \big(\vec{\nabla}\cdot\vec{\omega}\big)\vec{\mathrm{v}} - \big(\vec{\mathrm{v}}\cdot\vec{\nabla}\big)\vec{\omega} - \big(\vec{\nabla}\cdot\vec{\mathrm{v}}\big)\vec{\omega}.$$

Since the divergence of the vorticity field $\vec{\nabla}\cdot\vec{\omega}(t,\vec{r})$ vanishes, the previous two equations yield

$$\frac{\partial\vec{\omega}(t,\vec{r})}{\partial t} + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\omega}(t,\vec{r}) - \big[\vec{\omega}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r}) = -\big[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]\vec{\omega}(t,\vec{r}) - \frac{\vec{\nabla}\mathcal{P}(t,\vec{r})\times\vec{\nabla}\rho(t,\vec{r})}{\rho(t,\vec{r})^2}. \tag{IV.22}$$

While it is tempting to introduce the material derivative $\mathrm{D}\vec{\omega}/\mathrm{D}t$ on the left hand side of this equation, for the first two terms, we rather define the whole left member to be a new derivative

$$\boxed{\frac{\mathcal{D}_{\vec{\mathrm{v}}}\vec{\omega}(t,\vec{r})}{\mathcal{D}t} \equiv \frac{\partial\vec{\omega}(t,\vec{r})}{\partial t} + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\omega}(t,\vec{r}) - \big[\vec{\omega}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r})} \tag{IV.23a}$$

or equivalently

$$\frac{\mathcal{D}_{\vec{\mathrm{v}}}\vec{\omega}(t,\vec{r})}{\mathcal{D}t} \equiv \frac{\mathrm{D}\vec{\omega}(t,\vec{r})}{\mathrm{D}t} - \big[\vec{\omega}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r}). \tag{IV.23b}$$

We shall refer to $\mathcal{D}_{\vec{\mathrm{v}}}/\mathcal{D}t$ as the *comoving time derivative*, for reasons that will be explained at the end of this Section.

Using this definition, Eq. (IV.22) reads

$$\frac{\mathcal{D}_{\vec{\mathrm{v}}}\vec{\omega}(t,\vec{r})}{\mathcal{D}t} = -\big[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]\vec{\omega}(t,\vec{r}) - \frac{\vec{\nabla}\mathcal{P}(t,\vec{r})\times\vec{\nabla}\rho(t,\vec{r})}{\rho(t,\vec{r})^2}. \tag{IV.24}$$

In the particular case of a barotropic fluid—recall that we also assumed that it is ideal and only has conservative volume forces—this becomes

$$\boxed{\frac{\mathcal{D}_{\vec{\mathrm{v}}}\vec{\omega}(t,\vec{r})}{\mathcal{D}t} = -\big[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]\vec{\omega}(t,\vec{r}).} \tag{IV.25}$$

Thus, the comoving time-derivative of the vorticity is parallel to itself.

From Eq. (IV.25), one deduces at once that if $\vec{\omega}(t,\vec{r})$ vanishes at some time t, it remains zero—which is the differential formulation of corollary (IV.20).

Invoking the continuity equation (III.9), the volume expansion rate $\vec{\nabla}\cdot\vec{\mathrm{v}}$ on the right hand side of Eq. (IV.25) can be replaced by $-(1/\rho)\mathrm{D}\rho/\mathrm{D}t$. For scalar fields, material derivative and comoving time-derivative coincide, which leads to the compact form

$$\frac{\mathcal{D}_{\vec{\mathrm{v}}}}{\mathcal{D}t}\left[\frac{\vec{\omega}(t,\vec{r})}{\rho(t,\vec{r})}\right] = \vec{0} \tag{IV.26}$$

for perfect barotropic fluids with conservative volume forces. That is, anticipating on the discussion of the comoving time derivative hereafter, $\vec{\omega}/\rho$ evolves in the fluid flow in the same way as the separation between two material neighboring points: the ratio is "frozen" in the fluid evolution.

Comoving time derivative

To understand the meaning of the comoving time derivative $\mathcal{D}_{\vec{\mathrm{v}}}/\mathcal{D}t$, let us come back to Fig. II.1 depicting the positions at successive times t and $t+\delta t$ of a small material vector $\delta\vec{\ell}(t)$. By definition

of the material derivative, the change in $\delta\vec{\ell}$ between these two instants—as given by the trajectories of the two material points which are at $\vec{r}$ resp. $\vec{r}+\delta\vec{\ell}(t)$ at time t—is

$$\delta\vec{\ell}(t+\delta t)-\delta\vec{\ell}(t)=\frac{\mathrm{D}\delta\vec{\ell}(t)}{\mathrm{D}t}\delta t.$$

On the other hand, displacing the origin of $\delta\vec{\ell}(t+\delta t)$ to let it coincide with that of $\delta\vec{\ell}(t)$, one sees

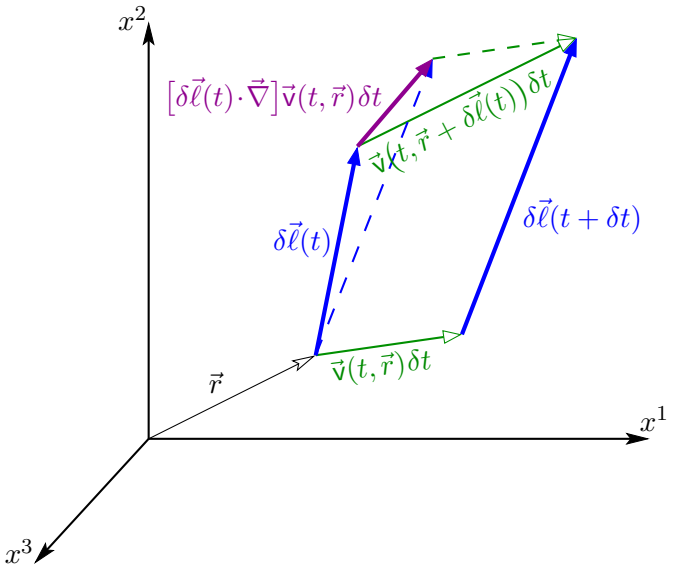


Figure IV.6 – Positions of a material line element $\delta\vec{\ell}$ at successive times t and $t+\delta t$.

on Fig. IV.6 that this change equals

$$\delta\vec{\ell}(t+\delta t)-\delta\vec{\ell}(t)=[\delta\vec{\ell}(t)\cdot\vec{\nabla}]\vec{\mathrm{v}}(t,\vec{r})\delta t.$$

Equating both results and dividing by δt, one finds $\dfrac{\mathrm{D}\delta\vec{\ell}(t)}{\mathrm{D}t}=[\delta\vec{\ell}(t)\cdot\vec{\nabla}]\vec{\mathrm{v}}(t,\vec{r})$, i.e. precisely

$$\frac{\mathcal{D}_{\vec{\mathrm{v}}}\delta\vec{\ell}(t)}{\mathcal{D}t}=\vec{0}. \tag{IV.27}$$

Thus, the comoving time derivative of a material vector, which moves with the fluid, vanishes. In turn, the comoving time derivative at a given instant t of an arbitrary vector measures its rate of change with respect to a material vector with which it coincides at time t.

This interpretation suggests—this can be proven more rigorously—what the action of the comoving time derivative on a scalar field should be. In that case, $\mathcal{D}_{\vec{\mathrm{v}}}/\mathcal{D}t$ should coincide with the material derivative, which already accounts for all changes—due to non-stationarity and convective transport—affecting material points in their motion. This justifies a posteriori our using $\mathcal{D}_{\vec{\mathrm{v}}}\rho/\mathcal{D}t=\mathrm{D}\rho/\mathrm{D}t$ above.

More generally, the comoving time derivative introduced in Eq. (IV.23a) may be rewritten as

$$\frac{\mathcal{D}_{\vec{\mathrm{v}}}}{\mathcal{D}t}(\,\cdot\,)\equiv\frac{\partial}{\partial t}(\,\cdot\,)+\mathcal{L}_{\vec{\mathrm{v}}}(\,\cdot\,), \tag{IV.28}$$

where $\mathcal{L}_{\vec{\mathrm{v}}}$ denotes the *Lie derivative* along the velocity field $\vec{\mathrm{v}}(\vec{r})$, whose action on an arbitrary vector field $\vec{\omega}(\vec{r})$ is precisely (time plays no role here)

$$\mathcal{L}_{\vec{\mathrm{v}}}\,\vec{\omega}(\vec{r})\equiv[\vec{\mathrm{v}}(\vec{r})\cdot\vec{\nabla}]\vec{\omega}(\vec{r})-[\vec{\omega}(\vec{r})\cdot\vec{\nabla}]\vec{\mathrm{v}}(\vec{r}),$$

while it operates on an arbitrary scalar field $\rho(\vec{r})$ according to

$$\mathcal{L}_{\vec{\mathrm{v}}}\,\rho(\vec{r})\equiv[\vec{\mathrm{v}}(\vec{r})\cdot\vec{\nabla}]\rho(\vec{r}).$$

More information on the Lie derivative, including its operation on 1-forms or more generally on $\binom{m}{n}$-tensors—from which the action of the comoving time derivative follows—, can be found e.g. in Ref. [17, Chap. 3.1–3.5].

IV.4 Potential flows

According to Lagrange's theorem (IV.20), every flow of a perfect barotropic fluid with conservative volume forces which is everywhere irrotational at a given instant remains irrotational at every time.

Focusing accordingly on the *incompressible* and irrotational motion of an ideal fluid with conservative volume forces, which is also referred to as a *potential flow*[xlvii], the dynamical equations can be recast such that the main one is a linear partial differential equation for the *velocity potential* (§ IV.4.1), for which there exist mathematical results (§ IV.4.2). Two-dimensional potential flows are especially interesting, since one may then introduce a complex velocity potential—and the corresponding complex velocity—, which is a holomorphic function (§ IV.4.3). This allows one to use the full power of complex analysis so as to devise flows around obstacles with various geometries by combining "elementary" solutions and deforming them.

IV.4.1 Equations of motion in potential flows

Using a known result from vector analysis, a vector field whose curl vanishes everywhere on a simply connected domain of $\mathbb{R}^3$ can be written as the gradient of a scalar field. Thus, in the case of an irrotational flow $\vec{\nabla}\times\vec{\mathrm{v}}(t,\vec{r})=\vec{0}$, the velocity field can be expressed as

$$\boxed{\vec{\mathrm{v}}(t,\vec{r})=-\vec{\nabla}\varphi(t,\vec{r})} \tag{IV.29}$$

with $\varphi(t,\vec{r})$ the so-called *velocity potential*.[xlviii].

Remarks:

∗ The minus sign in definition (IV.29) is purely conventional. While the choice adopted here is not universal, it has the advantage of being directly analogous to the convention in electrostatics ($\vec{E}=-\vec{\nabla}\Phi_{\text{Coul.}}$) or Newtonian gravitation physics ($\vec{g}=-\vec{\nabla}\Phi_{\text{Newt.}}$).

∗ Since Lagrange's theorem does not hold in a dissipative fluid, in which vorticity can be created or annihilated (Sec. V.5), the rationale behind the definition of the velocity potential disappears.

Using the velocity potential (IV.29) and the relation $\vec{a}_V=-\vec{\nabla}\Phi$ expressing that the volume forces are conservative, the Euler equation (III.20) reads

$$-\frac{\partial\vec{\nabla}\varphi(t,\vec{r})}{\partial t}+\vec{\nabla}\left\{\frac{\left[\vec{\nabla}\varphi(t,\vec{r})\right]^2}{2}+\Phi(t,\vec{r})\right\}=-\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}).$$

Assuming that the flow is also incompressible, and thus ρ constant, this becomes

$$-\frac{\partial\vec{\nabla}\varphi(t,\vec{r})}{\partial t}+\vec{\nabla}\left\{\frac{\left[\vec{\nabla}\varphi(t,\vec{r})\right]^2}{2}+\frac{\mathcal{P}(t,\vec{r})}{\rho}+\Phi(t,\vec{r})\right\}=\vec{0}. \tag{IV.30}$$

or equivalently

$$\boxed{-\frac{\partial\varphi(t,\vec{r})}{\partial t}+\frac{\left[\vec{\nabla}\varphi(t,\vec{r})\right]^2}{2}+\frac{\mathcal{P}(t,\vec{r})}{\rho}+\Phi(t,\vec{r})=C(t),} \tag{IV.31}$$

where $C(t)$ denotes a function of time only.

Eventually, expressing the incompressibility condition [cf. Eq. (II.15)] $\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})=0$ leads to the *Laplace equation*[w] for the velocity potential φ

$$\boxed{\triangle\varphi(t,\vec{r})=0.} \tag{IV.32}$$

[xlvii] *Potentialströmung* [xlviii] *Geschwindigkeitspotential*

[w] P.-S. (de) Laplace, 1749–1827

Equations (IV.29), (IV.31) and (IV.32) are the three equations of motion governing potential flows. Since the Laplace equation is partial differential, it is still necessary to specify the corresponding boundary conditions.

In agreement with the discussion in § III.3.2 c, there are two types of conditions: at walls or obstacles, which are impermeable to the fluid; and "at infinity"—for a flow in an unbounded domain of space—, where the fluid flow is generally assumed to be uniform. Choosing a proper reference frame $\mathcal{R}$, this uniform motion of the fluid may be turned into having a fluid at rest. Denoting by $\mathcal{S}(t)$ the material surface associated with walls or obstacles, which are assumed to be moving with velocity $\vec{v}_{\text{obs.}}$ with respect to $\mathcal{R}$, and by $\vec{e}_n(t,\vec{r})$ the unit normal vector to $\mathcal{S}(t)$ at a given point $\vec{r}$, the condition of vanishing relative normal velocity reads

$$-\vec{e}_n(t,\vec{r})\cdot\vec{\nabla}\varphi(t,\vec{r}) = \vec{e}_n(t,\vec{r})\cdot\vec{v}_{\text{obs.}}(t,\vec{r}) \quad \text{on} \quad \mathcal{S}(t). \tag{IV.33a}$$

In turn, the condition of rest at infinity reads

$$\varphi(t,\vec{r}) \underset{|\vec{r}|\to\infty}{\sim} K(t), \tag{IV.33b}$$

where in practice the scalar function $K(t)$ will be given.

Remarks:

∗ Since the Laplace equation (IV.32) is linear—the non-linearity of the Euler equation is in Eq. (IV.31), which becomes trivial once the spatial dependence of the velocity potential has been determined—, it will be possible to *superpose* the solutions of "simple" problems to obtain the solution for a more complicated geometry.

∗ In potential flows, the dependences on time and space are somewhat separated: The Laplace equation (IV.32) governs the spatial dependence of φ and thus $\vec{\mathrm{v}}$; meanwhile, time enters the boundary conditions (IV.33), and is thus used to fix the amplitude of the solution of the Laplace equation. In turn, when φ is known, relation (IV.31) gives the pressure field, where the integration "constant" $C(t)$ will also be fixed by boundary conditions.

IV.4.2 Mathematical results on potential flows

The *boundary value problem* consisting of the Laplace differential equation (IV.32) together with the boundary conditions on normal derivatives (IV.33) is called a *Neumann problem*[(x)] or boundary value problem of the second kind. For such problems, results on the existence and unicity of solutions have been established, which we shall now state without further proof.[(21)]

IV.4.2 a Potential flows in simply connected regions

The simplest case is that of a potential flow on a simply connected domain $\mathcal{D}$ of space. $\mathcal{D}$ may be unbounded, provided the condition at infinity is that the fluid should be at rest, Eq. (IV.33b).

> *On a simply connected domain, the Neumann problem* (IV.32)–(IV.33) *for the velocity potential admits a solution* $\varphi(t,\vec{r})$*, which is unique up to an additive constant. In turn, the flow velocity field* $\vec{\mathrm{v}}(t,\vec{r})$ *given by relation* (IV.29) *is unique.* (IV.34)

For a flow on a simply connected region, the relation (IV.29) between the flow velocity and its potential is "easily" invertible: fixing some reference position $\vec{r}_0$ in the domain, one may write

$$\varphi(t,\vec{r}) = \varphi(t,\vec{r}_0) - \int_{\vec{\gamma}} \vec{\mathrm{v}}(t,\vec{r}')\cdot \mathrm{d}\vec{\ell}(\vec{r}') \tag{IV.35}$$

where the line integral is taken along *any* path $\vec{\gamma}$ on $\mathcal{D}$ connecting the positions $\vec{r}_0$ and $\vec{r}$.

(21) The Laplace differential equation is dealt with in many textbooks, as e.g. in Ref. [18, Chapters 7–9], [19, Chapter 4], or [20, Chapter VII].

(x) C. Neumann, 1832–1925

That the line integral only depends on the path extremities $\vec{r}_0$, $\vec{r}$, not on the path itself, is clearly equivalent to Stokes' theorem stating that the circulation of velocity along *any* closed contour in the domain $\mathscr{D}$ is zero—it equals the flux of the vorticity, which is everywhere zero, through a surface delimited by the contour and entirely contained in $\mathscr{D}$.

Thus, $\varphi(t,\vec{r})$ is uniquely defined once the value $\varphi(t,\vec{r}_0)$, which is the arbitrary additive constant mentioned above, has been fixed.

This reasoning no longer holds in a multiply connected domain, as we now further discuss.

IV.4.2 b Potential flows in doubly connected regions

As a matter of fact, in a doubly (or a fortiori multiply) connected domain, there are by definition non-contractible closed paths. Consider for instance the domain $\mathscr{D}$ traversed by an infinite cylinder—which is not part of the domain—of Fig. IV.7. The path going from $\vec{r}_0$ to $\vec{r}_2$ along $\vec{\gamma}_{0\to2}$ and coming back to $\vec{r}_0$ along $\vec{\gamma}'_{0\to2}$ [(22)] cannot be continuously shrunk to a point without leaving $\mathscr{D}$. This opens the possibility that the line integral in relation (IV.35) could depend on the path connecting two points.

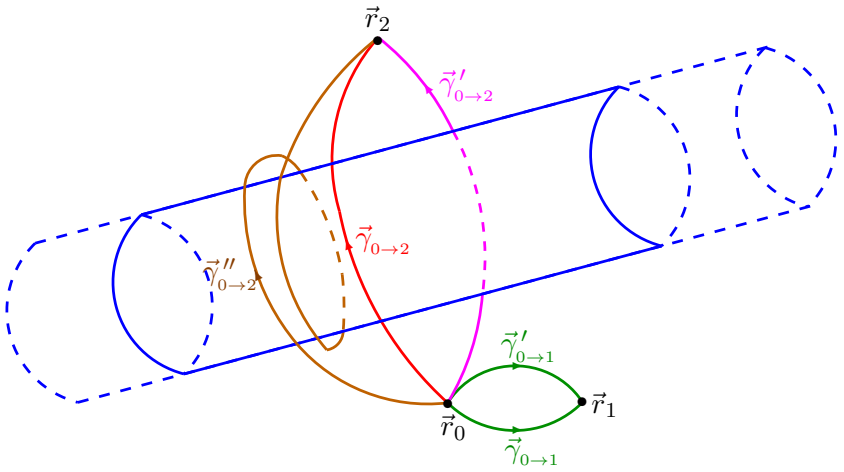


Figure IV.7

In a doubly connected domain $\mathscr{D}$, there is only a single "hole" that prevents closed paths from being homotopic to a point, i.e. contractible. Let $\Gamma(t)$ denote the circulation at time t of the velocity around a closed contour, with a given "positive" orientation, circling the hole *once*. One easily checks—e.g. invoking Stokes' theorem—that this circulation has the same value for all closed paths going only once around the hole with the same orientation, since they can be continuously deformed into each other without leaving $\mathscr{D}$. Accordingly, the "universal" value of the circulation $\Gamma(t)$ is also referred to as *cyclic constant* of the flow.

More generally, the circulation at time t of the velocity around a closed curve circling the hole n times and oriented in the positive resp. negative direction is $n\Gamma(t)$ resp. $-n\Gamma(t)$.

Going back to the line integral in Eq. IV.35, its value will generally depend on the path $\vec{\gamma}$ from $\vec{r}_0$ to $\vec{r}$—or more precisely, on the class, defined by the number of loops around the hole, of the path. Illustrating this idea on Fig. IV.7, while the line integral from $\vec{r}_0$ to $\vec{r}_2$ along the path $\vec{\gamma}_{0\to2}$ will have a given value $\mathcal{I}$, the line integral along $\vec{\gamma}'_{0\to2}$ will differ by one (say, positive) unit of $\Gamma(t)$ and be equal to $\mathcal{I}+\Gamma(t)$. In turn, the integral along $\vec{\gamma}''_{0\to2}$, which makes one negatively oriented loop more than $\vec{\gamma}_{0\to2}$ around the cylinder, takes the value $\mathcal{I}-\Gamma(t)$.

These preliminary discussions suggest that if the Neumann problem (IV.32)–(IV.33) for the velocity potential on a doubly connected domain admits a solution $\varphi(t,\vec{r})$, the latter will not be

(22) More precisely, if $\vec{\gamma}'_{0\to2}$ is parameterized by $\lambda\in[0,1]$ when going from $\vec{r}_0$ to $\vec{r}_2$, a path from $\vec{r}_2$ to $\vec{r}_0$ with the same geometric support—which is what is meant by "coming back along $\vec{\gamma}'_{0\to2}$"—is $\lambda\mapsto\vec{\gamma}'_{0\to2}(1-\lambda)$.

a scalar function in the usual sense, but rather a *multivalued* function, whose various values at a given position $\vec{r}$ at a fixed time t differ by an integer factor of the cyclic constant $\Gamma(t)$.

All in all, the following result holds *provided the cyclic constant* $\Gamma(t)$ *is known*, i.e. if its value at time t is part of the boundary conditions:

> *On a doubly connected domain, the Neumann problem* (IV.32)–(IV.33) *for the velocity potential with given cyclic constant* $\Gamma(t)$ *admits a solution* $\varphi(t,\vec{r})$*, which is unique up to an additive constant. The associated flow velocity field* $\vec{\mathsf{v}}(t,\vec{r})$ *is unique.* (IV.36)

The above wording does not specify the nature of the solution $\varphi(t,\vec{r})$:

- if $\Gamma(t) = 0$, in which case the flow is said to be *acyclic*, the velocity potential $\varphi(t,\vec{r})$ is a univalued function;
- if $\Gamma(t) \neq 0$, i.e. in a *cyclic flow*, the velocity potential $\varphi(t,\vec{r})$ is a multivalued function of its spatial argument. Yet as the difference between the various values at a given $\vec{r}$ is function of time only, the velocity field (IV.29) remains uniquely defined.

Remarks:

∗ Inspecting Eq. (IV.31), one might fear that the pressure field $\mathcal{P}(t,\vec{r})$ could be multivalued, reflecting the term $\partial\varphi(t,\vec{r})/\partial t$. Actually, however, Eq. (IV.31) is a first integral of Eq. (IV.30), in which the $\vec{r}$-independent multiples of $\Gamma(t)$ distinguishing the multiple values of $\varphi(t,\vec{r})$ disappear when the gradient is taken. That is, the term $\partial\varphi(t,\vec{r})/\partial t$ is to be taken with a grain of salt, since in fact it does not contain $\Gamma(t)$ nor its time derivative.

∗ In agreement with the first remark, the reader should remember that the velocity potential $\varphi(t,\vec{r})$ is just a useful auxiliary mathematical function,[(23)] yet the physical quantity is the velocity itself. Thus the possible multivaluedness of $\varphi(t,\vec{r})$ is not a real physical problem.

IV.4.3 Two-dimensional potential flows

We now focus on two-dimensional potential flows, for which the velocity field—and all other fields—only depend on two coordinates. The latter will either be Cartesian coordinates (x,y), which are naturally combined into a complex variable $z = x + \mathrm{i}y$, or polar coordinates (r,θ). Throughout this Section, the time variable t will not be denoted: apart from possibly influencing the boundary conditions, it plays no direct role in the determination of the velocity potential.

IV.4.3 a Complex flow potential and complex flow velocity

Let us first introduce a few useful auxiliary functions, which either simplify the description of two-dimensional potential flows, or allow one to "generate" such flows at will.

Stream function

Irrespective of whether the motion is irrotational or not, in an incompressible two-dimensional flow one can define a unique (up to an additive constant) *stream function*[(xlix)] $\psi(x,y)$ such that

$$\mathsf{v}^x(x,y) = -\frac{\partial\psi(x,y)}{\partial y}, \quad \mathsf{v}^y(x,y) = \frac{\partial\psi(x,y)}{\partial x} \tag{IV.37}$$

at every point (x,y). Indeed, when the above two relations hold, the incompressibility criterion $\vec{\nabla}\cdot\vec{\mathsf{v}}(x,y) = 0$ is fulfilled automatically.

[(23)] Like its cousins: gravitational potential $\Phi_{\text{Newt.}}$, electrostatic potential $\Phi_{\text{Coul.}}$, magnetic vector potential $\vec{A}\ldots$

[(xlix)] *Stromfunktion*

Remark: As in the case of the relation between the flow velocity field and the corresponding potential, Eq. (IV.29), the overall sign in the relation between $\vec{\mathsf{v}}(\vec{r})$ and $\psi(\vec{r})$ is conventional. Yet if one wishes to define the complex flow potential as in Eq. (IV.40) below, the relative sign of $\varphi(\vec{r})$ and $\psi(\vec{r})$ is fixed.

The stream function for a given planar fluid motion is such that the lines along which $\psi(\vec{r})$ is constant are precisely the streamlines of the flow.

> Let $\mathrm{d}\vec{x}(\lambda)$ denote a differential line element of a curve $\vec{x}(\lambda)$ of constant $\psi(\vec{r})$, i.e. a curve along which $\vec{\nabla}\psi = \vec{0}$. Then $\mathrm{d}\vec{x}(\lambda) \cdot \vec{\nabla}\psi\big(\vec{x}(\lambda)\big) = 0$ at every point on the line: using relations (IV.37), one recovers Eq. (I.15b) characterizing a streamline. □
>
> Stream functions are also defined in three-dimensional flows, yet in that case two of them are needed. More precisely, one can find two linearly independent functions $\psi_1(\vec{r})$, $\psi_2(\vec{r})$, such that the streamlines are the intersections of the surfaces of constant ψ_1 and of constant ψ_2. That is, they are such that the flow velocity obeys $\vec{\mathsf{v}}(\vec{r}) \propto \vec{\nabla}\psi_1(\vec{r}) \times \vec{\nabla}\psi_2(\vec{r})$, with an a priori position-dependent proportionality factor—which can be taken identically equal to unity in an incompressible flow.

Consider now a potential flow, that is a fluid motion which is incompressible and irrotational. In the two-dimensional case, the condition of vanishing vorticity reads

$$\omega^z(x,y) = \frac{\partial \mathsf{v}^y(x,y)}{\partial x} - \frac{\partial \mathsf{v}^x(x,y)}{\partial y} = 0,$$

which under consideration of relations (IV.37) gives

$$\boxed{\triangle\psi(x,y) = 0} \tag{IV.38a}$$

at every point (x,y). That is, the stream function obeys the Laplace equation—just like the velocity potential $\varphi(\vec{r})$.

A difference with $\varphi(\vec{r})$ arises with respect to the boundary conditions. At an obstacle or walls, modeled by a "surface" $\mathcal{S}$—in the plane $\mathbb{R}^2$, this surface is rather a curve—, the impermeability condition implies that the velocity is tangential to $\mathcal{S}$, i.e. $\mathcal{S}$ coincides with a streamline:

$$\psi(x,y) = \text{constant on } \mathcal{S} \tag{IV.38b}$$

For a flow on an unbounded domain, the velocity is assumed to be uniform at infinity, $\vec{\mathsf{v}}(x,y) \to \vec{\mathsf{v}}_\infty$, which is the case if

$$\psi(x,y) \underset{|\vec{r}|\to\infty}{\sim} \mathsf{v}_\infty^y\, x - \mathsf{v}_\infty^x\, y \tag{IV.38c}$$

with $\mathsf{v}_\infty^x, \mathsf{v}_\infty^y$ the components of $\vec{\mathsf{v}}_\infty$.

The boundary conditions (IV.38b)–(IV.38c) on the stream function are thus dissimilar from the corresponding conditions (IV.33a)–(IV.33b) on the velocity potential. In particular, the condition at an obstacle involves the stream function itself, instead of its derivative: the Laplace differential equation (IV.38a) with conditions (IV.38b)–(IV.38c) represents a *Dirichlet problem*,[(y)] or boundary value problem of the first kind, instead of a Neumann problem.

Complex flow potential

In the case of a two-dimensional potential flow, both the velocity potential $\varphi(x,y)$ and the stream function $\psi(x,y)$ are so-called *harmonic functions*, i.e. they are solutions to the Laplace differential equation, see Eqs. (IV.32) and (IV.38a). In addition, gathering Eqs. (IV.29) and (IV.37), one sees that they satisfy at every point (x,y) the identities

$$\frac{\partial\varphi(x,y)}{\partial x} = \frac{\partial\psi(x,y)}{\partial y}\ \big[= -\mathsf{v}^x(x,y)\big], \quad \frac{\partial\varphi(x,y)}{\partial y} = -\frac{\partial\psi(x,y)}{\partial x}\ \big[= -\mathsf{v}^y(x,y)\big]. \tag{IV.39}$$

[(y)] P. G. (Lejeune-)Dirichlet, 1805–1859

The relations between the partial derivatives of φ and ψ are precisely the Cauchy–Riemann equations obeyed by the corresponding derivatives of the real and imaginary parts of a holomorphic function of a complex variable $z = x + \mathrm{i}y$. That is, the identities (IV.39) suggest the introduction of a *complex (flow) potential*

$$\boxed{\phi(z) \equiv \varphi(x,y) + \mathrm{i}\psi(x,y) \quad \text{with} \quad z = x + \mathrm{i}y} \tag{IV.40}$$

which will automatically be holomorphic on the domain where the flow is defined. The functions φ and ψ are then said to be conjugate to each other. In line with that notion, the curves in the plane along which one of the functions is constant are the field lines of the other, and reciprocally.

Besides the complex potential $\phi(z)$, one also defines the corresponding *complex velocity* as the negative of its derivative, namely

$$\boxed{\mathsf{w}(z) \equiv -\frac{\mathrm{d}\phi(z)}{\mathrm{d}z} = \mathsf{v}^x(x,y) - \mathrm{i}\mathsf{v}^y(x,y)} \tag{IV.41}$$

where the second identity follows at once from the definition of ϕ and the relations between φ or ψ and the flow velocity. Like $\phi(z)$, the complex velocity $\mathsf{w}(z)$ is an analytic function of z.

IV.4.3 b Elementary two-dimensional potential flows

As a converse to the above construction of the complex potential, the real and imaginary parts of *any* analytic function of a complex variable are harmonic functions, i.e. any analytical function $\phi(z)$ defines a two-dimensional potential flow on its domain of definition. Accordingly, we now investigate a few "basic" complex potentials and the flows they describe.

Uniform flow

The simplest possibility is that of a linear complex potential:

$$\phi(z) = -\mathsf{v}\,\mathrm{e}^{-\mathrm{i}\alpha}z \quad \text{with} \quad \mathsf{v} \in \mathbb{R},\ \alpha \in \mathbb{R}. \tag{IV.42}$$

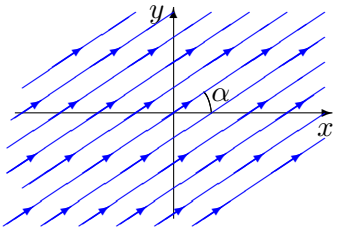


Figure IV.8

Using for instance Eq. (IV.41), this trivially leads to a uniform velocity field making an angle α with the x-direction,

$$\vec{\mathsf{v}}(x,y) = \big(\cos\alpha\,\vec{\mathrm{e}}_x + \sin\alpha\,\vec{\mathrm{e}}_y\big)\mathsf{v},$$

as illustrated in Fig. IV.8, in which a few streamlines are displayed, to which the equipotential lines (not shown) of $\varphi(x,y)$ are perpendicular.

Flow source or sink

Another flow with "simple" streamlines is that defined by the complex potential[(24)]

$$\phi(z) = -\frac{Q}{2\pi}\log(z - z_0) \quad \text{with} \quad Q \in \mathbb{R},\ z_0 \in \mathbb{C}. \tag{IV.43a}$$

The resulting complex flow velocity

$$\mathsf{w}(z) = \frac{Q}{2\pi(z - z_0)} \tag{IV.43b}$$

has a simple pole at $z = z_0$. Using polar coordinates (r,θ) centered on that pole, the flow velocity

[(24)] The reader unwilling to take the logarithm of a dimensionful quantity—which is a healthy reaction—may divide $z - z_0$ resp. r by a length in the potentials (IV.43a) and (IV.44a) resp. (IV.43d) and (IV.44c), or write the difference in Eq. (IV.46) as the logarithm of a quotient. She will however quickly convince herself that this does not affect the velocities (IV.43b) and (IV.44b).

is purely radial:

$$\vec{\mathrm{v}}(r,\theta) = \frac{Q}{2\pi r}\,\vec{\mathrm{e}}_r \tag{IV.43c}$$

as displayed in the left panel of Fig. IV.9 while the flow potential and the stream function are

$$\varphi(r,\theta) = -\frac{Q}{2\pi}\log r, \quad \psi(r,\theta) = -\frac{Q}{2\pi}\,\theta. \tag{IV.43d}$$

By computing the flux of velocity through a closed curve circling the pole—e.g. a circle centered on z_0, which is an equipotential of φ—, one finds that Q represents the mass flow rate through that curve. If Q is positive, there is a *source* of flow at z_0; if Q is negative, the pole at z_0 models a *sink* into which the fluid disappears.

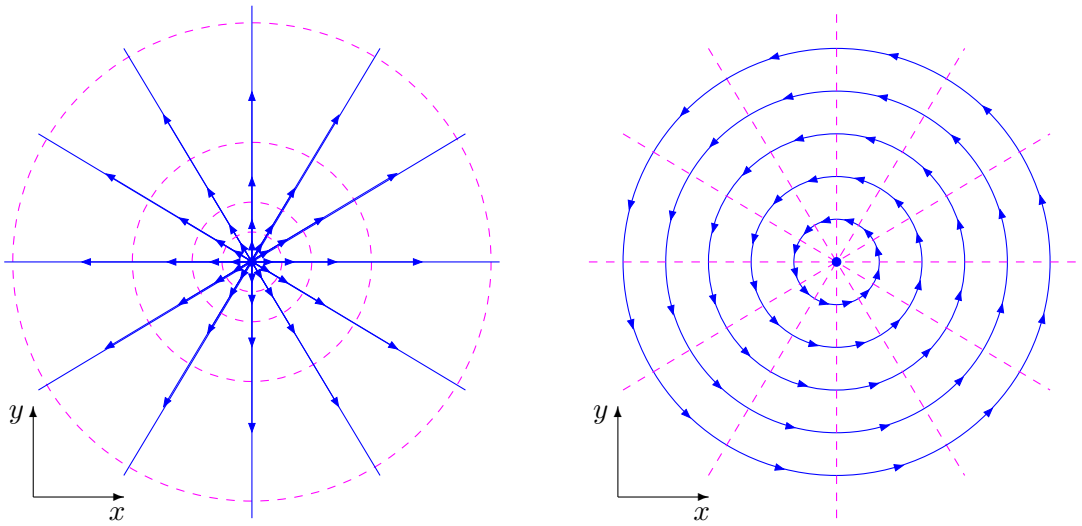


Figure IV.9 – Streamlines (full) and equipotential lines (dashed) for a flow source (IV.43c) (left) and a pointlike vortex (IV.44b) (right).

Pointlike vortex

The "conjugate" flow to the previous one, i.e. that for which φ and ψ are exchanged, corresponds to the complex potential[(24)]

$$\phi(z) = \frac{\mathrm{i}\Gamma}{2\pi}\log(z - z_0) \quad \text{with} \quad \Gamma \in \mathbb{R},\ z_0 \in \mathbb{C}. \tag{IV.44a}$$

Using as above polar coordinates (r,θ) centered on z_0, the flow velocity is purely tangential,

$$\vec{\mathrm{v}}(r,\theta) = \frac{\Gamma}{2\pi r}\,\vec{\mathrm{e}}_\theta, \tag{IV.44b}$$

as shown in Fig. IV.9 (right), where $\vec{\mathrm{e}}_\theta$ is a unit orthoradial basis vector at (r,θ). The complex potential (IV.44a) thus describes a vortex situated at z_0.

In turn, the velocity potential and stream function read

$$\varphi(r,\theta) = -\frac{\Gamma}{2\pi}\,\theta, \quad \psi(r,\theta) = \frac{\Gamma}{2\pi}\log r, \tag{IV.44c}$$

to be compared with those for a flow source, Eq. (IV.43d).

Remark: When writing down the complex velocity potentials (IV.43a) or (IV.44a), we left aside the issue of the (logarithmic!) branch point at $z = z_0$—and we did not specify which branch of the logarithm we consider. Now, either potential corresponds to a flow that is actually defined on a doubly connected region, since the velocity diverges at $z = z_0$. From the discussion in § IV.4.2 b, on such domains the potential is a multivalued object, yet this is irrelevant for the physical quantities,

namely the velocity field, which remains uniquely defined at each point. This is precisely what is illustrated here by the different branches of the logarithm, which differ by a constant integer multiple of $2\pi\mathrm{i}$ that does not affect the derivative.

Flow dipole

A further possible irrotational and incompressible two-dimensional flow is that defined by the complex potential

$$\phi(z) = \frac{\mu\,\mathrm{e}^{\mathrm{i}\alpha}}{z - z_0} \quad \text{with} \quad \mu \in \mathbb{R},\ \alpha \in \mathbb{R},\ z_0 \in \mathbb{C} \tag{IV.45a}$$

leading to the complex flow velocity

$$\mathsf{w}(z) = \frac{\mu\,\mathrm{e}^{\mathrm{i}\alpha}}{(z - z_0)^2}. \tag{IV.45b}$$

Again, both $\phi(z)$ and $\mathsf{w}(z)$ are singular at z_0.

Using polar coordinates (r, θ) centered on z_0, the flow velocity reads

$$\vec{\mathrm{v}}(r, \theta) = \frac{\mu}{r^2}\cos(\theta - \alpha)\,\vec{\mathrm{e}}_r + \frac{\mu}{r^2}\sin(\theta - \alpha)\,\vec{\mathrm{e}}_\theta, \tag{IV.45c}$$

which shows that the angle α gives the overall orientation of the flow with respect to the x-direction.

Setting for simplicity $\alpha = 0$ and coming back momentarily to Cartesian coordinates, the flow potential and stream function corresponding to Eq. (IV.45a) are

$$\varphi(x, y) = \frac{\mu x}{x^2 + y^2}, \quad \psi(x, y) = -\frac{\mu y}{x^2 + y^2}. \tag{IV.45d}$$

Thus, the streamlines are the curves $x^2 + y^2 = \text{const.} \times y$, i.e. they are circles centered on the y-axis

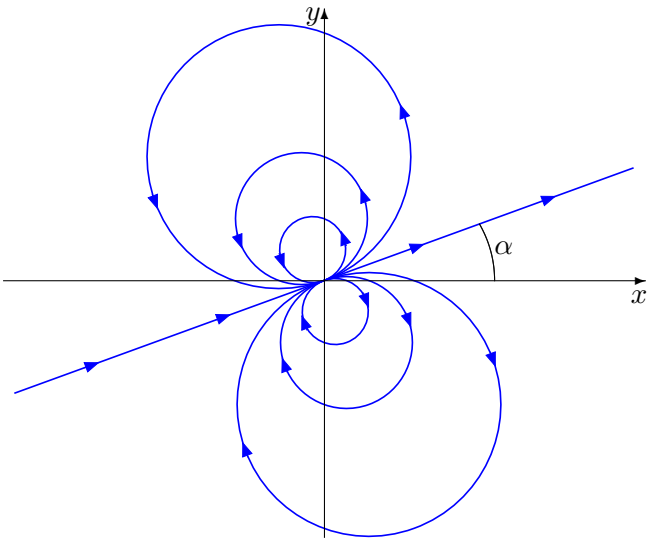


Figure IV.10 – Streamlines for a flow dipole (IV.45a) centered on the origin.

and tangent to the x-axis, as represented in Fig. IV.10, where everything is tilted by an angle α.

One can check that the flow dipole (IV.45a) is actually the superposition of a pair of infinitely close source and sink with the same mass flow rate in absolute value:

$$\phi(z) = \lim_{\varepsilon \to 0} \frac{\mu}{2\varepsilon}\left[\log\left(z - z_0 + \varepsilon\,\mathrm{e}^{-\mathrm{i}\alpha}\right) - \log\left(z - z_0 - \varepsilon\,\mathrm{e}^{-\mathrm{i}\alpha}\right)\right]. \tag{IV.46}$$

This is clearly fully analogous to an electric dipole potential being the superposition of the potentials created by electric charges $+q$ and $-q$—and justifies the denomination "dipole flow".

One can similarly define higher-order multipoles: flow quadrupoles, octupoles, ..., for which the order of the pole of the velocity at z_0 increases (order 1 for a source or a sink, order 2 for a dipole, order 3 for a quadrupole, and so on).

Remarks:

∗ The complex flow potentials considered until now—namely those of uniform flows (IV.42), sources or sinks (IV.43a), pointlike vortices (IV.44a), and dipoles (IV.45a) or multipoles—and their superpositions are the only two-dimensional flows valid on an unbounded domain.

> As a matter of fact, demanding that the flow velocity $\vec{\mathsf{v}}(\vec{r})$ should be uniform at infinity and the complex velocity $\mathsf{w}(z)$ analytic except at a finite number of singularities—say only one, at z_0, to simplify the argumentation—, then $\mathsf{w}(z)$ may be expressed as a superposition of integer powers of $1/(z-z_0)$:
>
> $$\mathsf{w}(z) = \sum_{p=0}^{\infty} \frac{a_{-p}}{(z-z_0)^p}, \tag{IV.47a}$$
>
> since any positive power of $(z-z_0)$ would be unbounded when $|z| \to \infty$. Integrating over z, see Eq. (IV.41), the allowed complex potentials are of the form
>
> $$\phi(z) = -a_0 z - a_{-1}\log(z-z_0) + \sum_{p=1}^{\infty} \frac{p\, a_{-p-1}}{(z-z_0)^p}. \tag{IV.47b}$$

∗ Conversely, the reader can check—by computing the integral of $\mathsf{w}(z)$ along a contour at infinity—that the total mass flow rate and circulation of the velocity field for a given flow are respectively the real and imaginary parts of the residue a_{-1} in the Laurent series of its complex velocity $\mathsf{w}(z)$, i.e. are entirely governed by the source/sink term (IV.43a) and vortex term (IV.44a) in the complex potential.

∗ Eventually, the singularities that arise in the flow velocity will in practice not be a problem, since these points will not be part of the physical flow, as we shall see on an example in § IV.4.3 c.

Flow inside or around a corner

As a last example, consider the complex flow potential

$$\phi(z) = A\,\mathrm{e}^{-\mathrm{i}\alpha}(z-z_0)^n \quad \text{with} \quad A \in \mathbb{R},\ \alpha \in \mathbb{R},\ n \geq \frac{1}{2},\ z_0 \in \mathbb{C}. \tag{IV.48a}$$

Except in the case $n = 1$, this potential cannot represent a flow on an unbounded domain, since one easily checks that the velocity is unbounded as $|z|$ goes to infinity. The interest of this potential lies rather the behavior in the vicinity of $z = z_0$.

As a matter, writing down the flow potential and the stream function in a system of polar coordinates centered on z_0,

$$\varphi(r,\theta) = A\,r^n\cos(n\theta-\alpha), \quad \psi(r,\theta) = A\,r^n\sin(n\theta-\alpha) \tag{IV.48b}$$

shows that they both are (π/n)-periodic functions of the polar angle θ. Thus the flow on the domain $\mathscr{D}$ delimited by the streamlines $\psi(r,\alpha)$ and $\psi(r,\alpha+\pi/n)$ is isolated from the motion in the remainder of the complex plane. One may therefore assume that there are walls along these two streamlines, and that the complex potential (IV.48a) describes a flow between them.

For $n = 1$, one recovers the uniform flow (IV.42)—in which we are free to put a wall along any streamline, restricting the domain $\mathscr{D}$ to a half plane instead of the whole plane. If $n > 1$, π/n is smaller than π and the domain $\mathscr{D}$ is comprised between a half-plane; in that case, the fluid motion is a flow *inside* a corner. On the other hand, for $\frac{1}{2} \leq n < 1$, $\pi/n > \pi$, so that the motion is a flow *past* a corner.

The streamlines for the flows obtained with six different values for n are displayed in Fig. IV.11, namely two flows in corners with angles $\pi/3$ and $2\pi/3$, a uniform flow in the upper half plane, two

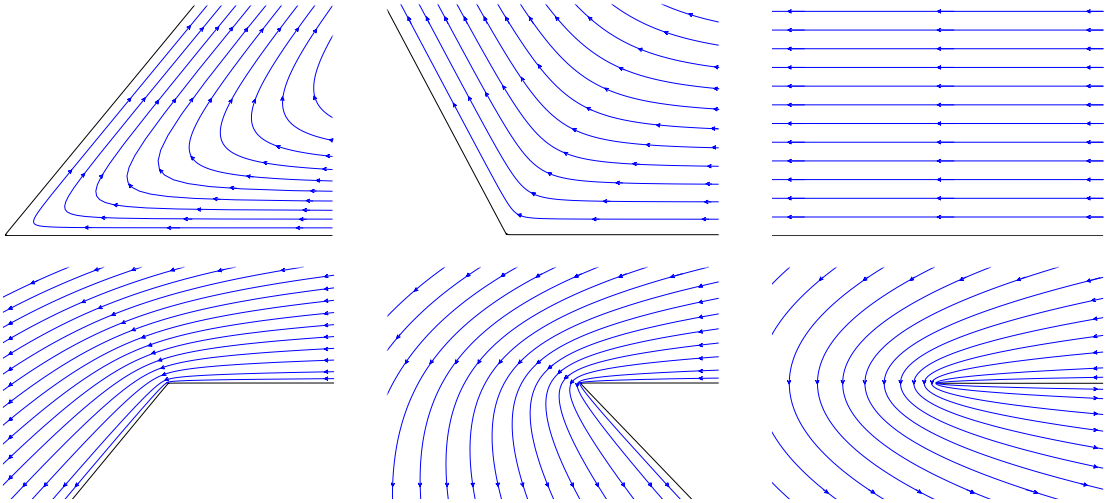


Figure IV.11 – Streamlines for the flow defined by potential (IV.48a) with from top to bottom and from left to right $n = 3$, $\frac{3}{2}$, 1, $\frac{3}{4}$, $\frac{3}{5}$ and $\frac{1}{2}$.

flows past corners with inner angles $2\pi/3$ and $\pi/3$, and a flow past a flat plaque, corresponding respectively to $n = 3$, $\frac{3}{2}$, 1, $\frac{3}{4}$, $\frac{3}{5}$ and $\frac{1}{2}$.

IV.4.3 c Two-dimensional flows past a cylinder

Thanks to the linearity of the Laplace differential equations, one may add "elementary" solutions of the previous paragraph to obtain new solutions, which describe possible two-dimensional flows. We now present two examples, which represent flows coming from infinity, where they are uniform, and falling on a cylinder—either immobile or rotating around its axis.

Acyclic flow

Let us superpose the complex potentials for a uniform flow (IV.42) along the x-direction and a flow dipole (IV.45a) situated at the origin and making an angle $\alpha = \pi$ with the vector $\vec{e}_x$:

$$\phi(z) = -\mathsf{v}_\infty \left(z + \frac{R^2}{z} \right), \tag{IV.49a}$$

where the dipole strength μ is written as $R^2\mathsf{v}_\infty$. Adopting polar coordinates (r, θ), this ansatz leads to the velocity potential and stream function

$$\varphi(r,\theta) = -\mathsf{v}_\infty \left(r + \frac{R^2}{r} \right) \cos\theta, \quad \psi(r,\theta) = -\mathsf{v}_\infty \left(r - \frac{R^2}{r} \right) \sin\theta. \tag{IV.49b}$$

One sees that the circle $r = R$ is a line of constant ψ, i.e. a streamline. This means that the flow outside that circle is decoupled from that inside. In particular, one may assume that the space inside the circle is filled by a solid obstacle, a "cylinder",[(25)] without changing the flow characteristics on $\mathbb{R}^2$ deprived from the disk $r < R$. The presence of this obstacle has the further advantage that it "hides" the singularity of the potential or the resulting velocity at $z = 0$, by cleanly removing it from the domain over which the flow is defined. This is illustrated, together with the streamlines for this flow, in Fig. IV.12.

From the complex potential (IV.49a) follows at once the complex velocity

$$\mathsf{w}(z) = \mathsf{v}_\infty \left(1 - \frac{R^2}{z^2} \right), \tag{IV.50a}$$

[(25)] The denomination is motivated by the fact that even though the flow characteristics depend on two spatial coordinates only, the actual flow might take in place in a three-dimensional space, in which case the obstacle is an infinite circular cylinder.

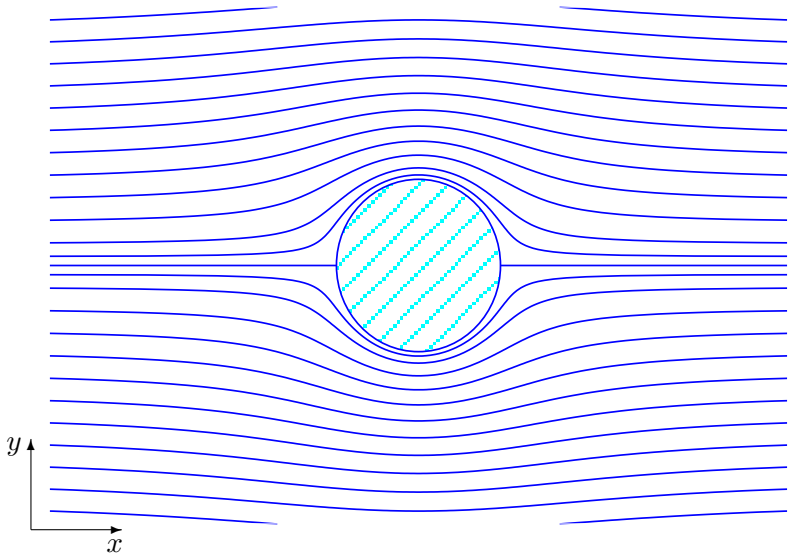


Figure IV.12 – Streamlines for the acyclic potential flow past a cylinder (IV.49a).

which in polar coordinates gives

$$\vec{\mathrm{v}}(r,\theta) = \mathrm{v}_\infty \left[\left(1 - \frac{R^2}{r^2}\right) \cos\theta\, \vec{\mathrm{e}}_r - \left(1 + \frac{R^2}{r^2}\right) \sin\theta\, \vec{\mathrm{e}}_\theta \right]. \tag{IV.50b}$$

The latter is purely tangential for $r = R$, in agreement with the fact that the cylinder surface is a streamline. The flow velocity even fully vanishes at the points with $r = R$ and $\theta = 0$ or π, which are thus *stagnation points*.[(l)]

Assuming that the motion is stationary, one can calculate the force exerted on the cylinder by the flowing fluid. Invoking the Bernoulli equation (IV.11)—which holds since the flow is steady and incompressible—and using the absence of vorticity, which leads to the constant being the same throughout the flow, one obtains

$$\mathcal{P}(\vec{r}) + \frac{1}{2}\rho \vec{\mathrm{v}}(\vec{r})^2 = \mathcal{P}_\infty + \frac{1}{2}\rho \mathrm{v}_\infty^2,$$

where $\mathcal{P}_\infty$ denotes the pressure at infinity. That is, at each point on the surface of the cylinder

$$\mathcal{P}(R,\theta) = \mathcal{P}_\infty + \frac{1}{2}\rho\big[\mathrm{v}_\infty^2 - \vec{\mathrm{v}}(R,\theta)^2\big] = \mathcal{P}_\infty + \frac{1}{2}\rho \mathrm{v}_\infty^2 \big(1 - 4\sin^2\theta\big),$$

where the second identity follows from Eq. (IV.50b). The resulting stress vector on the vector at a given θ is directed radially towards the cylinder center, $\vec{T}_s(R,\theta) = -\mathcal{P}(R,\theta)\, \vec{\mathrm{e}}_r(R,\theta)$. Integrating over $\theta \in [0, 2\pi]$, the total force on the cylinder due to the flowing fluid simply vanishes—in conflict with the intuition. This phenomenon is known as *d'Alembert paradox*.[(z)]

> The intuition according to which the moving fluid should exert a force on the immobile obstacle is good. What we find here is a failure of the perfect-fluid model, which is in that case too idealized, by allowing the fluid to slip without friction along the obstacle.

Cyclic flow

To the flow profile which was just considered, we add a pointlike vortex (IV.44a) situated at the origin

$$\phi(z) = -\mathrm{v}_\infty \left(z + \frac{R^2}{z} \right) + \frac{\mathrm{i}\Gamma}{2\pi} \log \frac{z}{R}, \tag{IV.51a}$$

[(l)] *Staupunkte*

[(z)] J. le Rond d'Alembert, 1717–1783

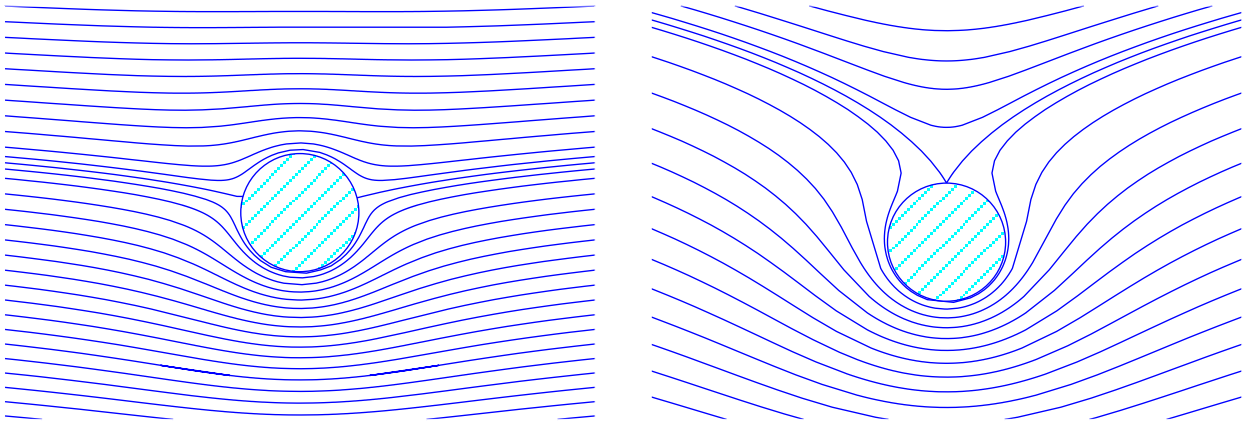


Figure IV.13 – Streamlines for the cyclic potential flow past a (rotating) cylinder (IV.51a) with $\Gamma/(4\pi R \mathsf{v}_\infty) = 0.25$ (left) or 1 (right).

where we have divided z by R in the logarithm to have a dimensionless argument, although this plays no role for the velocity. Comparing with the acyclic flow, which models fluid motion around a motionless cylinder, the complex potential may be seen as a model for the flow past a rotating cylinder, as in the case of the Magnus effect (§ IV.2.2 d).

Adopting polar coordinates (r, θ), the velocity potential and stream function read

$$\varphi(r,\theta) = -\mathsf{v}_\infty \left(r + \frac{R^2}{r} \right) \cos\theta - \frac{\Gamma}{2\pi}\,\theta, \quad \psi(r,\theta) = -\mathsf{v}_\infty \left(r - \frac{R^2}{r} \right) \sin\theta + \frac{\Gamma}{2\pi} \log\frac{r}{R}, \tag{IV.51b}$$

so that the circle $r = R$ remains a streamline, delimiting a fixed obstacle.

The resulting velocity field reads in complex form

$$\mathsf{w}(z) = \mathsf{v}_\infty \left(1 - \frac{R^2}{z^2} \right) - \frac{\mathrm{i}\Gamma}{2\pi z}, \tag{IV.52a}$$

and in polar coordinates

$$\vec{\mathsf{v}}(r,\theta) = \mathsf{v}_\infty \left[\left(1 - \frac{R^2}{r^2} \right) \cos\theta\, \vec{\mathrm{e}}_r - \left(1 + \frac{R^2}{r^2} - \frac{\Gamma}{2\pi r \mathsf{v}_\infty} \right) \sin\theta\, \vec{\mathrm{e}}_\theta \right]. \tag{IV.52b}$$

The latter is again purely tangential for $r = R$, in agreement with the fact that the cylinder surface is a streamline.

One easily checks that when the strength of the vortex is not too large, namely $\Gamma \leq 4\pi R \mathsf{v}_\infty$, the flow has stagnations points on the surface of the cylinder—two if the inequality holds in the strict sense, a single degenerate point if $\Gamma = 4\pi R\, \mathsf{v}_\infty$—, as illustrated in Fig. IV.13. If $\Gamma > 4\pi R \mathsf{v}_\infty$, the flow defined by the complex potential (IV.51a) still has a stagnation point, yet now away from the surface of the rotating cylinder, as exemplified in Fig. IV.14.

In either case, repeating the same calculation based on the Bernoulli equation as for the acyclic flow allows one to derive the force exerted by the fluid on the cylinder. The resulting force no longer vanishes, but equals $-\Gamma \rho \mathsf{v}_\infty\, \vec{\mathrm{e}}_y$ on a unit length of the cylinder, where ρ is the mass density of the fluid and $\vec{\mathrm{e}}_y$ the unit basis vector in the y-direction. This is in line with the arguments presented in § IV.2.2 d.

IV.4.3 d Conformal deformations of flows

A further possibility to build two-dimensional potential flows is to "distort" the elementary solutions of § IV.4.3 b, or linear combinations of these building blocks. Such deformations may however not be arbitrary, since they must preserve the orthogonality at each point in the fluid of

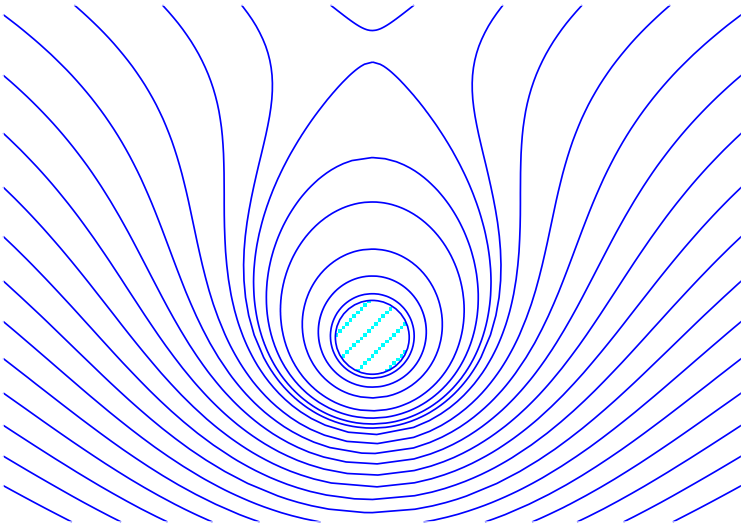


Figure IV.14 – Streamlines for the cyclic potential flow past a (rotating) cylinder (IV.51a) with $\Gamma/(4\pi R \mathsf{v}_\infty) = 4$.

the streamline (with constant ψ) and the equipotential line (constant φ) passing through that point. Besides rotations and dilations—which do not distort the profile of the solution, and are actually already taken into account in the solutions of § IV.4.3 b—, the generic class of transformations of the (complex) plane that preserve angles locally is that of *conformal maps*.

As recalled in Appendix B.4, such conformal mappings—between open subsets of the complex planes of variables z and Z—are defined by any holomorphic function $Z = f(z)$ whose derivative is everywhere non-zero and by its inverse F. If $\phi(z)$ denotes an arbitrary complex flow potential on the z-plane, then $\Phi(Z) \equiv \phi(F(Z))$ is a flow potential on the Z-plane. Applying the chain rule, the associated complex flow velocity is $\mathsf{w}(F(Z))F'(Z)$, where F' denotes the derivative of F.

A first example is to consider the trivial uniform flow with potential $\phi(z) = A\,z$, and the conformal mapping $z \mapsto Z = f(z) = z^{1/n}$ with $n \geq \frac{1}{2}$. The resulting complex flow potential on the Z-plane is $\Phi(Z) = -A\,Z^n$.

Except in the trivial case $n = 1$, $f(z)$ is singular at $z = 0$, where f' vanishes, so that the mapping is non-conformal: cutting a half-line ending at $z = 0$, f maps the complex plane deprived from this half-line onto an angular sector delimited by half-lines making an angle π/n—as already seen in § IV.4.3 b.

Joukowsky transform

A more interesting set of conformally deformed fluid flows consists of those provided by the use of the *Joukowsky transform*[(aa)]

$$Z = f(z) = z + \frac{R_\mathrm{J}^2}{z} \tag{IV.53}$$

where $R_\mathrm{J} \in \mathbb{R}$.

The mapping (IV.53) is obviously holomorphic in the whole complex z-plane deprived of the origin—which a single pole—, and has 2 points $z = \pm R_\mathrm{J}$ at which f' vanishes. These two singular points correspond in the Z-plane to algebraic branch points of the reciprocal function $z = F(Z)$ at $Z = \pm 2R_\mathrm{J}$. To remove them, one introduces a branch cut along the line segment $|X| \leq 2R_\mathrm{J}$. On the open domain $\mathcal{U}$ consisting of the complex Z-plane deprived from that line segment, F is holomorphic and conformal. One checks that the cut line segment is precisely the image by f of

[(aa)] Н. Е. Жуковский = N. E. Zhukovsky, 1847–1921

the circle $|z| = R_J$ in the complex z-plane. Thus, f and F provide a bijective mapping between the exterior of the circle $|z| = R_J$ in the z-plane and the domain $\mathcal{U}$ in the Z-plane.

Another property of the Joukowsky transform is that the singular points $z = \pm R_J$ are zeros of f' of order 1, so that angles are locally multiplied by 2. That is, every continuously differentiable curve going through $z = \pm R_J$ is mapped by f on a curve through $Z = \pm 2R_J$ with an angular point, i.e. a discontinuous derivative, there.

Consider first the circle $\mathcal{C}(0, R)$ in the z-plane of radius $R > R_J$ centered on the origin; it can be parameterized as

$$\mathcal{C}(0, R) = \{z = R\,e^{i\vartheta},\ 0 \le \vartheta \le 2\pi\}.$$

Its image in the Z-plane by the Joukowsky transform (IV.53) is the set of points such that

$$Z = \left(R + \frac{R_J^2}{R}\right)\cos\vartheta + i\left(R - \frac{R_J^2}{R}\right)\sin\vartheta,\ 0 \le \vartheta \le 2\pi,$$

that is, the ellipse centered on the origin $Z = 0$ with semi-major resp. semi-minor axis $R + R_J^2/R$ resp. $R - R_J^2/R$ along the X- resp. Y-direction. **figure needed?** Accordingly, the flows past a circular cylinder studied in § IV.4.3 c can be deformed by f into flows past elliptical cylinders, where the angle between the ellipse major axis and the flow velocity far from the cylinder may be chosen at will.

Bibliography for Chapter IV

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- Faber [1] Chapters 1.7, 2.8–2.9, 4.1–4.12;
- Feynman [10, 11] Chapter 40;
- Guyon *et al.* [2] Chapters 5.3–5.4, 6.1–6.3, 6.5–6.6 & 7-1–7.3;
- Landau–Lifshitz [4, 5] Chapter I § 3, 5, 8–11;
- Sommerfeld [7, 8] Chapters II § 6,7 and IV § 18,19.

CHAPTER V

Non-relativistic dissipative flows

The dynamics of Newtonian fluids is entirely governed by a relatively simple set of equations, namely the continuity equation (III.9), the Navier–Stokes equation (III.32), and—when phenomena related with temperature gradients become relevant—the energy conservation equation (III.37). As in the case of perfect fluids, there are a priori more unknown dynamical fields than equations, so that an additional relation has to be provided, either a kinematic constraint or an equation of state. In this chapter and the next two ones, a number of simple solutions of these equations are presented, together with big classes of phenomena that are accounted in various more or less simplified situations.

With the exception of the static-fluid case, in which the only novelty with respect to the hydrostatics of perfect fluids is precisely the possible transport of energy by heat conduction (§ V.1.1), the motions of interest in the present chapter are mostly laminar flows in which viscous effects play an important role while heat transport is negligible. Thus, the role of the no-slip condition at a boundary of the fluid is illustrated with a few chosen examples of stationary motions within idealized geometrical setups (Sec. V.1).

By introducing flow-specific characteristic length and velocity scales, the Navier–Stokes equation can be rewritten in a form involving only dimensionless variables and fields, together with parameters—like for instance the Reynolds number. These parameters quantify the relative importance of the several physical effects likely to play a role in a motion (Sec. V.2).

According to the value of the dimensionless numbers entering the dynamical equations, the latter may possibly be simplified. This leads to simpler equations with limited domain of validity, yet which become more easily tractable, as exemplified by the case of flows in which shear viscous

effects predominate over the influence of inertia (Sec. V.3). Another simplified set of equations can be derived to describe the fluid motion in the thin layer close to a boundary of the flow, in which the influence of this boundary plays a significant role (Sec V.4).

Eventually, the viscosity-induced modifications to the dynamical evolution of vorticity are discussed in Sec. V.5.

V.1 Statics and steady laminar flows of a Newtonian fluid

In this Section, we first write down the equations governing the statics of a Newtonian fluid (§ V.1.1), then we investigate a few idealized stationary laminar fluid motions, in which the velocity field is entirely driven by the no-slip condition at boundaries (§ V.1.2–V.1.4).

V.1.1 Static Newtonian fluid

Consider a motionless $[\vec{v}(t,\vec{r}) = \vec{0}]$ Newtonian fluid in an external gravitational potential $\Phi(\vec{r})$— or more generally, submitted to conservative volume forces such that $\vec{f}_V(t,\vec{r}) = -\rho(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r})$. The three coupled equations (III.9), (III.32) and (III.37) respectively simplify to

$$\frac{\partial \rho(t,\vec{r})}{\partial t} = 0, \tag{V.1a}$$

which means that the mass density $\rho(t,\vec{r})$ is time independent,

$$\vec{\nabla}\mathcal{P}(t,\vec{r}) = -\rho(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}), \tag{V.1b}$$

identical to the fundamental equation (IV.2) governing the hydrostatics of a perfect fluid, and

$$\frac{\partial e(t,\vec{r})}{\partial t} = \vec{\nabla}\cdot\big[\kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r})\big], \tag{V.1c}$$

which describes the transport of energy without macroscopic fluid motion, i.e. non-convectively, thanks to *heat conduction*.

Given an equation of state relating the internal energy density to the temperature, Eq. (V.1c) can become an equation for $T(t,\vec{r})$ only, in particular if the various thermodynamic and transport coefficients involved are assumed to be uniform across the fluid.

V.1.2 Plane Couette flow

In the example of this Section and the next two ones (§ V.1.3–V.1.4), we consider steady, incompressible, laminar flows, in absence of significant volume forces. Since the mass density ρ is fixed, thus known, only four equations are needed to determine the flow velocity $\vec{v}(\vec{r})$ and pressure $\mathcal{P}(\vec{r})$, the simplest possibility being to use the continuity and Navier–Stokes equations. In the stationary and incompressible regime, these become

$$\vec{\nabla}\cdot\vec{v}(\vec{r}) = 0 \tag{V.2a}$$

$$\big[\vec{v}(\vec{r})\cdot\vec{\nabla}\big]\vec{v}(\vec{r}) = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}(\vec{r}) + \nu\triangle\vec{v}(\vec{r}), \tag{V.2b}$$

with ν the kinematic shear viscosity, assumed to be the same throughout the fluid.

The so-called (plane) *Couette flow*[(ab)] is, in its idealized version, the motion of a viscous fluid between two infinitely extended plane plates, as represented in Fig. V.1, where the lower plate is at rest, while the upper one moves in its own plane with a constant velocity $\vec{u}$. It will be assumed that the same pressure $\mathcal{P}_\infty$ holds "at infinity" in any direction in the (x,z)-plane.

[(ab)] M. Couette, 1858–1943

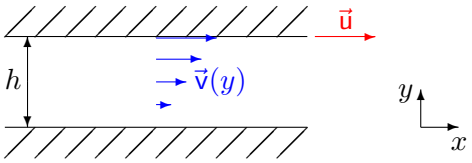

Figure V.1 – Setup of the plane Couette flow.

As the flow is assumed to be laminar, the geometry of the problem is invariant under arbitrary translations in the (x, z)-plane. This is automatically taken into account by the ansatz $\vec{\mathsf{v}}(\vec{r}) = \mathsf{v}(y)\,\vec{\mathrm{e}}_x$ for the flow velocity. Inserting this form in Eqs. (V.2) yields

$$\frac{\partial \mathsf{v}(y)}{\partial x} = 0, \tag{V.3a}$$

$$\mathsf{v}(y)\frac{\partial \mathsf{v}(y)}{\partial x}\,\vec{\mathrm{e}}_x = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}(\vec{r}) + \nu\frac{\mathrm{d}^2\mathsf{v}(y)}{\mathrm{d}y^2}\,\vec{\mathrm{e}}_x. \tag{V.3b}$$

With the ansatz for $\vec{\mathsf{v}}(\vec{r})$, the first equation is automatically fulfilled, while the term on the left hand side of the second equation vanishes. Projecting the latter on the y and z directions thus yields $\partial\mathcal{P}(\vec{r})/\partial y = 0$—expressing the assumed absence of sizable effects from gravity—and $\partial\mathcal{P}(\vec{r})/\partial z = 0$—since the problem is independent of z. Along the x direction, one finds

$$\frac{\partial \mathcal{P}(\vec{r})}{\partial x} = \eta\frac{\mathrm{d}^2\mathsf{v}(y)}{\mathrm{d}y^2}. \tag{V.4}$$

Since the right member of this equation is independent of x and z, a straightforward integration gives $\mathcal{P}(\vec{r}) = \alpha(y)x + \beta(y)$, where the functions α, β only depend on y. These functions are determined by the boundary conditions: from $\mathcal{P}(x{=}-\infty) = \mathcal{P}(x{=}\infty) = \mathcal{P}_\infty$ follow $\alpha(y) = 0$, $\beta(y) = \mathcal{P}_\infty$, and Eq. (V.4) eventually simplifies to

$$\frac{\mathrm{d}^2\mathsf{v}(y)}{\mathrm{d}y^2} = 0.$$

This yields $\mathsf{v}(y) = \gamma y + \delta$, with two integration constants γ and δ that are again fixed by the boundary conditions. At each plate, the relative velocity of the fluid with respect to the plate must vanish:

$$\mathsf{v}(y{=}0) = 0, \qquad \mathsf{v}(y{=}h) = |\vec{\mathsf{u}}|,$$

leading to $\delta = 0$ and $\gamma = |\vec{u}|/h$. All in all, the velocity thus depends linearly on y

$$\vec{\mathsf{v}}(\vec{r}) = \frac{y}{h}\,\vec{\mathsf{u}} \quad \text{for } 0 \le y \le h.$$

Consider now a surface element $\mathrm{d}^2\mathcal{S}$. The contact force $\mathrm{d}^2\vec{F}_s$ exerted on it by the fluid follows from the Cauchy stress tensor, whose Cartesian components (III.28c) here read

$$\sigma^{ij}(\vec{r}) = -\mathcal{P}(\vec{r})\delta^{ij} + \eta\left[\frac{\partial \mathsf{v}^i(\vec{r})}{\partial x_j} + \frac{\partial \mathsf{v}^j(\vec{r})}{\partial x_i}\right] \cong \begin{pmatrix} -\mathcal{P}_\infty & \eta\frac{|\vec{\mathsf{u}}|}{h} & 0 \\ \eta\frac{|\vec{\mathsf{u}}|}{h} & -\mathcal{P}_\infty & 0 \\ 0 & 0 & -\mathcal{P}_\infty \end{pmatrix}.$$

The force per unit surface on the motionless plate at $y = 0$, corresponding to a unit normal vector $\vec{\mathrm{e}}_n(\vec{r}) = \vec{\mathrm{e}}_y$, is

$$\frac{\mathrm{d}^2\vec{F}_s(\vec{r})}{\mathrm{d}^2\mathcal{S}} = \vec{T}_s(\vec{r}) = \left[\sum_{i,j=1}^{3}\sigma^{ij}(\vec{r})\,\vec{\mathrm{e}}_i \otimes \vec{\mathrm{e}}_j\right]\cdot\vec{\mathrm{e}}_y = \sum_{i,j=1}^{3}\sigma^{ij}(\vec{r})\big(\vec{\mathrm{e}}_j\cdot\vec{\mathrm{e}}_y\big)\vec{\mathrm{e}}_i = \begin{pmatrix} \eta\frac{|\vec{\mathsf{u}}|}{h} \\ -\mathcal{P}_\infty \\ 0 \end{pmatrix}.$$

Due to the friction exerted by the fluid, the lower plate is dragged by the flow in the (positive) x direction.

Remark: The tangential stress on the lower plate is $\eta\,\vec{\mathsf{u}}/h$, proportional to the shear viscosity: measuring the tangential stress with known $|\vec{\mathsf{u}}|$ and h yields a measurement of η. In practice, this measurement rather involves the more realistic cylindrical analog to the above plane flow, the so-called *Couette–Taylor flow*.[(ac)]

V.1.3 Plane Poiseuille flow

Let us now consider the flow of a Newtonian fluid between two motionless plane plates with a finite length along the x direction—yet still infinitely extended along the z direction—, as illustrated in Fig. V.2. The pressure is assumed to be different at both ends of the plates in the x direction, leading to the presence of a pressure gradient along x.

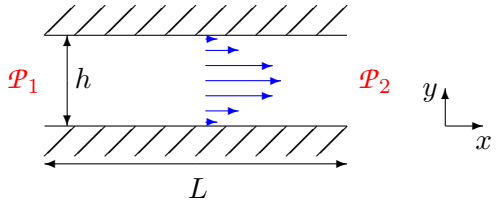

Figure V.2 – Flow between two motionless plates for $\mathcal{P}_1 > \mathcal{P}_2$, i.e. $\delta\mathcal{P} > 0$.

Assuming for the flow velocity $\vec{\mathsf{v}}(\vec{r})$ the same form $\mathsf{v}(y)\,\vec{e}_x$, independent of x, as in the case of the plane Couette flow, the equations of motion governing $\mathsf{v}(y)$ and pressure $\mathcal{P}(\vec{r})$ are the same as in the previous § V.1.2, namely Eqs. (V.3)–(V.4). The boundary conditions are however different. Thus, $\mathcal{P}_1 \neq \mathcal{P}_2$ results in a finite constant pressure gradient along x, $\alpha = \partial\mathcal{P}(\vec{r})/\partial x = -\delta\mathcal{P}/L \neq 0$, with $\delta\mathcal{P} \equiv \mathcal{P}_1 - \mathcal{P}_2$ the pressure drop. Equation (V.4) then leads to

$$\mathsf{v}(y) = -\frac{1}{2\eta}\frac{\delta\mathcal{P}}{L}y^2 + \gamma y + \delta,$$

with γ and δ two new constants.

The "no-slip" boundary conditions for the velocity at the two plates read

$$\mathsf{v}(y{=}0) = 0, \qquad \mathsf{v}(y{=}h) = 0,$$

which leads to $\delta = 0$ and $\gamma = \dfrac{1}{2\eta}\dfrac{\delta\mathcal{P}}{L}h$. The flow velocity thus has the parabolic profile

$$\mathsf{v}(y) = \frac{1}{2\eta}\frac{\delta\mathcal{P}}{L}\big[y(h-y)\big] \quad \text{for } 0 \leq y \leq h, \tag{V.5}$$

directed along the direction of the pressure gradient.

Remark: The flow velocity (V.5) becomes clearly problematic in the limit $\eta \to 0$! Tracing the problem back to its source, the equations of motion (V.3) cannot hold with a finite pressure gradient along the x direction and a vanishing viscosity. One quickly checks that the only possibility in the case of a perfect fluid is to drop one of the assumptions, either incompressibility or laminarity.

V.1.4 Hagen–Poiseuille flow

The previous two examples involved plates with an infinite length in at least one direction, thus were idealized constructions. In contrast, an experimentally realizable fluid motion is that of the *Hagen–Poiseuille flow*,[(ad)] in which a Newtonian fluid flows under the influence of a pressure gradient in a cylindrical tube with finite length L and radius a (Fig. V.3). Again, the motion is assumed to be steady, incompressible and laminar.

[(ac)] G. I. Taylor, 1886–1975 [(ad)] G. Hagen, 1797–1884

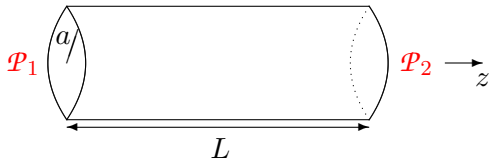

Figure V.3 – Setup of the Hagen–Poiseuille flow.

Using cylindrical coordinates, the ansatz $\vec{\mathsf{v}}(\vec{r}) = \mathsf{v}(r)\,\vec{\mathrm{e}}_z$ with $r = \sqrt{x^2+y^2}$ satisfies the continuity equation $\vec{\nabla}\cdot\vec{\mathsf{v}}(\vec{r}) = 0$ and gives for the incompressible Navier–Stokes equation

$$\vec{\nabla}\mathcal{P}(\vec{r}) = \eta\triangle\vec{\mathsf{v}}(\vec{r}) \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \dfrac{\partial\mathcal{P}(\vec{r})}{\partial x} = \dfrac{\partial\mathcal{P}(\vec{r})}{\partial y} = 0 \\ \dfrac{\partial\mathcal{P}(\vec{r})}{\partial z} = \eta\left[\dfrac{\partial^2\mathsf{v}(r)}{\partial x^2} + \dfrac{\partial^2\mathsf{v}(r))}{\partial y^2}\right] = \eta\left[\dfrac{\mathrm{d}^2\mathsf{v}(r)}{\mathrm{d}r^2} + \dfrac{1}{r}\dfrac{\mathrm{d}\mathsf{v}(r)}{\mathrm{d}r}\right]. \end{array} \right. \tag{V.6}$$

The right member of the equation in the second line is independent of z, implying that the pressure gradient along the z direction is constant. Using the boundary conditions yields

$$\frac{\partial\mathcal{P}(\vec{r})}{\partial z} = -\frac{\delta\mathcal{P}}{L},$$

with $\delta\mathcal{P} \equiv \mathcal{P}_1 - \mathcal{P}_2$. The z component of the Navier–Stokes equation (V.6) thus becomes

$$\frac{\mathrm{d}^2\mathsf{v}(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\mathsf{v}}{\mathrm{d}r} = -\frac{\delta\mathcal{P}}{\eta L}. \tag{V.7}$$

As always, this linear differential equation is solved in two successive steps, starting with the associated homogeneous equation. To find the general solution of the latter, one may introduce $\chi(r) \equiv \mathrm{d}\mathsf{v}(r)/\mathrm{d}r$, which obeys the simpler equation

$$\frac{\mathrm{d}\chi(r)}{\mathrm{d}r} + \frac{\chi(r)}{r} = 0.$$

The generic solution is $\ln\chi(r) = -\ln r + \text{const.}$, i.e. $\chi(r) = A/r$ with A a constant. This then leads to $\mathsf{v}(r) = A\ln r + B$ with B an additional constant.

A particular solution of the inhomogeneous equation (V.7) is $\mathsf{v}(r) = Cr^2$ with $C = -\delta\mathcal{P}/4\eta L$. The general solution of Eq. (V.7) is then given by

$$\mathsf{v}(r) = A\ln r + B - \frac{\delta\mathcal{P}}{4\eta L}r^2,$$

where the two integration constants still need to be determined.

To have a regular flow velocity at $r = 0$, the constant A should vanish. In turn, the boundary condition at the tube wall, $\mathsf{v}(r{=}a) = 0$, determines the value of the constant $B = (\delta\mathcal{P}/4\eta L)a^2$. All in all, the velocity profile thus reads

$$\mathsf{v}(r) = \frac{\delta\mathcal{P}}{4\eta L}\left(a^2 - r^2\right) \quad \text{for } r \le a. \tag{V.8}$$

This is again parabolic, with $\vec{\mathsf{v}}$ pointing in the same direction as the pressure drop.

The mass flow rate across the tube cross section follows from a straightforward integration:

$$Q = \int_0^a \rho\mathsf{v}(r)\,2\pi r\,\mathrm{d}r = 2\pi\rho\frac{\delta\mathcal{P}}{4\eta L}\int_0^a \left(a^2 r - r^3\right)\mathrm{d}r = 2\pi\rho\frac{\delta\mathcal{P}}{4\eta L}\frac{a^4}{4} = \frac{\pi\rho a^4}{8\eta}\frac{\delta\mathcal{P}}{L}. \tag{V.9}$$

This result, known as *Hagen–Poiseuille law* (or equation), shows that the mass flow rate is proportional to the pressure drop per unit length.

Remarks:

∗ The Hagen–Poiseuille law only holds under the assumption that the flow velocity vanishes at the tube walls. The experimental confirmation of the law—which was actually deduced from experiment

by Hagen (1839) and Poiseuille (1840)—is thus a proof of the validity of the no-slip assumption for the boundary condition.

∗ The mass flow rate across the tube cross section may be used to define the average flow velocity such that $Q = \pi a^2 \rho \langle \mathsf{v} \rangle$ with

$$\langle \mathsf{v} \rangle \equiv \frac{1}{\pi a^2} \int_0^a \mathsf{v}(r)\, 2\pi r \, \mathrm{d}r = \frac{1}{2} \mathsf{v}(r=0).$$

The Hagen–Poiseuille law then expresses a proportionality between the pressure drop per unit length and $\langle \mathsf{v} \rangle$ in a laminar flow.

> Viewing $\delta\mathcal{P}/L$ as the "generalized force" driving the motion, the corresponding "response" $\langle \mathsf{v} \rangle$ of the fluid is thus linear.

The relation is quite different in the case of a *turbulent* flow with the same geometry: for instance, measurements by Reynolds [21] gave $\delta\mathcal{P}/L \propto \langle \mathsf{v} \rangle^{1.722}$.

V.2 Dynamic similarity

The incompressible motion of a Newtonian fluid is governed by the kinetic condition $\vec{\nabla} \cdot \vec{\mathsf{v}}(t, \vec{r}) = 0$, the continuity equation (III.9), and the incompressible Navier–Stokes equation (III.33). In order to determine the relative influence of the various terms of the latter, it is often convenient to consider dimensionless forms of the equation, which leads to the introduction of a variety of dimensionless numbers.

For instance, the influence of the fluid mass density ρ and shear viscosity η, which are uniform throughout the fluid, on a flow in the absence of volume forces is entirely encoded in the Reynolds number (§ V.2.1). Allowing for volume forces, either due to gravity or to inertial forces, their relative importance is controlled by similar dimensionless parameters (§ V.2.2).

Let L_c resp. v_c be a characteristic length resp. velocity scale for a given flow. Since the Navier–Stokes equation itself does not involve any parameter with the dimension of a length or a velocity, both scales are controlled by "geometry", i.e. by the boundary conditions for the specific problem under consideration. Thus, L_c may be the size (diameter, side length) of a tube in which the fluid flows or of an obstacle around which the fluid moves. In turn, v_c may be the uniform velocity far from such an obstacle.

With the help of L_c and v_c, one can rescale the physical quantities in the problem, so as to obtain dimensionless quantities, which will hereafter be denoted with *:

$$\vec{r}^* \equiv \frac{\vec{r}}{L_c}, \quad \vec{\mathsf{v}}^* \equiv \frac{\vec{\mathsf{v}}}{\mathsf{v}_c}, \quad t^* \equiv \frac{t}{L_c/\mathsf{v}_c}, \quad \mathcal{P}^* \equiv \frac{\mathcal{P} - \mathcal{P}_0}{\rho \mathsf{v}_c^2}, \tag{V.10}$$

where $\mathcal{P}_0$ is some characteristic value of the (unscaled) pressure.

V.2.1 Reynolds number

Consider first the incompressible Navier–Stokes equation in the absence of external volume forces. Rewriting it in terms of the dimensionless variables and fields (V.10) yields

$$\frac{\partial \vec{\mathsf{v}}^*(t^*, \vec{r}^*)}{\partial t^*} + \left[\vec{\mathsf{v}}^*(t^*, \vec{r}^*) \cdot \vec{\nabla}^*\right] \vec{\mathsf{v}}^*(t^*, \vec{r}^*) = -\vec{\nabla}^* \mathcal{P}^*(t^*, \vec{r}^*) + \frac{\eta}{\rho \mathsf{v}_c L_c} \triangle^* \vec{\mathsf{v}}^*(t^*, \vec{r}^*), \tag{V.11}$$

with $\vec{\nabla}^*$ resp. $\triangle^*$ the gradient resp. Laplacian with respect to the reduced position variable $\vec{r}^*$. Besides the reduced variables and fields, this equation involves a single dimensionless parameter, the inverse of the *Reynolds number*

$$\boxed{\mathrm{Re} \equiv \frac{\rho \mathsf{v}_c L_c}{\eta} = \frac{\mathsf{v}_c L_c}{\nu}.} \tag{V.12}$$

This number measures the relative importance of inertia and viscous friction forces on a fluid element or a body immersed in the moving fluid: at large resp. small Re, viscous effects are negligible resp. predominant.

Remark: As stated above Eq. (V.10), both L_c and v_c are controlled by the geometry and boundary conditions. The Reynolds number—and every similar dimensionless we shall introduce hereafter—is thus a characteristic of a given flow, not of the fluid.

Law of similitude[(li)]

The solutions for the dynamical fields $\vec{\mathsf{v}}^*$, $\mathcal{P}^*$ at fixed boundary conditions and geometry—specified in terms of dimensionless ratios of geometrical lengths—are functions of the independent variables t^*, $\vec{r}^*$, and of the Reynolds number:

$$\vec{\mathsf{v}}^*(t^*,\vec{r}^*) = \vec{\mathsf{f}}_1^*(t^*,\vec{r}^*,\mathrm{Re}), \qquad \mathcal{P}^*(t^*,\vec{r}^*) = \mathsf{f}_2^*(t^*,\vec{r}^*,\mathrm{Re}), \tag{V.13}$$

with $\vec{\mathsf{f}}_1^*$ resp. f_2^* a vector resp. scalar function. The "physical" flow velocity and pressure fields are then given by

$$\vec{\mathsf{v}}(t,\vec{r}) = \mathsf{v}_c\vec{\mathsf{f}}_1^*\left(\frac{\mathsf{v}_c t}{L_c},\frac{\vec{r}}{L_c},\mathrm{Re}\right), \qquad \mathcal{P}(t,\vec{r}) = \mathcal{P}_0 + \rho\mathsf{v}_c^2\mathsf{f}_2^*\left(\frac{\mathsf{v}_c t}{L_c},\frac{\vec{r}}{L_c},\mathrm{Re}\right).$$

These equations underlie the use of fluid dynamical simulations with experimental models at a reduced scale, yet possessing the same (rescaled) geometry. Let L_c, v_c resp. L_M, v_M be the characteristic lengths and velocities of the real-size flow resp. of the reduced-scale experimental flow; for simplicity, we assume that the same fluid is used in both cases. If $\mathsf{v}_\mathrm{M}/\mathsf{v}_c = L_c/L_\mathrm{M}$, the Reynolds number for the experimental model is the same as for the real-size fluid motion: both flows then admit the same solutions $\vec{\mathsf{v}}^*$ and $\mathcal{P}^*$, and are said to be *dynamically similar*.

Remark: The functional relationships between the "dependent variables" $\vec{\mathsf{v}}^*$, $\mathcal{P}^*$ and the "independent variables" t^*, $\vec{r}^*$ and a dimensionless parameter (Re) represent a simple example of the more general (Vaschy[(ae)]–)Buckingham[(af)] π-theorem [22] in *dimensional analysis*, see Appendix C.

V.2.2 Other dimensionless numbers

If the fluid motion is likely to be influenced by gravity, the corresponding volume force density $\vec{f}_V = -\rho\vec{g}$ (for a uniform gravity field) must be taken into account in the right member of the incompressible Navier–Stokes equation (III.33). Accordingly, if the latter is written in dimensionless form as in the previous paragraph, there is an additional term on the right hand side of Eq. (V.11), proportional to $1/\mathrm{Fr}^2$, with

$$\mathrm{Fr} \equiv \frac{\mathsf{v}_c}{\sqrt{gL_c}} \tag{V.14}$$

the *Froude number*.[(ag)] This dimensionless parameter measures the relative size of inertial and gravitational effects in the flow, the latter being important when Fr is small.

In the presence of gravity, the dimensionless dynamical fields $\vec{\mathsf{v}}^*$, $\mathcal{P}^*$ become functions of the reduced variables t^*, $\vec{r}^*$ controlled by both parameters Re and Fr.

The Navier–Stokes equation (III.32) holds in an inertial frame. In a non-inertial reference frame, there come additional terms, which may be expressed as fictive force densities on the right hand side, and come in addition to the "physical" volume force density $\vec{f}_V$. In the case of a reference frame in uniform rotation (with respect to an inertial frame) with angular velocity $\vec{\Omega}_0$, there are thus two extra contributions corresponding to centrifugal and Coriolis forces, namely $\vec{f}_\mathrm{cent.} = -\rho\vec{\nabla}\left[-\frac{1}{2}\left(\vec{\Omega}_0\times\vec{r}\right)^2\right]$ and $\vec{f}_\mathrm{Cor.} = -2\rho\vec{\Omega}_0\times\vec{\mathsf{v}}$, respectively.

[(li)] *Ähnlichkeitsgesetz*

[(ae)] A. Vaschy, 1857–1899 [(af)] E. Buckingham, 1867–1940 [(ag)] W. Froude, 1810–1879

The relative importance of the latter in a given flow can be estimated with dimensionless numbers. Thus, denoting $\Omega_0 \equiv |\vec{\Omega}_0|$, the *Ekman number*[(ah)]

$$\mathrm{Ek} \equiv \frac{\eta}{\rho \Omega_0 L_c^2} = \frac{\nu}{\Omega_0 L_c^2} \tag{V.15}$$

measures the relative size of (shear) viscous and Coriolis forces, with the latter predominating over the former when $\mathrm{Ek} \ll 1$.

One may also wish to compare the influences of the convective and Coriolis terms in the Navier–Stokes equation. This is done with the help of the *Rossby number*[(ai)]

$$\mathrm{Ro} \equiv \frac{\mathsf{v}_c}{\Omega_0 L_c} \tag{V.16}$$

which is small when the effect of the Coriolis force is the dominant one.

Remark: Quite obviously, the Reynolds (V.12), Ekman (V.15), and Rossby (V.16) numbers obey the simple identity

$$\mathrm{Ro} = \mathrm{Re} \cdot \mathrm{Ek}.$$

V.3 Flows at small Reynolds number

This Section deals with incompressible fluid motions at small Reynolds number $\mathrm{Re} \ll 1$, i.e. flows in the regime in which shear viscous effects predominate over those of inertia in the Navier–Stokes equation. Such fluid motions are also referred to as *Stokes flows* or *creeping flows*.[(lii)]

V.3.1 Physical relevance. Equations of motion

Flows of very different nature may exhibit a small Reynolds number (V.12), because the latter combines physical quantities whose value can vary by many orders of magnitude in Nature.[(26)] A few examples of creeping flows are listed hereafter:

- The motion of fluids past microscopic bodies; the small value of the Reynolds number then reflects the smallness of the length scale L_c; for instance:
  - In water ($\eta \approx 10^{-3}$ Pa·s i.e. $\nu \approx 10^{-6}$ m$^2\cdot$s^{-1}), a bacteria of size $L_c \approx 5$ μm "swims" with velocity $\mathsf{v}_c \approx 10$ μm$\cdot$s^{-1}, so that $\mathrm{Re} \approx 5 \cdot 10^{-5}$ for the motion of the water past the bacteria: if the bacteria stops propelling itself, the friction exerted by the water brings it immediately to rest.[(27)] Similarly, creeping flows are employed to describe the motion of reptiles in sand—or more precisely, the flow of sand a past an undulating reptile [24].
  - The motion of a fluid past a suspension of small size (Brownian) particles. This will be studied at further length in § V.3.2.
- The slow-velocity motion of geological material: in that case, the small value of v_c and the large shear viscosity compensate the possibly large value of the typical length scale L_c.

[(26)] This is mostly true of the characteristic length and velocity scales and of the shear viscosity; in (non-relativistic) fluids, the mass density is always of the same order of magnitude, up to a factor 10^3.

[(27)] A longer discussion of the motion of bacteria—from a physicist's point of view—, together with the original formulation of the "scallop theorem", can be found in Ref. [23].

[(lii)] *schleichende Strömungen*

[(ah)] V. Ekman, 1874–1954 [(ai)] C.-G. Rossby, 1898–1957

For example, the motion of the Earth's mantle[28] with $L_c \approx 100$ km, $\mathsf{v}_c \approx 10^{-5}\ \mathrm{m\cdot s^{-1}}$, $\rho \approx 5\cdot 10^3\ \mathrm{kg\cdot m^{-3}}$ and $\eta \approx 10^{22}\ \mathrm{Pa\cdot s}$ corresponds to a Reynolds number $\mathrm{Re} \approx 5\cdot 10^{-19}$.

Note that the above examples all represent incompressible flows. For the sake of simplicity, we shall also only consider steady motions.

V.3.1 a Stokes equation

Physically, a small Reynolds number means that the influence of inertia is much smaller than that of shear viscosity. That is, the convective term $(\vec{\mathsf{v}}\cdot\vec{\nabla})\vec{\mathsf{v}}$ in the Navier–Stokes equation is negligible with respect to the viscous contribution. Assuming additionally stationarity—which allows us to drop the time variable—and incompressibility, the Navier–Stokes equation (III.32) simplifies to the *Stokes equation*

$$\boxed{\vec{\nabla}\mathcal{P}(\vec{r}) = \eta\triangle\vec{\mathsf{v}}(\vec{r}) + \vec{f}_V(\vec{r}).} \tag{V.17}$$

This constitutes a linearization of the incompressible Navier–Stokes equation.

Using the relation

$$\vec{\nabla}\times\big[\vec{\nabla}\times\vec{c}(\vec{r})\big] = \vec{\nabla}\big[\vec{\nabla}\cdot\vec{c}(\vec{r})\big] - \triangle\vec{c}(\vec{r}) \tag{V.18}$$

valid for any vector field $\vec{c}(\vec{r})$, the incompressibility condition, and the definition of vorticity, the Stokes equation (in the absence of external volume forces) can be rewritten as

$$\vec{\nabla}\mathcal{P}(\vec{r}) = -\eta\vec{\nabla}\times\vec{\omega}(\vec{r}). \tag{V.19}$$

As a result, the pressure satisfies the differential Laplace equation

$$\triangle\mathcal{P}(\vec{r}) = 0. \tag{V.20}$$

In practice, however, this equation is not the most useful one, because the boundary conditions in a flow are mostly given in terms of the flow velocity, in particular at walls or obstacles, not of the pressure.

Taking the curl of Eq. (V.19) and invoking again relation (V.18) remembering that the vorticity vector is itself already a curl, one finds

$$\triangle\vec{\omega}(\vec{r}) = \vec{0}, \tag{V.21}$$

i.e. the vorticity also obeys the Laplace equation. We shall see in Sec. V.5 that the more general dynamical equation governing the dynamics of vorticity in Newtonian fluids indeed yields Eq. (V.21) in the case of stationary flows at small Reynolds number.

V.3.1 b Properties of the solutions of the Stokes equation

Thanks to the linearity of the Stokes equation (V.17), its solutions possess various properties:[29]

- Uniqueness of the solution at fixed boundary conditions.
- Additivity of the solutions: if $\vec{\mathsf{v}}_1$ and $\vec{\mathsf{v}}_2$ are solutions of Eq. (V.17) with respective boundary conditions, then the sum $\lambda_1\vec{\mathsf{v}}_1 + \lambda_2\vec{\mathsf{v}}_2$ with real numbers λ_1, λ_2 is also a solution, for a problem with adequate boundary conditions.
  Physically, the multiplying factors should not be too large, to ensure that the Reynolds number of the new problem remains small. The multiplication of the velocity field $\vec{\mathsf{v}}(\vec{r})$ by a constant λ represents a change in the mass flow rate, while the streamlines (I.15) remain unchanged.

[28] From the mass density, the shear viscosity and the typical speed of sound $c_s \approx 5000\ \mathrm{m\cdot s^{-1}}$ of *transverse* waves—i.e. shear waves, that may propagate in a solid, but not in a fluid—, one constructs a characteristic time scale $t_{\mathrm{mantle}} = \eta/\rho c_s^2 \approx 3000$ years. For motions with a typical duration $t_c \ll t_{\mathrm{mantle}}$, the Earth's mantle behaves like a deformable solid: for instance, with respect to the propagation of sound waves following an earthquake. On the other hand, for motions on a "geological" time scale $t_c \gg t_{\mathrm{mantle}}$, the mantle may be modeled as a fluid.

[29] Proofs can be found e.g. in Ref. [2, Chapter 8.2.3].

The dimensionless velocity field $\vec{\mathsf{v}}^*$ associated with the two solutions $\vec{\mathsf{v}}(\vec{r})$ and $\lambda\vec{\mathsf{v}}(\vec{r})$ is the same, provided the differing characteristic velocities v_c resp. $\lambda\mathsf{v}_c$ are used. In turn, these define different values of the Reynolds number. For these solutions, $\vec{\mathsf{v}}^*$ as given by Eq. (V.13) is thus independent of the parameter Re, and thereby only depends on the variable $\vec{r}^*$: $\vec{\mathsf{v}} = \mathsf{v}_c\vec{f}\big(\vec{r}/L_c\big)$. This also holds for the corresponding dimensionless pressure $\mathcal{P}^*$.

Using dimensional arguments only, the tangential stress is $\eta\partial\mathsf{v}_i/\partial x_j \sim \eta\mathsf{v}_c/L_c$, so that the friction force on an object of typical linear size[30] L_c is proportional to $\eta\mathsf{v}_c L_c$. This result will now be illustrated on an explicit example [cf. Eq. (V.26)], for which the computation can be performed analytically.

V.3.2 Stokes flow past a sphere

Consider a sphere with radius R immersed in a fluid with mass density ρ and shear viscosity η, which far from the sphere flows with uniform velocity $\vec{\mathsf{v}}_\infty$, as sketched in Fig. V.4. The goal is to determine the force exerted by the moving fluid on the sphere, which necessitates the calculation of the pressure and flow velocity. Given the geometry of the problem, a system of spherical coordinates (r, θ, φ) centered on the sphere center will be used.

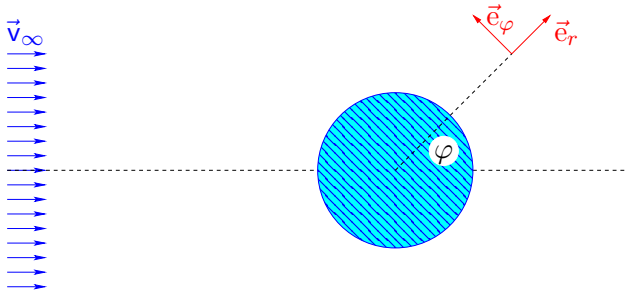

Figure V.4 – Stokes flow past a sphere.

The Reynolds number $\mathrm{Re} = \rho|\vec{\mathsf{v}}_\infty|R/\eta$ is assumed to be small, so that the motion in the vicinity of the sphere can be modeled as a creeping flow, which is further taken to be incompressible. For the flow velocity, one looks for a stationary solution of the equations of motion of the form $\vec{\mathsf{v}}(\vec{r}) = \vec{\mathsf{v}}_\infty + \vec{\mathsf{u}}(\vec{r})$, with the boundary condition $\vec{\mathsf{u}}(\vec{r}) \to \vec{0}$ when $|\vec{r}| \to \infty$. In addition, the usual impermeability and no-slip conditions hold at the surface of the sphere, resulting in the requirement $\vec{\mathsf{u}}(|\vec{r}|=R) = -\vec{\mathsf{v}}_\infty$.

Using the linearity of the equations of motion for creeping flows, $\vec{\mathsf{u}}$ obeys the equations

$$\triangle\big[\vec{\nabla}\times\vec{\mathsf{u}}(\vec{r})\big] = \vec{0}, \tag{V.22a}$$

which reflects Eq. (V.21), and

$$\vec{\nabla}\cdot\vec{\mathsf{u}}(\vec{r}) = 0, \tag{V.22b}$$

which comes from the incompressibility condition.

The latter equation is automatically satisfied if $\vec{\mathsf{u}}(\vec{r})$ is the curl of some vector field $\vec{V}(\vec{r})$. Using dimensional considerations, the latter should depend linearly on the only explicit velocity scale in the problem, namely $\vec{\mathsf{v}}_\infty$. Accordingly, one makes the ansatz[31]

$$\vec{V}(\vec{r}) = \vec{\nabla}\times\big[f(r)\,\vec{\mathsf{v}}_\infty\big] = \vec{\nabla}f(r)\times\vec{\mathsf{v}}_\infty,$$

[30] As noted in the introduction to Sec. V.2, the characteristic length and velocity scales in a flow are precisely determined by the boundary conditions.

[31] The simpler guesses $\vec{\mathsf{u}}(\vec{r}) = f(r)\,\vec{\mathsf{v}}_\infty$ or $\vec{\mathsf{u}}(\vec{r}) = \vec{\nabla}f(r)\times\vec{\mathsf{v}}_\infty$ are both unsatisfactory: the velocity $\vec{\mathsf{u}}(\vec{r})$ is then always parallel resp. orthogonal to $\vec{\mathsf{v}}_\infty$, so that $\vec{\mathsf{v}}(\vec{r})$ cannot vanish everywhere at the surface of the sphere.

with $f(r)$ a function of $r = |\vec{r}|$, i.e. f only depends on the distance from the sphere: apart from the direction of $\vec{\mathrm{v}}_\infty$, which is already accounted for in the ansatz, there is no further preferred spatial direction, so that f should be spherically symmetric.

Relation (V.18) together with the identity $\vec{\nabla} \cdot [f(r)\,\vec{\mathrm{v}}_\infty] = \vec{\nabla} f(r) \cdot \vec{\mathrm{v}}_\infty$ then yield

$$\vec{\mathrm{u}}(\vec{r}) = \vec{\nabla} \times \vec{V}(\vec{r}) = \vec{\nabla}\big[\vec{\nabla} f(r) \cdot \vec{\mathrm{v}}_\infty\big] - \triangle f(r)\,\vec{\mathrm{v}}_\infty. \tag{V.23}$$

The first term on the right hand side has a vanishing curl, and thus does not contribute when inserting $\vec{\mathrm{u}}(\vec{r})$ in equation (V.22a):

$$\vec{\nabla} \times \vec{\mathrm{u}}(\vec{r}) = -\vec{\nabla} \times \big[\triangle f(r)\,\vec{\mathrm{v}}_\infty\big] = -\vec{\nabla}\big[\triangle f(r)\big] \times \vec{\mathrm{v}}_\infty,$$

so that

$$\triangle\Big(\vec{\nabla}\big[\triangle f(r)\big]\Big) \times \vec{\mathrm{v}}_\infty = \vec{0}.$$

Since $f(r)$ does not depend on the azimuthal and polar angles, the vector $\triangle\big(\vec{\nabla}\big[\triangle f(r)\big]\big)$ at position $\vec{r}$ is directed along the radial direction; as thus, it cannot be everywhere parallel to $\vec{\mathrm{v}}_\infty$. Therefore, $\triangle\big(\vec{\nabla}\big[\triangle f(r)\big]\big)$ must vanish identically for the above equation to hold. One can check—for instance using components—the identity $\triangle\big(\vec{\nabla}\big[\triangle f(r)\big]\big) = \vec{\nabla}\big(\triangle[\triangle f(r)]\big)$, so that the equation obeyed by $f(r)$ becomes

$$\triangle[\triangle f(r)] = \text{const.}$$

The integration constant must be zero, since it is a fourth derivative of $f(r)$, while the velocity $\vec{\mathrm{u}}(\vec{r})$, which according to Eq. (V.23) depends on the second derivatives, must vanish as $r \to \infty$. One thus has

$$\triangle[\triangle f(r)] = 0.$$

In spherical coordinates, the Laplacian reads

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2},$$

with ℓ an integer that depends on the angular dependence of the function: given the spherical symmetry of the problem for f, one should take $\ell = 0$. Making the ansatz $\triangle f(r) = C/r^\alpha$, the equation $\triangle[\triangle f(r)] = 0$ is only satisfied for $\alpha = 0$ or 1. Using Eq. (V.23) and the condition $\vec{\mathrm{u}}(\vec{r}) \to \vec{0}$ for $r \to \infty$, only $\alpha = 1$ is possible.

The general solution of the linear ordinary differential equation

$$\triangle f(r) = \frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} = \frac{C}{r} \tag{V.24a}$$

is then given by

$$f(r) = A + \frac{B}{r} + \frac{C}{2}r, \tag{V.24b}$$

where the first two terms in the right member represent the most general of the associated homogeneous equation, while the third term is a particular solution of the inhomogeneous equation.

Equations (V.23) and (V.24) lead to the velocity field

$$\begin{aligned}\vec{\mathrm{u}}(\vec{r}) &= \vec{\nabla}\left[\left(-B\frac{\vec{r}}{r^3} + \frac{C}{2}\frac{\vec{r}}{r}\right) \cdot \vec{\mathrm{v}}_\infty\right] - \frac{C}{r}\,\vec{\mathrm{v}}_\infty = -B\frac{\vec{\mathrm{v}}_\infty - 3\big(\vec{\mathrm{e}}_r \cdot \vec{\mathrm{v}}_\infty\big)\,\vec{\mathrm{e}}_r}{r^3} + \frac{C}{2}\frac{\vec{\mathrm{v}}_\infty - \big(\vec{\mathrm{e}}_r \cdot \vec{\mathrm{v}}_\infty\big)\,\vec{\mathrm{e}}_r}{r} - \frac{C}{r}\,\vec{\mathrm{v}}_\infty \\ &= -B\frac{\vec{\mathrm{v}}_\infty - 3\big(\vec{\mathrm{e}}_r \cdot \vec{\mathrm{v}}_\infty\big)\,\vec{\mathrm{e}}_r}{r^3} - \frac{C}{2}\frac{\vec{\mathrm{v}}_\infty + \big(\vec{\mathrm{e}}_r \cdot \vec{\mathrm{v}}_\infty\big)\,\vec{\mathrm{e}}_r}{r},\end{aligned}$$

which vanishes at $|\vec{r}| \to \infty$, fulfilling one of the boundary conditions. The other boundary condition $\vec{\mathrm{u}}(|\vec{r}|{=}R) = -\vec{\mathrm{v}}_\infty$ at the surface of the sphere translates into

$$\left(1 - \frac{B}{R^3} - \frac{C}{2R}\right)\vec{\mathrm{v}}_\infty + \left(\frac{3B}{R^3} - \frac{C}{2R}\right)\big(\vec{\mathrm{e}}_r \cdot \vec{\mathrm{v}}_\infty\big)\,\vec{\mathrm{e}}_r = \vec{0}.$$

This identity must hold irrespective of the orientation of $\vec{e}_r$, which requires both prefactors of $\vec{v}_\infty$ and $\vec{e}_r$ identically vanish, leading to $B = R^3/4$ and $C = 6B/R^2 = 3R/2$ and thereby to

$$\vec{v}(\vec{r}) = \vec{v}_\infty - \frac{3R}{4r}\left[\vec{v}_\infty + \left(\vec{e}_r \cdot \vec{v}_\infty\right)\vec{e}_r\right] - \frac{R^3}{4r^3}\left[\vec{v}_\infty - 3\left(\vec{e}_r \cdot \vec{v}_\infty\right)\vec{e}_r\right]. \tag{V.25}$$

Inserting this flow velocity in the Stokes equation (V.17) gives the pressure

$$\mathcal{P}(\vec{r}) = \frac{3}{2}\eta R \frac{\vec{e}_r \cdot \vec{v}_\infty}{r^2} + \text{const},$$

where the unspecified constant is the value of the pressure at infinity, which may be given as an extra boundary condition.

Using $\mathcal{P}(\vec{r})$ and the derivatives)of the velocity field $\vec{v}(\vec{r})$, one can then compute the mechanical stress (III.29) at a point on the surface of the sphere. The total force exerted by the flow on the latter follows from integrating the mechanical stress over the whole surface, and equals

$$\vec{F} = 6\pi R\eta\, \vec{v}_\infty. \tag{V.26}$$

This result is referred as *Stokes' law*. Inverting the point of view, a sphere moving with velocity $\vec{v}_{\text{sphere}}$ in a fluid at rest undergoes a friction force $-6\pi R\eta \vec{v}_{\text{sphere}}$.

Remarks:

∗ For the potential flow of a perfect fluid past a sphere with radius R, the flow velocity is[(32)]

$$\vec{v}(\vec{r}) = \vec{v}_\infty + \frac{R^3}{2r^3}\left[\vec{v}_\infty - 3\left(\vec{e}_r \cdot \vec{v}_\infty\right)\vec{e}_r\right].$$

That is, the velocity varies much faster in the vicinity of the sphere than for the Stokes flow (V.25): in the latter case, momentum is transported not only convectively but also by viscosity, which redistributes it over a wider region.

The approximation of a flow at small Reynolds number, described by the Stokes equation, actually only holds in the vicinity of the sphere. Far from it, the flow is much less viscous.

∗ In the limit $\eta \to 0$, corresponding to a perfect fluid, the force (V.26) exerted by the flow on the sphere vanishes: this is again the *d'Alembert paradox* encountered in § IV.4.3 c.

∗ The proportionality factor between the sphere velocity and the friction force it experiences is called the *mobility*[(liii)] μ. According to Stokes' law (V.26), for a sphere in the creeping-flow regime one has $\mu = 1/(6\pi R\eta)$.
In his famous article on Brownian motion [25], A. Einstein related this mobility with the diffusion coefficient D of a suspension of small spheres in a fluid at rest:

$$D = \mu k_B T = \frac{k_B T}{6\pi R\eta}.$$

This formula (*Stokes–Einstein equation*) was checked experimentally by J. Perrin, which allowed the latter to determine a value of the Avogadro constant and to prove the "discontinuous structure of matter" [26].

[(32)] The proof can be found e.g. in Landau–Lifshitz [4, 5] § 10 problem 2.

[(liii)] *Beweglichkeit, Mobilität*

V.4 Boundary layer

The Reynolds number defined in § V.2.1, which quantifies the relative importances of viscous and inertial effects in a given flow, involves characteristic length and velocity scales L_c, v_c, that depend on the geometry of the fluid motion. When the flow involves an obstacle, as was the case in the example presented in § V.3.2, a natural choice when studying the details of the fluid motion in the vicinity of the obstacle is to adopt the typical size R of the latter as characteristic length L_c defining the Reynolds number.

Far from the obstacle, however, it is no longer obvious that R is really relevant. For L_c, a better choice might be the distance to the obstacle—or to any other wall or object present in the problem. Such a characteristic length gives a Reynolds number which can be orders of magnitude larger than the value computed with L_c. That is, even if the flow is viscous (small Re) close to the obstacle, far from it the motion could still be to a large extent inviscid (large Re) and thus well approximated by a perfect-fluid description.

The above argumentation suggests that viscous effects may not be relevant throughout the whole fluid, but only in the region(s) in the vicinity of walls or obstacles. This is indeed the case, and the corresponding region surrounding walls or obstacles is referred to as *boundary layer*.[liv] In the latter—which often turns out to be rather thin—, the velocity grows rapidly from its vanishing value at the surface of objects (no-slip condition) to the finite value it takes far from them and which is mostly imposed by the boundary conditions "at infinity".

In this Section, we shall first illustrate on an example flow the existence of the boundary layer, computing in particular its typical width (§ V.4.1). The latter can also be estimated in a more general approach to the description of the fluid motion inside the boundary layer, as sketched in § V.4.2.

V.4.1 Flow in the vicinity of a wall set impulsively in motion

Consider an incompressible Newtonian fluid with uniform kinematic shear viscosity ν situated in the upper half-space $y > 0$, which is at rest for $t < 0$. The volume forces acting on the fluid are supposed to be negligible.

At $t = 0$, the plane $y = 0$ is suddenly set in uniform motion parallel to itself, with constant velocity $\vec{\mathsf{u}}(t>0) = \mathsf{u}\vec{\mathrm{e}}_x$. As a consequence, the fluid in the vicinity of the plane is being dragged along; thanks to the viscous forces, the motion is transfered to the next fluid layers. The resulting flow is assumed to be laminar, with a fluid velocity everywhere parallel to $\vec{\mathrm{e}}_x$.

The invariance of the problem geometry under translations in the x- or z-directions justifies an ansatz $\vec{\mathsf{v}}(t,\vec{r}) = \mathsf{v}(t,y)\vec{\mathrm{e}}_x$—which automatically fulfills the incompressibility condition—, and similarly for the pressure field. That is, there are no gradients along the x- and z-directions. As a result, the component of the incompressible Navier–Stokes equation (III.33) along the x-direction reads

$$\frac{\partial \mathsf{v}(t,y)}{\partial t} = \nu \frac{\partial^2 \mathsf{v}(t,y)}{\partial y^2}. \tag{V.27a}$$

The boundary conditions to this equation are on the one hand the no-slip requirement at the moving plane, namely

$$\mathsf{v}(t, y{=}0) = \mathsf{u} \quad \text{for } t > 0; \tag{V.27b}$$

on the other hand, the fluid infinitely far from the moving plane remains unperturbed, i.e.

$$\lim_{y\to\infty} \mathsf{v}(t,y) = 0 \quad \text{for } t > 0. \tag{V.27c}$$

Eventually, there is the initial condition

$$\mathsf{v}(t{=}0, y) = 0 \quad \forall y > 0. \tag{V.27d}$$

[liv] *Grenzschicht*

The equations (V.27) governing the fluid motion involve only two dimensionful quantities, namely the plane velocity u and the fluid kinematic viscosity ν. With their help, one can construct a characteristic time ν/u^2 and a characteristic length ν/u in a unique manner, up to numerical factors. Invoking dimensional arguments, one thus sees that the fluid velocity is necessarily of the form

$$\mathsf{v}(t,y) = \mathsf{u}\mathsf{f}_1\left(\frac{\mathsf{u}^2 t}{\nu}, \frac{\mathsf{u}y}{\nu}\right),$$

with f_1 a dimensionless function of dimensionless variables. Since t and y are independent, so are their reduced versions $\mathsf{u}^2 t/\nu$ and $\mathsf{u}y/\nu$. Instead of the latter, one may adopt the equivalent set $\mathsf{u}^2 t/\nu$, $\xi \equiv y/(2\sqrt{\nu t})$, i.e. write

$$\mathsf{v}(t,y) = \mathsf{u}\mathsf{f}_2\left(\frac{\mathsf{u}^2 t}{\nu}, \frac{y}{2\sqrt{\nu t}}\right),$$

with f_2 again a dimensionless function.

The whole problem (V.27) is clearly linear in u, since the involved differential equations—continuity equation and Navier–Stokes equation (V.27a)—are linear; this allows us to exclude any dependence of f_2 on the variable $\mathsf{u}^2 t/\nu$, so that the solution is actually of the form

$$\mathsf{v}(t,y) = \mathsf{u}\mathsf{f}\left(\frac{y}{2\sqrt{\nu t}}\right) \tag{V.28}$$

with f dimensionless, and dependent on a single reduced variable.

Inserting the latter ansatz in Eq. (V.27a) leads after some straightforward algebra to the ordinary differential equation

$$\mathsf{f}''(\xi) + 2\xi\,\mathsf{f}'(\xi) = 0, \tag{V.29a}$$

with f', f'' the first two derivatives of f. Meanwhile, the boundary conditions (V.27b)–(V.27c) become

$$\mathsf{f}(0) = 1 \quad , \quad \lim_{\xi\to\infty} \mathsf{f}(\xi) = 0. \tag{V.29b}$$

The corresponding solution is

$$\mathsf{f}(\xi) = \operatorname{erfc}(\xi) = 1 - \operatorname{erf}(\xi), \tag{V.30}$$

where erf denotes the (*Gauss*) *error function*, defined as[(33)]

$$\operatorname{erf}(\xi) \equiv \frac{2}{\sqrt{\pi}} \int_0^\xi \mathrm{e}^{-\upsilon^2}\,\mathrm{d}\upsilon \tag{V.31}$$

while erfc is the *complementary error function*[(33)]

$$\operatorname{erfc}(\xi) \equiv \frac{2}{\sqrt{\pi}} \int_\xi^\infty \mathrm{e}^{-\upsilon^2}\,\mathrm{d}\upsilon. \tag{V.32}$$

All in all, the solution of the problem (V.27) is thus

$$\mathsf{v}(t,y) = \mathsf{u}\left[1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right]. \tag{V.33}$$

For $\xi = 2$, $\operatorname{erf}(2) = 0.99532\ldots$, i.e. $\operatorname{erfc}(2) \approx 0.005$. That is, at given t, the magnitude of the velocity at

$$y = \delta_l(t) \equiv 4\sqrt{\nu t} \tag{V.34}$$

is reduced by a factor 200 with respect to its value at the moving plane. This length $\delta_l(t)$ is a typical measure for the width of the *boundary layer* over which momentum is transported from the plane into the fluid, i.e. the region in which the fluid viscosity plays a role.

[(33)] The reader interested in its properties can have a look at the *NIST Handbook of mathematical functions* [27], Chapter 7.

The width (V.34) of the boundary layer increases with the square root of time: this is the typical behavior expected for a diffusive process—which is understandable since Eq. (V.27a) is nothing but the classical *diffusion equation*.

Remarks:

∗ The width (V.34) of the boundary layer also increases with the (kinematic) shear viscosity ν: the latter characterizes the efficiency of momentum transport, and when it is larger, the influence of the plane motion at a given time after it has started extends further away from the plane.

∗ At every given fixed y, the velocity (V.33) tends asymptotically to u as $t \to \infty$, which was to be expected.

∗ The above problem is often referred to as *first Stokes problem* or *Rayleigh problem*.[aj] In the *second Stokes problem*, the plane is not set impulsively into motion, it oscillates sinusoidally in its own plane with a frequency ω. In that case, the amplitude of the induced fluid oscillations decrease "only" exponentially with the distance to the plane, and the typical extent of the region affected by shear viscous effects is $\sqrt{\nu/\omega}$.

V.4.2 Modeling of the flow inside the boundary layer

As argued in the introduction to the present Section, the existence of a "small" boundary layer, to which the effects induced by viscosity in the vicinity of an obstacle—more specifically, the influence of the no-slip condition at the boundaries—are confined, can be argued to be a rather general feature.[34] Taking its existence as granted, we shall now model the fluid motion inside such a boundary layer.

For simplicity, we consider a steady incompressible two-dimensional flow past an obstacle of typical size L_c, in the absence of relevant volume forces. At each point of the surface of the obstacle, the curvature radius is assumed to be large with respect to the local width δ_l of the boundary layer. That is, using local Cartesian (x, y) coordinates with x resp. y parallel resp. orthogonal to the surface, the boundary layer has a large size—of order L_c—along the x-direction, while it is much thinner—of order δ_l—along y. For the sake of brevity, the variables (x, y) of the various dynamical fields v_x, v_y, $\mathcal{P}$ will be omitted.

figure needed!

For the fluid inside the boundary layer, the equations of motion are on the one hand the incompressibility condition $\vec{\nabla} \cdot \vec{\mathsf{v}} = 0$, i.e.

$$\frac{\partial \mathsf{v}_x}{\partial x} + \frac{\partial \mathsf{v}_y}{\partial y} = 0, \tag{V.35a}$$

and on the other hand the incompressible Navier–Stokes equation (III.33). Projected on the x- and y-directions, the latter reads

$$\left(\mathsf{v}_x \frac{\partial}{\partial x} + \mathsf{v}_y \frac{\partial}{\partial y}\right) \mathsf{v}_x = -\frac{1}{\rho} \frac{\partial \mathcal{P}}{\partial x} + \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \mathsf{v}_x, \tag{V.35b}$$

$$\left(\mathsf{v}_x \frac{\partial}{\partial x} + \mathsf{v}_y \frac{\partial}{\partial y}\right) \mathsf{v}_y = -\frac{1}{\rho} \frac{\partial \mathcal{P}}{\partial y} + \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \mathsf{v}_y. \tag{V.35c}$$

Since the boundary layer is much extended along the tangential direction than along the normal one, the range of x values is much larger than that of y values. To obtain dimensionless variables taking their values over a similar interval, one defines

[34] Note however that it does not exist for every possible flow.

[aj] J. W. Strutt, Lord Rayleigh, 1842–1919

$$x^* \equiv \frac{x}{L_c} \quad , \quad y^* \equiv \frac{y}{\delta_l} \tag{V.36}$$

where the typical extent in the normal direction, i.e. the width of the boundary layer

$$\delta_l \ll L_c \tag{V.37}$$

has to be determined by requiring that both x^*, y^* should be of order unity.

Remarks:

∗ In realistic cases, the width δ_l may actually depend on the position x along the flow boundary, yet this complication is ignored here.

∗ If the local radius of curvature of the boundary is not much larger than the width δ_l of the boundary layer, one should use curvilinear coordinates x^1 (tangential to the boundary) and x^2 instead of Cartesian ones, yet within that alternative coordinate system the remainder of the derivation still holds.

Similarly, the dynamical fields are rescaled to yield dimensionless fields:

$$\mathsf{v}_x^* \equiv \frac{\mathsf{v}_x}{\mathsf{v}_\infty} \quad , \quad \mathsf{v}_y^* \equiv \frac{\mathsf{v}_y}{\mathsf{u}} \quad , \quad \mathcal{P}^* \equiv \frac{\mathcal{P}}{\rho \mathsf{v}_\infty^2}, \tag{V.38}$$

where, in order to account for the expectation that the normal velocity v_y is (in average) much smaller than the tangential one v_x which is of order v_∞ at the outer edge of the boundary layer, a second velocity scale

$$\mathsf{u} \ll \mathsf{v}_\infty, \tag{V.39}$$

was introduced, such that v_x^*, v_y^*, and $\mathcal{P}^*$ are of order unity. These fields are functions of the dimensionless variables (x^*, y^*), although this shall not be written explicitly.

Eventually, the Reynolds number corresponding to the motion along x is

$$\mathrm{Re} \equiv \frac{L_c \mathsf{v}_\infty}{\nu}. \tag{V.40}$$

With the help of definitions (V.36)–(V.40), the equations of motion (V.35) can be recast in a dimensionless form:

$$\frac{\partial \mathsf{v}_x^*}{\partial x^*} + \frac{L_c}{\delta_l} \frac{\mathsf{u}}{\mathsf{v}_\infty} \frac{\partial \mathsf{v}_y^*}{\partial y^*} = 0; \tag{V.41a}$$

$$\mathsf{v}_x^* \frac{\partial \mathsf{v}_x^*}{\partial x^*} + \frac{L_c}{\delta_l} \frac{\mathsf{u}}{\mathsf{v}_\infty} \mathsf{v}_y^* \frac{\partial \mathsf{v}_x^*}{\partial y^*} = -\frac{\partial \mathcal{P}^*}{\partial x^*} + \frac{1}{\mathrm{Re}} \frac{L_c^2}{\delta_l^2} \left(\frac{\partial^2 \mathsf{v}_x^*}{\partial y^{*2}} + \frac{\delta_l^2}{L_c^2} \frac{\partial^2 \mathsf{v}_x^*}{\partial x^{*2}} \right), \tag{V.41b}$$

$$\frac{\mathsf{u}}{\mathsf{v}_\infty} \mathsf{v}_x^* \frac{\partial \mathsf{v}_y^*}{\partial x^*} + \frac{L_c}{\delta_l} \frac{\mathsf{u}^2}{\mathsf{v}_\infty^2} \mathsf{v}_y^* \frac{\partial \mathsf{v}_y^*}{\partial y^*} = -\frac{L_c}{\delta} \frac{\partial \mathcal{P}^*}{\partial y^*} + \frac{1}{\mathrm{Re}} \frac{L_c^2}{\delta_l^2} \frac{\mathsf{u}}{\mathsf{v}_\infty} \left(\frac{\partial^2 \mathsf{v}_y^*}{\partial y^{*2}} + \frac{\delta_l^2}{L_c^2} \frac{\partial^2 \mathsf{v}_y^*}{\partial x^{*2}} \right), \tag{V.41c}$$

Consider first the continuity equation (V.41a). It will only yield a non-trivial constraint on the flow if both terms have the same order of magnitude, which is possible if

$$\frac{L_c}{\delta_l} \frac{\mathsf{u}}{\mathsf{v}_\infty} = 1, \tag{V.42}$$

yielding a first condition on the unknown characteristic quantities δ_l and u.

In turn, a second constraint comes from the dimensionless Navier–Stokes equation (V.41b) along the tangential direction. In the boundary layer, by definition, the effects from inertia encoded in the convective term and those of viscosity are of the same magnitude, which necessitates that the prefactor of the viscous term should be of order unity. This suggests the condition

$$\frac{L_c^2}{\delta_l^2}\frac{1}{\mathrm{Re}} = 1. \tag{V.43}$$

Equations (V.42)–(V.43) are then easily solved, yielding for the unknown quantities characterizing the flow along the direction normal to the boundary

$$\delta_l = \frac{L_c}{\sqrt{\mathrm{Re}}} \quad , \quad \mathsf{u} = \frac{\mathsf{v}_\infty}{\sqrt{\mathrm{Re}}}. \tag{V.44}$$

As in the first or second Stokes problems, see e.g. Eq. (V.34), the width of the boundary layer is proportional to the square root of the kinematic viscosity ν.

Substituting the conditions (V.42)–(V.43) in the system of equations (V.41) and keeping only the leading terms, one eventually obtains

$$\frac{\partial \mathsf{v}_x^*}{\partial x^*} + \frac{\partial \mathsf{v}_y^*}{\partial y^*} = 0; \tag{V.45a}$$

$$\mathsf{v}_x^*\frac{\partial \mathsf{v}_x^*}{\partial x^*} + \mathsf{v}_y^*\frac{\partial \mathsf{v}_x^*}{\partial y^*} = -\frac{\partial \mathcal{P}^*}{\partial x^*} + \frac{\partial^2 \mathsf{v}_x^*}{\partial y^{*2}}, \tag{V.45b}$$

$$\frac{\partial \mathcal{P}^*}{\partial y^*} = 0. \tag{V.45c}$$

These equations constitute the simplified system, first by written down by Prandtl,[(ak)] that describes the fluid motion in a laminar boundary layer—where the laminarity assumption is hidden in the use of the typical length scale L_c imposed by geometry, rather than of a smaller one driven by turbulent patterns.

V.5 Vortex dynamics in Newtonian fluids

The equations derived in § IV.3.2 regarding the behavior of vorticity in a perfect fluid are easily generalized to the case of a Newtonian fluid.

V.5.1 Vorticity transport in Newtonian fluids

As was done with the Euler equation when going from Eq. (III.18) to the Eq. (III.20), one may rewrite the convective term in the Navier–Stokes equation (III.32) as $\big(\vec{\mathsf{v}}\cdot\vec{\nabla}\big)\vec{\mathsf{v}} = \frac{1}{2}\vec{\nabla}\big(\vec{\mathsf{v}}^2\big) + \vec{\mathsf{v}}\times\vec{\omega}$. Assuming then that the volume forces are conservative, i.e. $\vec{f}_V = -\rho\vec{\nabla}\Phi$, and taking the rotational curl, one easily finds

$$\frac{\partial\vec{\omega}(t,\vec{r})}{\partial t} - \vec{\nabla}\times\big[\vec{\mathsf{v}}(t,\vec{r})\times\vec{\omega}(t,\vec{r})\big] = -\frac{\vec{\nabla}\mathcal{P}(t,\vec{r})\times\vec{\nabla}\rho(t,\vec{r})}{\rho(t,\vec{r})^2} + \nu\triangle\vec{\omega}(t,\vec{r}), \tag{V.46}$$

which generalizes Eq. (IV.21) to the case of Newtonian fluids. Note that even without assuming that the flow is incompressible, the term involving the bulk viscosity has already dropped out from the problem.

As in § IV.3.2, the second term in the left member can be further transformed, which leads to the equivalent forms

$$\frac{\mathrm{D}\vec{\omega}(t,\vec{r})}{\mathrm{D}t} = \big[\vec{\omega}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}) - \big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\vec{\omega}(t,\vec{r}) - \frac{\vec{\nabla}\mathcal{P}(t,\vec{r})\times\vec{\nabla}\rho(t,\vec{r})}{\rho(t,\vec{r})^2} + \nu\triangle\vec{\omega}(t,\vec{r}), \tag{V.47a}$$

involving the material derivative $\mathrm{D}\vec{\omega}/\mathrm{D}t$, or else

$$\frac{\mathcal{D}_{\vec{\mathsf{v}}}\vec{\omega}(t,\vec{r})}{\mathcal{D}t} = -\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\vec{\omega}(t,\vec{r}) - \frac{\vec{\nabla}\mathcal{P}(t,\vec{r})\times\vec{\nabla}\rho(t,\vec{r})}{\rho(t,\vec{r})^2} + \nu\triangle\vec{\omega}(t,\vec{r}), \tag{V.47b}$$

which makes use of the comoving time-derivative (IV.23a).

[(ak)] L. Prandtl, 1875–1953

The right hand side of this equation simplifies in various cases. In the particular of a barotropic fluid, the second term vanishes. In an incompressible flow, the first two terms are zero.

As we shall illustrate on an example, the viscous term, proportional to the Laplacian of vorticity, is of diffusive nature, and tends to spread out the vorticity lines over a larger region.

V.5.2 Diffusion of a rectilinear vortex

As example of application of the equation of motion introduced in the previous Section, let us consider the two-dimensional motion in the (x, y)-plane of an incompressible Newtonian fluid with conservative forces, in which there is at $t = 0$ a rectilinear vortex along the z-axis:

$$\vec{\omega}(t{=}0, \vec{r}) = \frac{\Gamma_0}{2\pi r}\delta(z)\vec{\mathrm{e}}_z \tag{V.48}$$

with r the distance from the z-axis. Obviously, the circulation around any curve circling this vortex once is simply Γ_0.

At time $t > 0$, this vortex will start diffusing, with its evolution governed by Eq. (V.46). Given the symmetry of the problem round the z-axis, which suggests the use of cylindrical coordinates (r, θ, z), the vorticity vector will remain parallel to $\vec{\mathrm{e}}_z$, and its magnitude should only depend on r:

$$\vec{\omega}(t, \vec{r}) = \omega^z(t, r)\vec{\mathrm{e}}_z \tag{V.49}$$

This results in a velocity field $\vec{\mathsf{v}}(t, \vec{r})$ in the (x, y)-plane, in the orthoradial direction. As a consequence the convective derivative in the left hand side of Eq. (V.47a) vanishes since $\vec{\omega}(t, r)$ has no gradient along $\vec{\mathrm{e}}_\theta$. Similarly, the term $(\vec{\omega}\cdot\vec{\nabla})\vec{\mathsf{v}}$ also vanishes, since the velocity is independent of z. Eventually, the term involving $\vec{\nabla}\cdot\vec{\mathsf{v}}$ vanishes thanks to the assumed incompressibility. All in all, the vorticity thus obeys the diffusion equation

$$\frac{\partial\omega^z(t, r)}{\partial t} = \nu\triangle\omega^z(t, r) = \nu\left[\frac{\partial^2\omega^z(t, r)}{\partial r^2} + \frac{1}{r}\frac{\partial\omega^z(t, r)}{\partial r}\right], \tag{V.50}$$

with the initial condition (V.48).

The problem is clearly linear in Γ_0, so that the solution $\omega^z(t, r)$ should be proportional to Γ_0, without any further dependence on Γ_0. This leaves the kinematic viscosity ν as only dimensionful parameter available in the problem: using a dimensional reasoning similar to that made in the study of the first Stokes problem (§ V.4.1), there is a single relevant dimensionless variable, namely $\xi = r^2/(\nu t)$, combining the time and space variables. The only ansatz respecting the dimensional requirements is then

$$\omega^z(t, r) = \frac{\Gamma_0}{\nu t}\mathsf{f}(\xi), \quad \text{with } \xi \equiv \frac{r^2}{\nu t} \tag{V.51}$$

with f a dimensionless function. Inserting this ansatz into Eq. (V.50) leads to the ordinary differential equation

$$\mathsf{f}(\xi) + \xi\mathsf{f}'(\xi) + 4\big[\mathsf{f}'(\xi) + \xi\mathsf{f}''(\xi)\big] = 0. \tag{V.52}$$

A first integration yields

$$\xi\mathsf{f}(\xi) + 4\xi\mathsf{f}'(\xi) = \text{const.}$$

In order to satisfy the initial condition, the integration constant should be zero, leaving with the linear differential equation $\mathsf{f}(\xi) + 4\mathsf{f}'(\xi) = 0$, which is readily integrated to yield

$$\mathsf{f}(\xi) = C\,\mathrm{e}^{-\xi/4},$$

that is

$$\omega^z(t, r) = \frac{\Gamma_0}{\nu t}C\,\mathrm{e}^{-r^2/(4\nu t)}, \tag{V.53}$$

with C an integration which still has to be fixed.

To determine the latter, let us consider the circulation of the velocity at time t around a circle $\mathscr{C}_R$ of radius R centered on the axis $z = 0$:

$$\Gamma(t, R) = \oint_{\mathscr{C}_R} \vec{\mathrm{v}}(t, \vec{r}) \cdot \mathrm{d}\vec{\ell} = \int_0^R \int_0^{2\pi} \omega^z(t, r)\, r\, \mathrm{d}r\, \mathrm{d}\theta = 2\pi \int_0^R \omega^z(t, r)\, r\, \mathrm{d}r\, \mathrm{d}\theta, \tag{V.54}$$

where the second identity follows from the Stokes theorem while the third is trivial. Inserting the solution (V.53) yields

$$\Gamma(t, R) = 4\pi\Gamma_0 C \big[1 - \mathrm{e}^{-R^2/(4\nu t)}\big],$$

showing the C should equal $1/4\pi$ to yield the proper circulation at $t = 0$. All in all, the vorticity field in the problem reads

$$\omega^z(t, r) = \frac{\Gamma_0}{4\pi\nu t}\, \mathrm{e}^{-r^2/(4\nu t)}. \tag{V.55}$$

That is, the vorticity extends over a region of typical width $\delta(t) = \sqrt{4\nu t}$, which increases with time: one recognizes the characteristic diffusive behavior, proportional to $\sqrt{t}$—as well as the typical $\sqrt{\nu}$ dependence of the size of the region affected by viscous effects, encountered in Sec. V.4.

The vorticity (V.55) leads to the circulation around a circle of radius R

$$\Gamma(t, R) = \Gamma_0 \big[1 - \mathrm{e}^{-R^2/(4\nu t)}\big], \tag{V.56}$$

which at given R decreases with time, in contrast to the perfect-fluid case, in which the circulation would be conserved.

Eventually, one can also easily compute the velocity field associated with the expanding vortex, namely

$$\vec{\mathrm{v}}(t, \vec{r}) = \frac{\Gamma_0}{2\pi r} \big[1 - \mathrm{e}^{-r^2/(4\nu t)}\big] \frac{\vec{\mathrm{e}}_\theta}{r}, \tag{V.57}$$

where $|\vec{\mathrm{e}}_\theta| = r$.

Bibliography for Chapter V

- National Committee for Fluid Mechanics film & film notes on *Rotating flows*, *Low Reynolds Number Flow*, *Fundamentals of Boundary Layers* and *Vorticity*;
- Faber [1] Chapters 6.6, 6.9 and 6.11;
- Guyon *et al.* [2] Chapters 4.5, 7.3.2, 9 & 10.1–10.4;
- Landau–Lifshitz [4, 5] Chapter II § 17–20 & 24, Chapter IV § 39 and Chapter VIII § 79;
- Sommerfeld [7, 8] Chapters II § 10, III § 16 and VII § 35.

CHAPTER VI

Waves in non-relativistic fluids

A large class of solutions of the equations of motion (III.9), (III.18) and (III.35)—in the case of a perfect fluid—, or (III.9), (III.32), (III.37)—for a Newtonian fluid—, is that of waves. Quite generically, this denomination designates "perturbations" of some "unperturbed" fluid motion, which will also be referred to as "background flow".

In more mathematical terms, the starting point is a set of fields $\{\rho_0(t,\vec{r}), \vec{\mathrm{v}}_0(t,\vec{r}), \mathcal{P}_0(t,\vec{r})\}$ solving the equations of motion, that represent the background flow, either ideal or dissipative. The wave itself consists of a second set of fields $\{\delta\rho(t,\vec{r}), \delta\vec{\mathrm{v}}(t,\vec{r}), \delta\mathcal{P}(t,\vec{r})\}$, which are added on top of the background ones, such that the resulting fields

$$\rho(t,\vec{r}) = \rho_0(t,\vec{r}) + \delta\rho(t,\vec{r}), \tag{VI.1a}$$

$$\mathcal{P}(t,\vec{r}) = \mathcal{P}_0(t,\vec{r}) + \delta\mathcal{P}(t,\vec{r}), \tag{VI.1b}$$

$$\vec{\mathrm{v}}(t,\vec{r}) = \vec{\mathrm{v}}_0(t,\vec{r}) + \delta\vec{\mathrm{v}}(t,\vec{r}) \tag{VI.1c}$$

are again solutions to the same equations of motion.

Different kinds of perturbations—triggered by some source which will not be specified hereafter, and is thus to be seen as an initial condition—can be considered, leading to different phenomena.

A first distinction, with which the reader is probably already familiar, is that between traveling waves, which propagate, and standing waves, which do not. Mathematically, in the former case the propagating quantity does not depend on space and time independently, but rather on a combination like (in a one-dimensional case) $x - c_\varphi t$, where c_φ denotes some propagation speed. In contrast, in standing waves the space and time dependences of the "propagating" quantity factorize. Hereafter, we shall mostly mention traveling waves.

Another difference is that between "small" and "large" perturbations or, in more technical terms, between linear and nonlinear waves. In the former case, which is that of sound waves (Sec. VI.1) or the simplest gravity-controlled surface waves in liquids (§ VI.3.1), the partial differential equation governing the propagation of the wave is linear—which means that nonlinear terms have been neglected. Quite obviously, nonlinearities of the dynamical equations—as e.g. the Euler equation—are the main feature of nonlinear waves, as for instance shock waves (VI.2) or solitons (§ VI.3.2).

VI.1 Sound waves

By definition, the phenomenon which in everyday life is referred to as "sound" consists of *small adiabatic pressure perturbations* around a background flow, where adiabatic actually means that the entropy remains constant. In the presence of such a wave, each point in the fluid undergoes alternative *compression* and *rarefaction* processes. That is, these waves are by construction (parts of) a compressible flow.

We shall first consider sound waves on a uniform perfect fluid at rest (§ VI.1.1).

What then? Doppler effect? Rarefaction waves? Eventually, we discuss how viscous effects in a Newtonian fluid lead to the absorption of sound waves (§ VI.1.4).

VI.1.1 Sound waves in a uniform fluid at rest

Assuming that there are no external forces, a trivial solution of the dynamical equations of perfect fluids is that with uniform and time independent mass density ρ_0 and pressure $\mathcal{P}_0$, with a vanishing flow velocity $\vec{\mathrm{v}}_0 = \vec{0}$. Assuming in addition that the particle number N_0 in the fluid is conserved, its total entropy has a fixed value S_0. These conditions will represent the background flow we consider hereafter.

With the various fields that were just specified, a perturbation (VI.1) of this background flow reads

$$\rho(t,\vec{r}) = \rho_0 + \delta\rho(t,\vec{r}), \tag{VI.2a}$$

$$\mathcal{P}(t,\vec{r}) = \mathcal{P}_0 + \delta\mathcal{P}(t,\vec{r}), \tag{VI.2b}$$

$$\vec{\mathrm{v}}(t,\vec{r}) = \vec{0} + \delta\vec{\mathrm{v}}(t,\vec{r}). \tag{VI.2c}$$

The necessary "smallness" of perturbations means for the mass density and pressure terms

$$|\delta\rho(t,\vec{r})| \ll \rho_0, \quad |\delta\mathcal{P}(t,\vec{r})| \ll \mathcal{P}_0. \tag{VI.2d}$$

Regarding the velocity, the background flow does not explicitly specify a reference scale, with which the perturbation should be compared. As we shall see below, the reference scale is actually implicitly contained in the equation(s) of state of the fluid under consideration, and the condition of small perturbation reads

$$|\delta\vec{\mathrm{v}}(t,\vec{r})| \ll c_s \tag{VI.2e}$$

with c_s the *speed of sound* in the fluid.

Inserting the fields (VI.2) in the equations of motion (III.9) and (III.18) and taking into account the uniformity and stationarity of the background flow, one finds

$$\frac{\partial\delta\rho(t,\vec{r})}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{\mathrm{v}}(t,\vec{r}) + \vec{\nabla}\cdot\big[\delta\rho(t,\vec{r})\,\delta\vec{\mathrm{v}}(t,\vec{r})\big] = 0, \tag{VI.3a}$$

$$\big[\rho_0 + \delta\rho(t,\vec{r})\big]\left\{\frac{\partial\delta\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \big[\delta\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\delta\vec{\mathrm{v}}(t,\vec{r})\right\} + \vec{\nabla}\delta\mathcal{P}(t,\vec{r}) = \vec{0}. \tag{VI.3b}$$

The required smallness of the perturbations will help us simplify these equations, in that we shall only keep the leading-order terms in an expansion in which we consider ρ_0, $\mathcal{P}_0$ as zeroth-order quantities while $\delta\rho(t,\vec{r})$, $\delta\mathcal{P}(t,\vec{r})$ and $\delta\vec{\mathrm{v}}(t,\vec{r})$ are small quantities of first order. Accordingly, the third term in the continuity equation is presumably much smaller than the other two, and may be left aside in a first approximation. Similarly, the contribution of $\delta\rho(t,\vec{r})$ and the convective term within the curly brackets on the left hand side of Eq. (VI.3b) may be dropped. The equations describing the coupled evolutions of $\delta\rho(t,\vec{r})$, $\delta\mathcal{P}(t,\vec{r})$ and $\delta\vec{\mathrm{v}}(t,\vec{r})$ are thus *linearized*:

$$\frac{\partial\delta\rho(t,\vec{r})}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{\mathrm{v}}(t,\vec{r}) = 0, \tag{VI.4a}$$

$$\rho_0\frac{\partial\delta\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \vec{\nabla}\delta\mathcal{P}(t,\vec{r}) = \vec{0}. \tag{VI.4b}$$

To obtain a closed system of equations, a further relation between the perturbations is needed. This will be provided by thermodynamics, i.e. by the implicit assumption that the fluid at rest is everywhere in a state in which its pressure $\mathcal{P}$ is function of mass density ρ, (local) entropy S, and (local) particle number N, i.e. that there exists a unique relation $\mathcal{P} = \mathcal{P}(\rho, S, N)$ which is valid at each point in the fluid and at every time. Expanding this relation around the (thermodynamic) point corresponding to the background flow, namely $\mathcal{P}_0 = \mathcal{P}(\rho_0, S_0, N_0)$, one may write

$$\mathcal{P}\big(\rho_0 + \delta\rho, S_0 + \delta S, N_0 + \delta N\big) = \mathcal{P}_0 + \left(\frac{\partial \mathcal{P}}{\partial \rho}\right)_{S,N} \delta\rho + \left(\frac{\partial \mathcal{P}}{\partial S}\right)_{\rho,N} \delta S + \left(\frac{\partial \mathcal{P}}{\partial N}\right)_{S,\rho} \delta N,$$

where the derivatives are to be evaluated at the point (ρ_0, S_0, N_0). Here, we wish to consider isentropic perturbations at constant particle number, i.e. both δS and δN vanish, leaving

$$\delta\mathcal{P} = \left(\frac{\partial \mathcal{P}}{\partial \rho}\right)_{S,N} \delta\rho.$$

For the partial derivative of the pressure, we introduce the notation

$$\boxed{c_s^2 \equiv \left(\frac{\partial \mathcal{P}}{\partial \rho}\right)_{S,N}} \tag{VI.5}$$

where both sides actually depend on ρ_0, S_0 and N_0, yielding

$$\delta\mathcal{P} = c_s^2\, \delta\rho.$$

This thermodynamic relation holds at each point of the fluid at each instant, so that one can now replace $\vec{\nabla}\delta\mathcal{P}(t,\vec{r})$ by $c_s^2\, \vec{\nabla}\delta\rho(t,\vec{r})$ in Eq. (VI.4b):

$$\rho_0 \frac{\partial \delta\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + c_s^2\, \vec{\nabla}\delta\rho(t,\vec{r}) = \vec{0}. \tag{VI.4c}$$

The equations (VI.4a), (VI.4c) for the perturbations $\delta\rho(t,\vec{r})$ and $\delta\vec{\mathrm{v}}(t,\vec{r})$ are linear first order partial differential equations. Thanks to the linearity, their solutions form a vector space—at least as long as no initial condition has been specified. One can for instance express the solutions as Fourier transforms, i.e. superpositions of plane waves, characterized by their (angular) frequency ω and their wave vector $\vec{k}$. Accordingly, we test the ansatz

$$\delta\rho(t,\vec{r}) = \widetilde{\delta\rho}(\omega,\vec{k})\, \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\vec{k}\cdot\vec{r}}, \qquad \delta\vec{\mathrm{v}}(t,\vec{r}) = \widetilde{\delta\vec{\mathrm{v}}}(\omega,\vec{k})\, \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\vec{k}\cdot\vec{r}}, \tag{VI.6}$$

with respective amplitudes $\widetilde{\delta\rho}$, $\widetilde{\delta\vec{\mathrm{v}}}$ that a priori depend on ω and $\vec{k}$ and are determined by the initial conditions for the problem. In turn, ω and $\vec{k}$ are not necessarily independent from each other, as we shall indeed find hereafter.

With this ansatz, Eqs. (VI.4) become

$$-\mathrm{i}\omega\widetilde{\delta\rho}(\omega,\vec{k}) + \mathrm{i}\rho_0\, \vec{k}\cdot\widetilde{\delta\vec{\mathrm{v}}}(\omega,\vec{k}) = 0 \tag{VI.7a}$$

$$-\mathrm{i}\omega\rho_0\, \widetilde{\delta\vec{\mathrm{v}}}(\omega,\vec{k}) + \mathrm{i}c_s^2\, \vec{k}\, \widetilde{\delta\rho}(\omega,\vec{k}) = \vec{0}. \tag{VI.7b}$$

From the second equation, the amplitude $\widetilde{\delta\vec{\mathrm{v}}}(\omega,\vec{k})$ is proportional to $\vec{k}$; in particular, it lies along the same direction. That is, the inner product $\vec{k}\cdot\widetilde{\delta\vec{\mathrm{v}}}$ simply equals the product of the norms of the two vectors.

Omitting from now on the $(\omega,\vec{k})$-dependence of the amplitudes, the inner product of Eq. (VI.7b) with $\vec{k}$—which does not lead to any loss of information—allows one to recast the system as

$$\begin{pmatrix} -\omega & \rho_0 \\ c_s^2\vec{k}^{\,2} & -\omega\rho_0 \end{pmatrix} \begin{pmatrix} \widetilde{\delta\rho} \\ \vec{k}\cdot\widetilde{\delta\vec{\mathrm{v}}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A first, trivial solution to this system is $\widetilde{\delta\rho} = 0$, $\widetilde{\delta\vec{\mathrm{v}}} = \vec{0}$, i.e. the absence of any perturbation. In order for non-trivial solutions to exist, the determinant $(\omega^2 - c_s^2\vec{k}^{\,2})\rho_0$ of the system should vanish.

This leads at once to the *dispersion relation*

$$\omega = \pm c_s |\vec{k}|. \tag{VI.8}$$

Denoting by $\vec{\mathrm{e}}_{\vec{k}}$ the unit vector in the direction of $\vec{k}$, the perturbations $\delta\rho(t,\vec{r})$ and $\delta\vec{\mathrm{v}}(t,\vec{r})$ defined by Eq. (VI.6), as well as $\delta\mathcal{P}(t,\vec{r}) = c_s^2\,\delta\rho(t,\vec{r})$, are all functions of $c_s t \pm \vec{r}\cdot\vec{\mathrm{e}}_{\vec{k}}$. These are thus *traveling waves*,[lv], that propagate with the phase velocity $\omega(\vec{k})/|\vec{k}| = c_s$, which is independent of $\vec{k}$. That is, c_s is the *speed of sound*, and the latter is the same for all frequencies.
For instance, for air at $T = 300$ K, the speed of sound is $c_s = 347\ \mathrm{m\cdot s^{-1}}$.

Air is a diatomic ideal gas, i.e. it has pressure $\mathcal{P} = Nk_\mathrm{B}T/\mathcal{V}$ and internal energy $U = \frac{5}{2}Nk_\mathrm{B}T$, giving

$$c_s^2 = \left(\frac{\partial\mathcal{P}}{\partial\rho}\right)_{S,N} = -\frac{\mathcal{V}^2}{mN}\left(\frac{\partial\mathcal{P}}{\partial\mathcal{V}}\right)_{S,N} = -\frac{\mathcal{V}^2}{mN}\left[-\frac{Nk_\mathrm{B}T}{\mathcal{V}^2} + \frac{Nk_\mathrm{B}}{\mathcal{V}}\left(\frac{\partial T}{\partial\mathcal{V}}\right)_{S,N}\right].$$

The thermodynamic relation $\mathrm{d}U = T\,\mathrm{d}S - \mathcal{P}\,\mathrm{d}\mathcal{V} + \mu\,\mathrm{d}N$ yields at constant entropy and particle number

$$\mathcal{P} = -\left(\frac{\partial U}{\partial\mathcal{V}}\right)_{S,N} = -\frac{5}{2}Nk_\mathrm{B}\left(\frac{\partial T}{\partial\mathcal{V}}\right)_{S,N} \qquad \text{i.e.} \qquad Nk_\mathrm{B}\left(\frac{\partial T}{\partial\mathcal{V}}\right)_{S,N} = -\frac{2\mathcal{P}}{5} = -\frac{2}{5}\frac{Nk_\mathrm{B}T}{\mathcal{V}}.$$

leading to $c_s^2 = \dfrac{7}{5}\dfrac{k_\mathrm{B}T}{m_\mathrm{air}}$, with $m_\mathrm{air} = 29/\mathcal{N}_\mathrm{A}\ \mathrm{g\cdot mol^{-1}}$. □

Remarks:

∗ Instead of c_s^2, one may use the fluid *isentropic compressibility*, defined as

$$\beta_S \equiv -\frac{1}{\mathcal{V}}\left(\frac{\partial\mathcal{V}}{\partial\mathcal{P}}\right)_S = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial\mathcal{P}}\right)_S, \tag{VI.9a}$$

to relate the variations of pressure and mass density $\delta\mathcal{P}$, $\delta\rho$. This compressibility is related to c_s (evaluated at $\rho = \rho_0$) by

$$\beta_S = \frac{1}{\rho_0 c_s^2} \quad \text{resp.} \quad c_s = \frac{1}{\sqrt{\rho_0\beta_S}}, \tag{VI.9b}$$

which shows that the sound velocity is larger in "stiffer" fluids, i.e. fluids with a smaller compressibility — as generally liquids with respect to gases.

∗ Taking the real parts of the complex quantities in the harmonic waves (VI.6), so as to obtain real-valued $\delta\rho$, $\delta\mathcal{P}$ and $\delta\vec{\mathrm{v}}$, one sees that these will be alternatively positive and negative, and in average—over a duration much longer than a period $2\pi/\omega$—zero. This in particular means that the successive compression and condensation ($\delta\mathcal{P} > 0$, $\delta\rho > 0$) or depression and rarefaction[lvi] ($\delta\mathcal{P} < 0$, $\delta\rho < 0$) processes do not lead to a resulting transport of matter.

∗ A single harmonic wave (VI.6) is a traveling wave. Yet if the governing equation or systems of equations is linear or has been linearized, as was done here, the superposition of harmonic waves is a valid solution. In particular, the superposition of two harmonic traveling waves with equal frequencies ω, opposite wave vectors $\vec{k}$—which is allowed by the dispersion relation (VI.8)—and equal amplitudes leads to a *standing wave*, in which the dependence on time and space is proportional to $\mathrm{e}^{\mathrm{i}\omega t}\cos(\vec{k}\cdot\vec{r})$.

Coming back to Eq. (VI.7b), the proportionality of $\widetilde{\delta\vec{\mathrm{v}}}(\omega,\vec{k})$ and $\vec{k}$ means that the sound waves in a fluid are *longitudinal*—in contrast to electromagnetic waves in vacuum, which are transversal waves.

The nonexistence of transversal waves in fluids reflects the absence of forces that would act against shear deformations so as to restore some equilibrium shape—shear viscous effects cannot play that role.

[lv] *fortschreitende Wellen* [lvi] *Verdünnung*

In contrast, there can be transversal sound waves in elastic solids, as e.g. the so-called S-modes (shear modes) in geophysics.

The inner product of Eq. (VI.7b) with $\vec{k}$, together with the dispersion relation (VI.8) and the collinearity of $\widetilde{\delta\vec{\mathrm{v}}}$ and $\vec{k}$, leads to the relation

$$\omega\rho_0\big|\vec{k}\big|\big|\widetilde{\delta\vec{\mathrm{v}}}\big| = c_s^2\,\big|\vec{k}\big|\widetilde{\delta\rho} \qquad \Leftrightarrow \qquad \frac{\big|\widetilde{\delta\vec{\mathrm{v}}}\big|}{c_s} = \frac{\widetilde{\delta\rho}}{\rho_0}$$

for the amplitudes of the perturbations. This justifies condition (VI.2e), which is then consistent with (VI.2d). Similarly, inserting the ansatz (VI.6) in Eq. (VI.3b), the terms within curly brackets become $-\mathrm{i}\omega\,\widetilde{\delta\vec{\mathrm{v}}} + \mathrm{i}\big(\vec{k}\cdot\widetilde{\delta\vec{\mathrm{v}}}\big)\widetilde{\delta\vec{\mathrm{v}}}$: again, neglecting the second with respect to the first is equivalent to requesting $\big|\widetilde{\delta\vec{\mathrm{v}}}\big| \ll c_s$.

Remark: Going back to Eqs. (VI.4a) and (VI.4c), the difference of the time derivative of the first one and the divergence of the second one leads to the known *wave equation*[35]

$$\frac{\partial^2\delta\rho(t,\vec{r})}{\partial t^2} - c_s^2\triangle\delta\rho(t,\vec{r}) = 0, \tag{VI.10a}$$

If the flow—including the background flow on which the sound wave develops, in case $\vec{\mathrm{v}}_0$ is not trivial as it was assumed here—is irrotational, so that one may write $\vec{\mathrm{v}}(t,\vec{r}) = -\vec{\nabla}\varphi(t,\vec{r})$, then the velocity potential φ also obeys the same equation

$$\frac{\partial^2\varphi(t,\vec{r})}{\partial t^2} - c_s^2\triangle\varphi(t,\vec{r}) = 0.$$

VI.1.2 Sound waves in a moving fluid

Doppler effect!

VI.1.3 Riemann problem. Rarefaction waves

Should be added at some point

VI.1.4 Absorption of sound waves

In chapter V, we only considered incompressible motions of Newtonian fluids, so that bulk viscosity could from the start play no role. The simplest example of compressible flow is that of sound waves. As in § VI.1.1, we consider small adiabatic perturbations of a fluid initially at rest and with uniform properties—which implies that external volume forces like gravity are neglected. Accordingly, the dynamical fields characterizing the fluid are

$$\rho(t,\vec{r}) = \rho_0 + \delta\rho(t,\vec{r}), \quad \mathcal{P}(t,\vec{r}) = \mathcal{P}_0 + \delta\mathcal{P}(t,\vec{r}), \quad \vec{\mathrm{v}}(t,\vec{r}) = \vec{0} + \delta\vec{\mathrm{v}}(t,\vec{r}), \tag{VI.11a}$$

with

$$|\delta\rho(t,\vec{r})| \ll \rho_0, \quad |\delta\mathcal{P}(t,\vec{r})| \ll \mathcal{P}_0, \quad \big|\delta\vec{\mathrm{v}}(t,\vec{r})\big| \ll c_s, \tag{VI.11b}$$

where c_s denotes the quantity which in the perfect-fluid case was found to coincide with the phase velocity of similar small perturbations, i.e. the "speed of sound", defined by Eq. (VI.5)

$$c_s^2 \equiv \left(\frac{\partial\mathcal{P}}{\partial\rho}\right)_{S,N}. \tag{VI.11c}$$

As in § VI.1.1, this partial derivative will allow us to relate the pressure perturbation $\delta\mathcal{P}$ to the variation of mass density $\delta\rho$.

[35] This traditional denomination is totally out of place in a chapter in which there are several types of waves, each of which has its own governing "wave equation". Yet historically, due to its role for electromagnetic or sound waves, it is *the* archetypal wave equation, while the equations governing other types of waves often have a specific name.

Remark: Anticipating on later findings, the perturbations must actually fulfill a further condition, related to the size of their spatial variations [cf. Eq. (VI.21)]. This is nothing but the assumption of "small gradients" that underlies the description of their propagation with the Navier–Stokes equation, i.e. with first-order dissipative fluid dynamics.

For the sake of simplicity, we consider a one-dimensional problem, i.e. perturbations propagating along the x-direction and independent of y and z—as are the properties of the underlying background fluid. Under this assumption, the continuity equation (III.9) reads

$$\frac{\partial\rho(t,x)}{\partial t}+\rho(t,x)\frac{\partial\delta\mathsf{v}(t,x)}{\partial x}+\delta\mathsf{v}(t,x)\frac{\partial\rho(t,x)}{\partial x}=0, \tag{VI.12a}$$

while the Navier–Stokes equation (III.32) becomes

$$\rho(t,x)\left[\frac{\partial\delta\mathsf{v}(t,x)}{\partial t}+\delta\mathsf{v}(t,x)\frac{\partial\delta\mathsf{v}(t,x)}{\partial x}\right]=-\frac{\partial\delta\mathcal{P}(t,x)}{\partial x}+\left(\frac{4}{3}\eta+\zeta\right)\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial x^2}. \tag{VI.12b}$$

Substituting the fields (VI.11a) in these equations and linearizing the resulting equations so as to keep only the leading order in the small perturbations, one finds

$$\frac{\partial\delta\rho(t,x)}{\partial t}+\rho_0\frac{\partial\delta\mathsf{v}(t,x)}{\partial x}=0, \tag{VI.13a}$$

$$\rho_0\frac{\partial\delta\mathsf{v}(t,x)}{\partial t}=-\frac{\partial\delta\mathcal{P}(t,x)}{\partial x}+\left(\frac{4}{3}\eta+\zeta\right)\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial x^2}. \tag{VI.13b}$$

In the second equation, the derivative $\partial(\delta\mathcal{P})/\partial x$ can be replaced by $c_s^2\,\partial(\delta\rho)/\partial x$. Let us in addition introduce the (traditional) notation[(36)]

$$\bar\nu\equiv\frac{1}{\rho_0}\left(\frac{4}{3}\eta+\zeta\right), \tag{VI.14}$$

so that Eq. (VI.13b) can be rewritten as

$$\rho_0\frac{\partial\delta\mathsf{v}(t,x)}{\partial t}+c_s^2\frac{\partial\delta\rho(t,x)}{\partial x}=\rho_0\bar\nu\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial x^2}. \tag{VI.15}$$

Subtracting c_s^2 times the derivative of Eq. (VI.13a) with respect to x from the time derivative of Eq. (VI.15) and dividing the result by ρ_0 then yields

$$\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial t^2}-c_s^2\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial x^2}=\bar\nu\frac{\partial^3\delta\mathsf{v}(t,x)}{\partial t\,\partial x^2}. \tag{VI.16a}$$

One easily checks that the mass density variation obeys a similar equation

$$\boxed{\frac{\partial^2\delta\rho(t,x)}{\partial t^2}-c_s^2\frac{\partial^2\delta\rho(t,x)}{\partial x^2}=\bar\nu\frac{\partial^3\delta\rho(t,x)}{\partial t\,\partial x^2}.} \tag{VI.16b}$$

In the perfect-fluid case $\bar\nu=0$, one recovers the traditional wave equation (VI.10a).

Equations (VI.16) are homogeneous linear partial differential equations, whose solutions can be written as superposition of plane waves. Accordingly, let us substitute the Fourier ansatz

$$\delta\rho(t,x)=\widetilde{\delta\rho}(\omega,k)\,\mathrm{e}^{-\mathrm{i}(\omega t-kx)} \tag{VI.17}$$

in Eq. (VI.16b). This yields after some straightforward algebra the dispersion relation

$$\boxed{\omega^2=c_s^2k^2-\mathrm{i}\omega k^2\bar\nu.} \tag{VI.18}$$

Obviously, ω and k cannot be simultaneously (non-zero) real numbers.

[(36)] Introducing the kinetic shear resp. bulk viscosity coefficients ν resp. ν', one has $\bar\nu=\frac{4}{3}\nu+\nu'$, hence the notation.

Let us assume $k \in \mathbb{R}$ and $\omega = \omega_r + \mathrm{i}\omega_i$, where ω_r, ω_i are real. The dispersion relation becomes

$$\omega_r^2 - \omega_i^2 + 2\mathrm{i}\omega_r\omega_i = c_s^2 k^2 - \mathrm{i}\omega_r k^2 \bar{\nu} + \omega_i k^2 \bar{\nu},$$

which can only hold if both the real and imaginary parts are separately equal. The identity between the imaginary parts reads (for $\omega_r \neq 0$)

$$\omega_i = -\frac{1}{2}\bar{\nu} k^2, \tag{VI.19}$$

which is always negative, since $\bar{\nu}$ is non-negative. This term yields in the Fourier ansatz (VI.17) an exponentially decreasing factor $\mathrm{e}^{-\mathrm{i}(\mathrm{i}\omega_i)t} = \mathrm{e}^{-\bar{\nu}k^2 t/2}$ which represents the *damping* or *absorption* of the sound wave. The perturbations with larger wave number k, i.e. corresponding to smaller length scales, are more dampened that those with smaller k. This is quite natural, since a larger k also means a larger gradient, thus an increased influence of the viscous term in the Navier–Stokes equation.

In turn, the identity between the real parts of the dispersion relation yields

$$\omega_r^2 = c_s^2 k^2 - \frac{1}{4}\bar{\nu}^2 k^4. \tag{VI.20}$$

This gives for the phase velocity $c_\varphi \equiv \omega/k$ of the traveling waves

$$c_\varphi^2 = c_s^2 - \frac{1}{4}\bar{\nu}^2 k^2. \tag{VI.21}$$

That is, the "speed of sound" actually depends on its wave number k, and is smaller for small wavelength, i.e. high-k, perturbations—which are also those which are more damped out.

Relation (VI.21) also shows that the whole linear description adopted below Eqs. (VI.12) requires that the perturbations should have a relatively large wavelength, namely such that $k \ll 2c_s/\bar{\nu}$, so that c_φ remain real-valued. This is equivalent to requesting that the dissipative term $\bar{\nu}\triangle\delta\mathsf{v} \sim k^2\bar{\nu}\delta\mathsf{v}$ in the Navier–Stokes equation (VI.13b) should be much smaller than the term describing the local acceleration, $\partial_t\delta\mathsf{v} \sim \omega\delta\mathsf{v} \sim c_s k\delta\mathsf{v}$.

Remarks:

$*$ Instead of considering "temporal damping" as was done above by assuming $k \in \mathbb{R}$ but $\omega \in \mathbb{C}$, one may investigate "spatial damping", i.e. assume $\omega \in \mathbb{R}$ and put the whole complex dependence in the wave number $k = k_r + \mathrm{i}k_i$. For (angular) frequencies ω much smaller than the inverse of the typical time scale $\tau_\nu \equiv \bar{\nu}/c_s^2$, one finds

$$\omega^2 \simeq c_s^2 k_r^2 \left(1 + \frac{3}{4}\omega^2\tau_\nu^2\right) \quad \Leftrightarrow \quad c_\varphi \equiv \frac{\omega}{k_r} \simeq c_s\left(1 + \frac{3}{8}\omega^2\tau_\nu^2\right)$$

i.e. the phase velocity increases with the frequency, and on the other hand

$$k_i \simeq \frac{\bar{\nu}\omega^2}{2c_s^3}. \tag{VI.22}$$

The latter relation is known as *Stokes' law of sound attenuation*, k_i representing the inverse of the typical distance over which the sound wave amplitude decreases, due to the factor $\mathrm{e}^{\mathrm{i}(\mathrm{i}k_i)x} = \mathrm{e}^{-k_i x}$ in the Fourier ansatz (VI.17). Larger frequencies are thus absorbed on a smaller distance from the source of the sound wave.

> Substituting $k = k_r + \mathrm{i}k_i = k_r(1 + \mathrm{i}\varkappa)$ in the dispersion relation (VI.18) and writing the identity of the real and imaginary parts, one obtains the system
>
> $$\begin{cases} 2\varkappa = \omega\tau_\nu(1 - \varkappa^2) \\ \omega^2 = c_s^2 k_r^2(1 + 2\omega\tau_\nu\varkappa - \varkappa^2) \end{cases}$$
>
> The first equation is a quadratic equation in $\varkappa$ that admits one positive and one negative solution: the latter can be rejected, while the former is $\varkappa \simeq \omega\tau_\nu/2 + \mathcal{O}\big((\omega\tau_\nu)^2\big)$. Inserting it in the second equation leads to the wanted results. □

An exact solution of the system of equations exists, yes it is neither enlightening mathematically, nor relevant from the physical point of view in the general case, as discussed in the next remark.

One may naturally also analyze the general case in which both ω and k are complex numbers. In any case, the phase velocity is given by $c_\varphi \equiv \omega/k_r$, although it is more difficult to recognize the physical content of the mathematical relations.

$*$ For air or water, the reduced kinetic viscosity $\bar{\nu}$ is of order 10^{-6}–10^{-5} $\mathrm{m}^2\cdot\mathrm{s}^{-1}$. With speeds of sound $c_s \simeq 300$–1500 $\mathrm{m}\cdot\mathrm{s}^{-1}$, this yields typical time scales τ_ν of order 10^{-12}–10^{-10} s. That is, the change in the speed of sound (VI.21), or equivalently deviations from the assumption $\omega\tau_\nu \ll 1$ underlying the attenuation coefficient (VI.22), become relevant for sound waves in the gigahertz/terahertz regime(!). This explains why measuring the bulk viscosity is a non-trivial task.

The wavelengths $c_s\tau_\nu$ corresponding to the above frequencies τ_ν^{-1} are of order 10^{-9}–10^{-7} m. This is actually not far from the value of the mean free path in classical fluids, so that the whole description as a continuous medium starts being questionable.

VI.2 Shock waves

When the amplitude of the perturbations considered in Sec. VI.1 cannot be viewed as small, as for instance if $|\delta\vec{\mathsf{v}}| \ll c_s$ does not hold, the linearization of the equations of motion (VI.3) is no longer licit, and the nonlinear terms of the Euler equation play a role.

A possibility is then that at a finite time t in the evolution of the fluid, a discontinuity in some of the fields may appear, referred to as shock wave.[lvii] How this may arise will be discussed in the case of a one-dimensional problem (§ VI.2.1). At a discontinuity, the differential formulation of the conservation laws derived in Chap. III no longer holds, and it becomes necessary to study the conservation of mass, momentum and energy across the surface of discontinuity associated with the shock wave (§ VI.2.2).

VI.2.1 Formation of a shock wave in a one-dimensional flow

As in § VI.1.1, we consider the propagation of an adiabatic perturbation of a background fluid at rest, in the absence of gravity or of other external volume forces. In the one-dimensional case, the dynamical equations (VI.3) read

$$\frac{\partial\rho(t,x)}{\partial t} + \rho(t,x)\frac{\partial\delta\mathsf{v}(t,x)}{\partial x} + \delta\mathsf{v}(t,x)\frac{\partial\rho(t,x)}{\partial x} = 0, \tag{VI.23a}$$

$$\rho(t,x)\left[\frac{\partial\delta\mathsf{v}(t,x)}{\partial t} + \delta\mathsf{v}(t,x)\frac{\partial\delta\mathsf{v}(t,x)}{\partial x}\right] + \frac{\partial\delta\mathcal{P}(t,x)}{\partial x} = 0. \tag{VI.23b}$$

The variation of the pressure $\delta\mathcal{P}(t,x)$ can again be expressed in terms of the variation in the mass density $\delta\rho(t,x)$ by invoking a Taylor expansion [cf. the paragraph between Eqs. (VI.4) and (VI.5)]. Since the perturbation of the background "flow" is no longer small, the thermodynamic state around which this Taylor expansion is performed is not necessarily that corresponding to the unperturbed fluid, but rather an arbitrary state, so that

$$\delta\mathcal{P}(t,x) \simeq c_s\big(\rho(t,x)\big)^2\delta\rho(t,x), \tag{VI.24}$$

where the speed of sound is that in the perturbed flow. When differentiating this identity, the derivative of $\delta\rho(t,x)$ with respect to x is also the derivative of $\rho(t,x)$, since the unperturbed fluid state is uniform. Accordingly, one may recast Eqs. (VI.23) as

$$\frac{\partial\rho(t,x)}{\partial t} + \rho(t,x)\frac{\partial\delta\mathsf{v}(t,x)}{\partial x} + \delta\mathsf{v}(t,x)\frac{\partial\rho(t,x)}{\partial x} = 0, \tag{VI.25a}$$

[lvii] *Stoßwelle*

$$\rho(t,x)\left[\frac{\partial\delta\mathsf{v}(t,x)}{\partial t}+\delta\mathsf{v}(t,x)\frac{\partial\delta\mathsf{v}(t,x)}{\partial x}\right]+c_s(\rho)^2\frac{\partial\rho(t,x)}{\partial x}=0, \tag{VI.25b}$$

which constitutes a system of two coupled partial differential equations for the two unknown fields $\rho(t,x)$ and $\delta\mathsf{v}(t,x)=\mathsf{v}(t,x)$.

To deal with these equations, one may assume that the mass density and the flow velocity have parallel dependences on time and space—as suggested by the fact that this property holds in the linearized case of sound waves, in which both $\rho(t,\vec{r})$ and $\vec{\mathsf{v}}(t,\vec{r})$ propagate with the same phase $(c_s|\vec{k}|t+\vec{k}\cdot\vec{r})$. Thus, the dependence of v on t and x is replaced with a functional dependence $\mathsf{v}\big(\rho(t,x)\big)$, with the known value $\mathsf{v}(\rho_0)=0$ corresponding to the unperturbed fluid at rest. Accordingly, the partial derivatives of the flow velocity with respect to t resp. x become

$$\frac{\partial\mathsf{v}(t,x)}{\partial t}=\frac{\mathrm{d}\mathsf{v}(\rho)}{\mathrm{d}\rho}\frac{\partial\rho(t,x)}{\partial t}\qquad\text{resp.}\qquad\frac{\partial\mathsf{v}(t,x)}{\partial x}=\frac{\mathrm{d}\mathsf{v}(\rho)}{\mathrm{d}\rho}\frac{\partial\rho(t,x)}{\partial x}.$$

The latter identities may then be inserted in Eqs. (VI.25). If one further multiplies Eq. (VI.25a) by $\rho(t,x)\,\mathrm{d}\mathsf{v}(\rho)/\mathrm{d}\rho$ and then subtracts Eq. (VI.25b) from the result, there comes

$$\left\{\rho^2\left[\frac{\mathrm{d}\mathsf{v}(\rho)}{\mathrm{d}\rho}\right]^2-c_s(\rho)^2\right\}\frac{\partial\rho(t,x)}{\partial x}=0,$$

that is, discarding the trivial solution of a uniform mass density,

$$\frac{\mathrm{d}\mathsf{v}(\rho)}{\mathrm{d}\rho}=\pm\frac{c_s(\rho)}{\rho}. \tag{VI.26}$$

The equations (VI.25)-(VI.26) are invariant under the simultaneous changes $\mathsf{v}\to-\mathsf{v}$, $x\to-x$, and $c_s\to-c_s$. Accordingly, one may restrict the discussion of Eq. (VI.26) to the case with a $+$ sign—the $-$ case amounts to considering a wave propagating in the opposite direction with the opposite velocity. The flow velocity is then formally given by

$$\mathsf{v}(\rho)=\int_{\rho_0}^{\rho}\frac{c_s(\rho')}{\rho'}\,\mathrm{d}\rho',$$

where we used $\mathsf{v}(\rho_0)=0$, while Eq. (VI.25b) can be rewritten as

$$\frac{\partial\rho(t,x)}{\partial t}+\big[\mathsf{v}\big(\rho(t,x)\big)+c_s\big(\rho(t,x)\big)\big]\frac{\partial\rho(t,x)}{\partial x}=0. \tag{VI.27}$$

Assuming that the mass density perturbation locally propagates as a traveling wave, i.e. making the ansatz[37] $\delta\rho(t,x)\propto f(x-c_wt)$ in Eq. (VI.27), then its phase velocity c_w will be given by $c_w=c_s(\rho)+\mathsf{v}$. Invoking Eq. (VI.26) then shows that $\mathrm{d}\mathsf{v}(\rho)/\mathrm{d}\rho>0$, so that c_w grows with increasing mass density: the denser regions in the fluid will propagate faster than the rarefied ones and possibly catch up with them—in case the latter where "in front" of the propagating perturbation—as illustrated in Fig. VI.1. In particular, there may arise after a finite amount of time a *discontinuity* of the function $\rho(t,x)$ at a given point x_0. The (propagating) point where this discontinuity takes place represents the front of a *shock wave*.

VI.2.2 Jump equations at a surface of discontinuity

To characterize the properties of a flow in the region of a shock wave, one needs first to specify the behavior of the physical quantities of relevance at the discontinuity, which is the object of this section. Generalizing the finding of the previous section in a one-dimensional setup, in which the discontinuity arises at a single (traveling) point, in the three-dimensional case there will be a whole *surface of discontinuity*,[lviii] that propagates in the unperturbed background fluid.

[37] This form is to be seen as the *local* form of the solution, not as a globally valid solution.

[lviii] *Unstetigkeitsfläche*

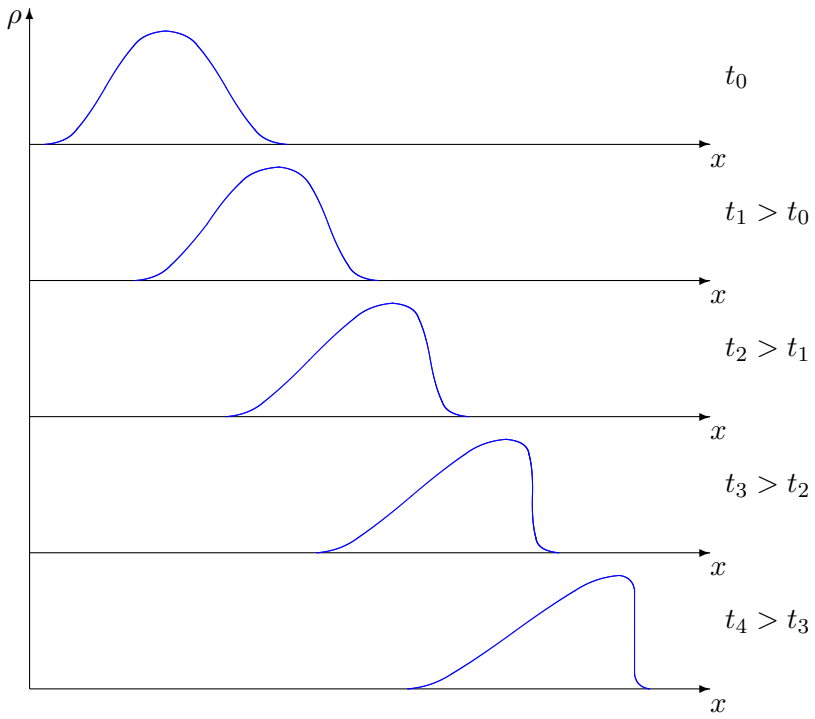

Figure VI.1 – Schematic representation of the evolution in time of the spatial distribution of dense and rarefied regions leading to a shock wave.

For the sake of brevity, the dependence on t and $\vec{r}$ of the various fields of interest will be omitted.

To describe the physics at the front of the shock wave, we adopt a comoving reference frame $\mathcal{R}$, which moves with the surface of discontinuity, and in this reference frame we consider a system of Cartesian coordinates (x^1, x^2, x^3) with the basis vector $\vec{e}_1$ perpendicular to the propagating surface. The region in front resp. behind the surface will be denoted by $(+)$ resp. $(-)$; that is, the fluid in which the shock waves propagates flows from the $(+)$- into the $(-)$-region: the former is upstream, the latter downstream.

The jump of a local physical quantity $g(\vec{r})$ across the surface of discontinuity is defined as

$$[\![g]\!] \equiv g_+ - g_-, \tag{VI.28}$$

where g_+ resp. g_- denotes the limiting value of g as $x^1 \to 0^+$ resp. $x^1 \to 0^-$. In case such a local quantity is actually continuous at the surface of discontinuity, then its jump across the surface vanishes.

At a surface of discontinuity $\mathcal{S}_\mathrm{d}$, the flux densities of mass, momentum, and energy across the surface, i.e. along the x^1-direction, must be continuous, so that mass, momentum, and energy remain locally conserved. These requirements are expressed by the *jump equations*[(lix)]

$$[\![\rho \mathsf{v}^1]\!] = 0, \tag{VI.29a}$$

$$[\![\mathbf{T}^{i1}]\!] = 0 \quad \forall i = 1, 2, 3, \tag{VI.29b}$$

$$\left[\!\!\left[\left(\frac{1}{2}\rho \vec{\mathsf{v}}^2 + e + \mathcal{P} \right) \mathsf{v}^1 \right]\!\!\right] = 0, \tag{VI.29c}$$

where the momentum flux density tensor has components $\mathbf{T}^{ij} = \mathcal{P} g^{ij} + \rho \mathsf{v}^i \mathsf{v}^j$ [see Eq. (III.21b)], with $g^{ij} = \delta^{ij}$ in the case of Cartesian coordinates.

(lix) *Sprunggleichungen*

The continuity of the mass flux density across the surface of discontinuity (VI.29a) can be recast as

$$(\rho \mathsf{v}^1)_- = (\rho \mathsf{v}^1)_+ \equiv j_1. \tag{VI.30}$$

A first, trivial solution arises if there is no flow of matter across the surface of discontinuity $\mathcal{S}_\mathrm{d}$, i.e. if $(\mathsf{v}^1)_+ = (\mathsf{v}^1)_- = 0$. In that case, Eq. (VI.29c) is automatically satisfied. Condition (VI.29b) for $i = 1$ becomes $[\![\mathcal{P}]\!] = 0$, i.e. the pressure is the same on both sides of $\mathcal{S}_\mathrm{d}$. Eventually, Eq. (VI.29b) with $i = 2$ or 3 holds automatically. All in all, there is no condition on the behavior of ρ, v^2 or v^3 across the surface of discontinuity—which means that these quantities may be continuous or not, in the latter case with an arbitrary jump.

If j_1 does not vanish, that is if some matter flows across $\mathcal{S}_\mathrm{d}$, then the jump equation for the component $\mathbf{T}^{21} = \rho \mathsf{v}^2 \mathsf{v}^1$ resp. $\mathbf{T}^{31} = \rho \mathsf{v}^3 \mathsf{v}^1$ leads to $[\![\mathsf{v}^2]\!] = 0$ resp. $[\![\mathsf{v}^3]\!] = 0$, i.e. the component v^2 resp. v^3 is continuous across the surface of discontinuity:

$$(\mathsf{v}^2)_- = (\mathsf{v}^2)_+ \quad \text{resp.} \quad (\mathsf{v}^3)_- = (\mathsf{v}^3)_+. \tag{VI.31}$$

In turn, rewriting the jump equation for $\mathbf{T}^{11} = \mathcal{P} + \rho(\mathsf{v}^1)^2$ with the help of j_1 yields

$$\mathcal{P}_- - \mathcal{P}_+ = j_1 \big[(\mathsf{v}^1)_+ - (\mathsf{v}^1)_-\big] = j_1^2 \left(\frac{1}{\rho_+} - \frac{1}{\rho_-}\right). \tag{VI.32}$$

Thus if $\rho_+ < \rho_-$, i.e. if the fluid is denser in the $(-)$-region downstream—as suggested by Fig. VI.1—, then $\mathcal{P}_- > \mathcal{P}_+$, while relation (VI.30) yields $(\mathsf{v}^1)_+ > (\mathsf{v}^1)_-$:[(38)]

$$\rho_- > \rho_+ \ , \quad \mathcal{P}_- > \mathcal{P}_+ \ , \quad (\mathsf{v}^1)_+ > (\mathsf{v}^1)_-. \tag{VI.33}$$

By invoking the necessary non-decrease of entropy, one can show (see Ref. [4], § 87) that these inequalities indeed hold.

Combining Eqs. (VI.30) and (VI.32) yields

$$\big[(\mathsf{v}^1)_+\big]^2 = \frac{j_1^2}{\rho_+^2} = \frac{\mathcal{P}_- - \mathcal{P}_+}{\rho_- - \rho_+} \frac{\rho_- \rho_+}{\rho_+^2} = \frac{\mathcal{P}_- - \mathcal{P}_+}{\rho_- - \rho_+} \frac{\rho_-}{\rho_+}$$

and similarly

$$\big[(\mathsf{v}^1)_-\big]^2 = \frac{\mathcal{P}_- - \mathcal{P}_+}{\rho_- - \rho_+} \frac{\rho_+}{\rho_-}.$$

If the jumps in pressure and mass density are small, one can show that their ratio is approximately the derivative $\partial \mathcal{P}/\partial \rho$, here at constant entropy and particle number, evaluated in the vicinity of the point where the jump occurs, i.e.

$$\big[(\mathsf{v}^1)_+\big]^2 \simeq \left(\frac{\partial \mathcal{P}}{\partial \rho}\right)_{S,N} \frac{\rho_-}{\rho_+} = \frac{\rho_-}{\rho_+} c_s^2, \qquad \big[(\mathsf{v}^1)_-\big]^2 \simeq \frac{\rho_+}{\rho_-} c_s^2.$$

With $\rho_- > \rho_+$ comes $(\mathsf{v}^1)_+ > c_s$ resp. $(\mathsf{v}^1)_- < c_s$ upstream resp. downstream of the shock front.[(39)] The former identity means that an observer comoving with the surface of discontinuity sees in front of her/him a fluid flowing with a supersonic velocity, that is, going temporarily back to a reference frame bound to the unperturbed fluid, the shock wave moves with a supersonic velocity.

[(38)] Conversely, $\rho_+ > \rho_-$ would lead to $\mathcal{P}_- < \mathcal{P}_+$ and $(\mathsf{v}^1)_+ < (\mathsf{v}^1)_-$.

[(39)] Here we are being a little sloppy: one should consider the right ($x^1 \to 0^+$) and left ($x^1 \to 0^-$) derivatives, corresponding respectively to the $(+)$ and $(-)$-regions, and thus find the associated speeds of sound $(c_s)_+$ and $(c_s)_-$ instead of a single c_s.

Invoking the continuity across $\mathcal{S}_\mathrm{d}$ of the product $\rho\mathsf{v}^1$ and of the components v^2, v^3 parallel to the surface of discontinuity, the jump equation (VI.29c) for the energy flux density simplifies to

$$\left[\!\!\left[\frac{1}{2}(\mathsf{v}^1)^2+\frac{e+\mathcal{P}}{\rho}\right]\!\!\right]=\frac{j_1^2}{2}\left(\frac{1}{\rho_+^2}-\frac{1}{\rho_-^2}\right)+\frac{e_++\mathcal{P}_+}{\rho_+}-\frac{e_-+\mathcal{P}_-}{\rho_-}=0.$$

Expressing j_1^2 with the help of Eq. (VI.32), one finds

$$\frac{\mathcal{P}_--\mathcal{P}_+}{2}\left(\frac{1}{\rho_+}+\frac{1}{\rho_-}\right)=\frac{w_-}{\rho_-}-\frac{w_+}{\rho_+} \tag{VI.34a}$$

with $w=e+\mathcal{P}$ the enthalpy density, or equivalently

$$\frac{\mathcal{P}_-+\mathcal{P}_+}{2}\left(\frac{1}{\rho_+}-\frac{1}{\rho_-}\right)=\frac{e_+}{\rho_+}-\frac{e_-}{\rho_-}. \tag{VI.34b}$$

Either of these equations represents a relation between the thermodynamic quantities on both sides of the surface of discontinuity, and defines in the space of the thermodynamic states of the fluid a so-called *shock adiabatic curve*, also referred to as *dynamical adiabatic curve*[lx] or *Hugoniot*[al] *adiabatic curve*, or *Rankine*[am]*–Hugoniot relation*.

More generally, Eqs. (VI.30)–(VI.34) relate the dynamical fields on both sides of the surface of discontinuity associated with a shock wave, and constitute the practical realization of the continuity conditions encoded in the jump equations (VI.29).

VI.3 Gravity waves

In this section, we investigate waves that are "driven" by gravity, in the sense that the latter is the main force that acts to bring back the perturbed fluid to its unperturbed, "background" state. Such perturbations are generically referred to as *gravity waves*.[lxi]

A first example is that of small perturbations at the free surface of a liquid originally at rest—the "waves" of everyday language. In that case, some external source, as e.g. wind or an earthquake, leads to a local rise of the fluid above its equilibrium level: gravity then acts against this rise and tends to bring back the liquid to its equilibrium position. In case the elevation caused by the perturbation is small compared to the sea depth, as well as in comparison to the perturbation wavelength, one has linear sea surface waves (§ VI.3.1). Another interesting case arises in shallow water, for perturbations whose horizontal extent is much larger than their vertical size, in which case one may find so-called *solitary waves* (§ VI.3.2).

Throughout this section, the flows—comprised of a background fluid at rest and the traveling perturbation—are supposed to be two-dimensional, with the x-direction along the propagation direction and the z-direction along the vertical, oriented upwards so that the acceleration due to gravity is $\vec{g}=-g\,\vec{e}_z$. The origin $z=0$ is taken at the bottom of the sea / ocean, which for the sake of simplicity is assumed to be flat.

VI.3.1 Linear sea surface waves

A surface wave is a perturbation of the altitude—with respect to the sea bottom—of the free surface of the sea. The latter is displaced by an amount $\delta h(t,x)$ from its equilibrium position h_0, corresponding to a fluid at rest with a horizontal free surface. These variations in the position of

[lx] *dynamische Adiabate* [lxi] *Schwerewellen*

[al] P. H. Hugoniot, 1851–1887 [am] W. J. M. Rankine, 1820–1872

the free surface signal the motion of the sea water, i.e. a flow, with a corresponding flow velocity throughout the sea $\vec{\mathsf{v}}(t,x,z)$.

We shall model this motion as vorticity-free, which allows us to introduce a velocity potential $\varphi(t,x,z)$, and assume that the mass density ρ of the sea water remains constant and uniform, i.e. we neglect its compressibility. The sea is supposed to occupy an unbounded region of space, which is a valid assumption if one is far from any coast.

Note that the "background flow" [see Eqs. (VI.1)] is characterized by a uniform constant ρ, a uniform velocity potential $\varphi = \varphi_0$—giving rise to a vanishing flow velocity—, and a pressure obeying Pascal's law (IV.4).

VI.3.1 a Equations of motion and boundary conditions

Under the above assumptions, the equations of motion read [cf. Eqs. (IV.31) and (IV.32)]

$$-\frac{\partial\varphi(t,x,z)}{\partial t} + \frac{\left[\vec{\nabla}\varphi(t,x,z)\right]^2}{2} + \frac{\mathcal{P}(t,x,z)}{\rho} + gz = \text{constant},\ ^{(40)} \tag{VI.35a}$$

where gz is the potential energy per unit mass of water, and

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right]\varphi(t,x,z) = 0. \tag{VI.35b}$$

To fully specify the problem, boundary conditions are still needed. As in the generic case for potential flow (Sec. IV.4), these will be Neumann boundary conditions, involving the derivative of the velocity potential.

- At the bottom of the sea, the water can have no vertical motion, corresponding to the usual impermeability condition; that is

$$\mathsf{v}_z(z{=}0) = -\left.\frac{\partial\varphi}{\partial z}\right|_{z=0} = 0. \tag{VI.36a}$$

- At the free surface of the sea, the vertical component v_z of the flow velocity equals the velocity of the surface, i.e. it equals the rate of change of the position of the (material!) surface:

$$-\left.\frac{\partial\varphi(t,x,z)}{\partial z}\right|_{z=h_0+\delta h(t,x)} = \frac{\mathrm{D}\delta h(t,x)}{\mathrm{D}t}.$$

  Using $\dfrac{\mathrm{D}}{\mathrm{D}t} = \dfrac{\partial}{\partial t} + \mathsf{v}_x\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial t} - \dfrac{\partial\varphi}{\partial x}\dfrac{\partial}{\partial x}$, this gives

$$\left[\frac{\partial\varphi(t,x,z)}{\partial z} + \frac{\partial\delta h(t,x)}{\partial t} - \frac{\partial\delta h(t,x)}{\partial x}\frac{\partial\varphi(t,x,z)}{\partial x}\right]_{z=h_0+\delta h(t,x)} = 0. \tag{VI.36b}$$

- At the free surface of the sea, the pressure on the water side—right below the surface—is directly related to that just above the surface. The latter is assumed to be constant and equal to some value $\mathcal{P}^{(0)}$, which represents for instance the atmospheric pressure "at sea level". As a first approximation—whose physical content will be discussed in the remark at the end of this paragraph—, the pressure is equal on both sides of the sea surface:

$$\mathcal{P}\big(t,x,z{=}h_0{+}\delta h(t,x)\big) = \mathcal{P}^{(0)}. \tag{VI.36c}$$

  Expressing the pressure with the help of Eq. (VI.35a), this condition may be recast as

$$\left[-\frac{\partial\varphi(t,x,z)}{\partial t} + \frac{\left[\vec{\nabla}\varphi(t,x,z)\right]^2}{2}\right]_{z=h_0+\delta h(t,x)} + g\,\delta h(t,x) = -\frac{\mathcal{P}^{(0)}}{\rho} - gh_0 + \text{constant}, \tag{VI.36d}$$

  where the whole right hand side of the equation represents a new constant.

(40) Here and in Eq. (VI.36d) and (VI.40b) below, the unspecified constant is in fact time dependent, yet this plays no role for the further calculations.

Hereafter we look for solutions consisting of a velocity potential $\varphi(t,x,z)$ and a surface profile $\delta h(t,x)$, as determined by Eqs. (VI.35) with conditions (VI.36).

Remark: The assumption of an identical pressure on both sides of an interface—either between two immiscible liquids, or between a liquid and a gas, as here—is generally *not* warranted, unless the interface happens to be flat. If there is the least curvature, the *surface tension* associated with the interface will lead to a larger pressure inside the concavity of the interface. Neglecting this effect—which we shall consider again in § VI.3.2—is valid only if the typical radius of curvature of the interface, which as we shall see below is the wavelength of the surface waves, is "large", especially with respect to the deformation scale δh.

VI.3.1 b Harmonic wave assumption

Since the domain on which the wave propagates is assumed to be unbounded in the x-direction, a natural ansatz for the solution of the Laplace equation (VI.35b) is that of a harmonic wave

$$\varphi(t,x,z) = f(z)\,\cos(kx-\omega t) \tag{VI.37}$$

propagating in the x-direction with a depth-dependent amplitude $f(z)$. Inserting this form in the Laplace equation yields the linear ordinary differential equation

$$\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2} - k^2 f(z) = 0,$$

whose obvious solution is $f(z) = a_1\,\mathrm{e}^{kz} + a_2\,\mathrm{e}^{-kz}$, with a_1 and a_2 two real constants.

The boundary condition (VI.36a) at the sea bottom $z=0$ gives $a_1 = a_2$, i.e.

$$\varphi(t,x,z) = C\cosh(kz)\cos(kx-\omega t), \tag{VI.38}$$

with C a real constant.

To make further progress with the equations of the system, and in particular to determine the profile of the free surface, further assumptions are needed, so as to obtain simpler equations. We shall now present a first such simplification, leading to linear waves. In § VI.3.2, another simplification—of a more complicated starting point—will be considered, which gives rise to (analytically tractable) nonlinear waves.

VI.3.1 c Linear waves

As in the case of sound waves, we now assume that the perturbations are "small", so as to be able to linearize the equations of motion and those expressing boundary conditions. Thus, we shall assume that the quadratic term $(\vec{\nabla}\varphi)^2$ is much smaller than $|\partial\varphi/\partial t|$, and that the displacement δh of the free surface from its rest position is much smaller than the equilibrium sea depth h_0.

> To fix ideas the "swell waves" observed far from any coast on the Earth oceans or seas have a typical wavelength λ of about 100 m and an amplitude δh_0 of 10 m or less—the shorter the wavelength, the smaller the amplitude—, while the typical sea/ocean depth h_0 is 1–5 km.

The assumption $(\vec{\nabla}\varphi)^2 \ll |\partial\varphi/\partial t|$ can on the one hand be made in Eq. (VI.35a), leading to

$$-\frac{\partial\varphi(t,x,z)}{\partial t} + \frac{\mathcal{P}(t,x,z)}{\rho} + gz = \frac{\mathcal{P}^{(0)}}{\rho} + gh_0, \tag{VI.39}$$

in which the right member represents the zeroth order, while the left member also contains first order terms, which must cancel each other for the identity to hold. On the other hand, taking also into account the assumption $|\delta h(t,x)| \ll h_0$, the boundary conditions (VI.36b) and (VI.36d) at the free surface of the sea can be rewritten as

$$\left.\frac{\partial\varphi(t,x,z)}{\partial z}\right|_{z=h_0} + \frac{\partial\delta h(t,x)}{\partial t} = 0 \tag{VI.40a}$$

and

$$-\frac{\partial\varphi(t,x,z)}{\partial t}\bigg|_{z=h_0} + g\,\delta h(t,x) = \text{constant}, \tag{VI.40b}$$

respectively. Together with the Laplace differential equation (VI.35b) and the boundary condition at the sea bottom (VI.36a), the two equations (VI.40) constitute the basis of the *Airy*[(an)] *linear wave theory*.

Combining the latter two equations yields at once the condition

$$\left[\frac{\partial^2\varphi(t,x,z)}{\partial t^2} + g\frac{\partial\varphi(t,x,z)}{\partial z}\right]_{z=h_0} = 0.$$

Using the velocity potential (VI.38), this relation reads

$$-\omega^2 C\cosh(kh_0)\cos(kx-\omega t) + gkC\sinh(kh_0)\cos(kx-\omega t) = 0,$$

resulting in the *dispersion relation*

$$\omega^2 = gk\tanh(kh_0). \tag{VI.41}$$

This relation becomes even simpler in two limiting cases:

- When $kh_0 \gg 1$, or equivalently $h_0 \gg \lambda$ where $\lambda = 2\pi/k$ denotes the wavelength, which represents the case of gravity waves at the surface of deep sea,[(41)] then $\tanh(kh_0) \simeq 1$. In that case, the dispersion relation simplifies to $\omega^2 = gk$: the phase and group velocity of the traveling waves are

  $$c_\varphi = \frac{\omega}{k} = \sqrt{\frac{g}{k}} \quad \text{and} \quad c_g = \frac{\mathrm{d}\omega(k)}{\mathrm{d}k} = \frac{1}{2}\sqrt{\frac{g}{k}} \tag{VI.42}$$

  respectively. Both are independent from the sea depth h_0, yet do depend on the angular wavenumber k of the wave, so that waves with different wavelengths propagate with different velocities.

- For $kh_0 \ll 1$, i.e. in the case of a shallow sea with $h_0 \ll \lambda$, the approximation $\tanh(kh_0) \simeq kh_0$ leads to the dispersion relation $\omega^2 = gh_0k^2$, i.e. to phase and group velocities

  $$c_\varphi = c_g = \sqrt{gh_0}, \tag{VI.43}$$

  independent from the wavelength λ, signaling the absence of dispersive behavior.

  This phase velocity decreases with decreasing water depth h_0. Accordingly, this might lead to an accumulation, similar to the case of a shock wave in Sec. VI.2, whose description however requires that one take into account the nonlinear terms in the equations, which have been discarded here. In particular, we have explicitly assumed $|\delta h(t,x)| \ll h_0$, in order to linearize the problem, so that considering the limiting case $h_0 \to 0$ is questionable.

  > In addition, a temptation when investigating the small-depth behavior $h_0 \to 0$ is clearly to describe the breaking of waves as they come to shore. Yet the harmonic ansatz (VI.38) assumes that the Laplace equation is considered on a horizontally unbounded domain, i.e. far from any coast, so again the dispersion relation (VI.41) may actually no longer be valid.

[(41)] The sea may not be "too deep", otherwise the assumed uniformity of the water mass density along the vertical direction in the unperturbed state does not hold. With $\lambda \simeq 100$ m, the inverse wave number is $k^{-1} \simeq 15$ m, so that $h_0 = 100$ m already represents a deep ocean; in comparison, the typical scale on which non-uniformities in the mass density are relevant is rather 1 km.

[(an)] G. B. Airy, 1801–1892

The boundary condition (VI.40b) provides us directly with the shape of the free surface of the sea, namely

$$\delta h(t,x) = \frac{1}{g}\frac{\partial\varphi(t,x,z)}{\partial t}\bigg|_{z=h_0} = \frac{\omega C}{g}\cosh(kh_0)\sin(kx-\omega t) \equiv \delta h_0 \sin(kx-\omega t),$$

with $\delta h_0 \equiv (\omega C/g)\cosh(kh_0)$ the amplitude of the wave, which must remain much smaller than h_0. The profile of the surface waves of Airy's linear theory—or rather its cross section—is thus a simple sinusoidal curve.

> This shape automatically suggests a generalization, which is a first step towards taking into account nonlinearities, such that the free surface profile is sum of (a few) harmonics $\sin(kx-\omega t)$, $\sin 2(kx-\omega t)$, $\sin 3(kx-\omega t)\ldots$ The approach leading to such a systematically expanded profile, which relies on a perturbative expansion to deal with the (still small) nonlinearities, is that of the *Stokes waves*.

The gradient of the potential (VI.38) yields (the components of) the flow velocity

$$\mathsf{v}_x(t,x,z) = \frac{kg}{\omega}\frac{\cosh(kz)}{\cosh(kh_0)}\delta h_0 \sin(kx-\omega t),$$
$$\mathsf{v}_z(t,x,z) = -\frac{kg}{\omega}\frac{\sinh(kz)}{\cosh(kh_0)}\delta h_0 \cos(kx-\omega t).$$

Integrating these functions with respect to time leads to the two functions

$$\mathsf{x}(t) = \mathsf{x}_0 + \frac{kg\,\delta h_0}{\omega^2}\frac{\cosh(kz)}{\cosh(kh_0)}\cos(kx-\omega t) = \mathsf{x}_0 + \frac{\delta h_0\cosh(kz)}{\sinh(kh_0)}\cos(kx-\omega t),$$
$$\mathsf{z}(t) = \mathsf{z}_0 + \frac{kg\,\delta h_0}{\omega^2}\frac{\sinh(kz)}{\cosh(kh_0)}\sin(kx-\omega t) = \mathsf{z}_0 + \frac{\delta h_0\sinh(kz)}{\sinh(kh_0)}\sin(kx-\omega t),$$

with x_0 and z_0 two integration constants. Choosing $\mathsf{x}_0 \simeq x$ and $\mathsf{z}_0 \simeq z$, if $\delta h_0 \ll k^{-1}$, these functions represent the components of the trajectory (pathline) of a fluid particle that is at time t in the vicinity of the point with coordinates (x,z), and whose velocity at that time is thus approximately the flow velocity $\vec{\mathsf{v}}(t,x,z)$. Since

$$\frac{[\mathsf{x}(t)-\mathsf{x}_0]^2}{\cosh^2(kz)} + \frac{[\mathsf{z}(t)-\mathsf{z}_0]^2}{\sinh^2(kz)} = \left[\frac{kg\,\delta h_0}{\omega^2\cosh(kh_0)}\right]^2 = \left[\frac{\delta h_0}{\sinh(kh_0)}\right]^2,$$

this trajectory is an ellipse, whose major and minor axes decrease with increasing depth $h_0 - z$. In the deep sea case $kh_0 \gg 1$, one can use the approximations $\sinh(kz) \simeq \cosh(kz) \simeq \mathrm{e}^{kz}/2$ for $1 \ll kz \lesssim kh_0$, which shows that the pathlines close to the sea surface are approximately circles.

Eventually, the pressure distribution in the sea follows from Eq. (VI.39) in which one uses the velocity potential (VI.38), resulting in

$$\mathcal{P}(t,x,z) = \mathcal{P}^{(0)} + \rho g(h_0-z) + \rho\frac{\partial\varphi(t,x,z)}{\partial t} = \mathcal{P}_0 + \rho g\left[h_0 - z + \delta h_0\frac{\cosh(kz)}{\cosh(kh_0)}\sin(kx-\omega t)\right].$$

The contribution $\mathcal{P}^{(0)} + \rho g(h_0 - z)$ is the usual hydrostatic one, corresponding to the unperturbed sea, while the effect of the surface wave is proportional to its amplitude δh_0 and decreases with increasing depth.

VI.3.2 Solitary waves

We now want to go beyond the linear limit considered in § VI.3.1 c for waves at the free surface of a liquid in a gravity field. To that extent, we shall take a few steps back, and first rewrite the dynamical equations of motion and the associated boundary conditions in a dimensionless form (§ VI.3.2 a). This formulation involves two independent parameters, and we shall focus on the limiting case where both are small—yet non-vanishing—and obey a given parametric relation. In that situation, the equation governing the shape of the free surface is the Korteweg–de Vries equation, which in particular describes solitary waves (§ VI.3.2 c).[(42)]

VI.3.2 a Dimensionless form of the equations of motion

As in § VI.3.1 c, the equations governing the dynamics of gravity waves at the surface of the sea are on the one hand the incompressibility condition

$$\vec{\nabla} \cdot \vec{\mathsf{v}}(t,\vec{r}) = 0, \tag{VI.44a}$$

and on the other hand the Euler equation

$$\frac{\partial \vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathsf{v}}(t,\vec{r}) \cdot \vec{\nabla}\right] \vec{\mathsf{v}}(t,\vec{r}) = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}(t,\vec{r}) - g\,\vec{e}_z. \tag{VI.44b}$$

The boundary conditions (VI.36) they obey are the absence of vertical velocity at the sea bottom

$$\mathsf{v}_z(t,x,z=0) = 0, \tag{VI.44c}$$

the identity of the sea vertical velocity with the rate of change of the surface altitude $h_0 + \delta h(t,x)$

$$\mathsf{v}_z\big(t,x,z=h_0+\delta h(t,x)\big) = \frac{\partial \delta h(t,x)}{\partial t} + \mathsf{v}_x(t,\vec{r})\frac{\partial \delta h(t,x)}{\partial x}, \tag{VI.44d}$$

and finally the existence of a uniform pressure at that free surface

$$\mathcal{P}\big(t,x,z=h_0+\delta h(t,x)\big) = \mathcal{P}_0.$$

In the sea at rest, the pressure field is given by the hydrostatic formula

$$\mathcal{P}_{\text{st.}}(t,x,z) = \mathcal{P}_0 + \rho g(h_0 - z).$$

Defining the "dynamical pressure" in the sea water as $\mathcal{P}_{\text{dyn.}} \equiv \mathcal{P} - \mathcal{P}_{\text{st.}}$, the two terms on the right hand side of the Euler equation (VI.44b) can be replaced by $-(1/\rho)\vec{\nabla}\mathcal{P}_{\text{dyn.}}$. In addition, the boundary condition at the free surface becomes

$$\mathcal{P}_{\text{dyn.}}\big(t,x,z=h_0+\delta h(t,x)\big) = \rho g \delta h(t,x). \tag{VI.44e}$$

Let us now recast Eqs. (VI.44) in a dimensionless form. For that extent, we introduce two characteristic lengths: L_c for long-wavelength motions along x or z, and δh_c for the amplitude of the surface deformation; for durations, we define a scale t_c, which will later be related to L_c with the help of a typical velocity. With these scales, we can construct dimensionless variables

$$t^* \equiv \frac{t}{t_c}, \quad x^* \equiv \frac{x}{L_c}, \quad z^* \equiv \frac{z}{L_c},$$

and fields:

$$\delta h^* \equiv \frac{\delta h}{\delta h_c}, \quad \mathsf{v}_x^* \equiv \frac{\mathsf{v}_x}{\delta h_c/t_c}, \quad \mathsf{v}_z^* \equiv \frac{\mathsf{v}_z}{\delta h_c/t_c}, \quad \mathcal{P}^* \equiv \frac{\mathcal{P}_{\text{dyn.}}}{\rho\,\delta h_c L_c/t_c^2}.$$

Considering the latter as functions of the reduced variables t^*, x^*, z^*, one can rewrite the equations (VI.44a)–(VI.44e). The incompressibility thus becomes

$$\frac{\partial \mathsf{v}_x^*}{\partial x^*} + \frac{\partial \mathsf{v}_z^*}{\partial z^*} = 0, \tag{VI.45a}$$

[(42)] This Section follows closely the Appendix A of Ref. [28].

and the Euler equation, projected successively on the x and z directions

$$\frac{\partial \mathsf{v}_x^*}{\partial t^*} + \varepsilon\left(\mathsf{v}_x^*\frac{\partial \mathsf{v}_x^*}{\partial x^*} + \mathsf{v}_z^*\frac{\partial \mathsf{v}_x^*}{\partial z^*}\right) = -\frac{\partial \mathcal{P}^*}{\partial x^*}, \tag{VI.45b}$$

and

$$\frac{\partial \mathsf{v}_z^*}{\partial t^*} + \varepsilon\left(\mathsf{v}_x^*\frac{\partial \mathsf{v}_z^*}{\partial x^*} + \mathsf{v}_z^*\frac{\partial \mathsf{v}_z^*}{\partial z^*}\right) = -\frac{\partial \mathcal{P}^*}{\partial z^*}, \tag{VI.45c}$$

where we have introduced the dimensionless parameter $\varepsilon \equiv \delta h_c/L_c$. In turn, the various boundary conditions are

$$\mathsf{v}_z^* = 0 \quad \text{at} \quad z^* = 0 \tag{VI.45d}$$

at the sea bottom, and at the free surface

$$\mathsf{v}_z^* = \frac{\partial \delta h^*}{\partial t^*} + \varepsilon \mathsf{v}_x^*\frac{\partial \delta h^*}{\partial x^*} \quad \text{at} \quad z^* = \delta + \varepsilon\,\delta h^* \tag{VI.45e}$$

with $\delta \equiv h_0/L_c$, and

$$\mathcal{P}^* = \frac{g t_c^2}{L_c}\delta h^* \quad \text{at} \quad z^* = \delta + \varepsilon\,\delta h^*.$$

Introducing the further dimensionless number

$$\mathrm{Fr} \equiv \frac{\sqrt{L_c/g}}{t_c},$$

the latter condition becomes

$$\mathcal{P}^* = \frac{1}{\mathrm{Fr}^2}\delta h^* \quad \text{at} \quad z^* = \delta + \varepsilon\,\delta h^*. \tag{VI.45f}$$

Inspecting these equations, one sees that the parameter ε controls the size of nonlinearities—cf. Eqs. (VI.45b), (VI.45c) and (VI.45e)—, while δ measures the depth of the sea in comparison to the typical wavelength L_c. Both parameters are a priori independent: δ is given by the physical setup we want to describe, while ε quantifies the amount of nonlinearity we include in the description.

To make progress, we shall from now on focus on gravity waves on shallow water, i.e. assume $\delta \ll 1$. In addition, we shall only consider small nonlinearities, $\varepsilon \ll 1$. To write down expansions in a consistent manner, we shall assume that the two small parameters are not of the same order, but rather that they obey $\varepsilon \sim \delta^2$. Calculations will be considered up to order $\mathcal{O}(\delta^3)$ or equivalently $\mathcal{O}(\delta\varepsilon)$.

For the sake of brevity, we now drop the superscript * from the dimensionless variables and fields.

VI.3.2 b Velocity potential

If the flow is irrotational, $\partial \mathsf{v}_x/\partial z = \partial \mathsf{v}_z/\partial x$, so that one may transform Eq. (VI.45b) into

$$\frac{\partial \mathsf{v}_x}{\partial t} + \varepsilon\left(\mathsf{v}_x\frac{\partial \mathsf{v}_x}{\partial x} + \mathsf{v}_z\frac{\partial \mathsf{v}_z}{\partial x}\right) + \frac{1}{\mathrm{Fr}^2}\frac{\partial \delta h}{\partial x} = 0. \tag{VI.46}$$

In addition, one may introduce a velocity potential $\varphi(t,x,z)$ such that $\vec{\mathsf{v}} = -\vec{\nabla}\varphi$. With the latter, the incompressibility condition (VI.45a) becomes the Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0. \tag{VI.47}$$

The solution for the velocity potential will be written as an infinite series in z

$$\varphi(t,x,z) = \sum_{n=0}^{\infty} z^n \varphi_n(t,x), \tag{VI.48}$$

with unknown functions $\varphi_n(t,x)$. Substituting this ansatz in the Laplace equation (VI.47) gives after some straightforward algebra

$$\sum_{n=0}^{\infty} z^n \left[\frac{\partial^2 \varphi_n(t,x)}{\partial x^2} + (n+1)(n+2)\varphi_{n+2}(t,x) \right] = 0.$$

In order for this identity to hold for arbitrary z—at least, for the values relevant for the flow—, each coefficient should individually vanish, i.e. the φ_n should obey the recursion relation

$$\varphi_{n+2}(t,x) = -\frac{1}{(n+1)(n+2)} \frac{\partial^2 \varphi_n(t,x)}{\partial x^2} \quad \text{for} \quad n \in \mathbb{N}. \tag{VI.49}$$

It is thus only necessary to determine φ_0 and φ_1 to know the whole series.

The boundary condition (VI.45d) at the bottom reads $\partial\varphi(t,x,z=0)/\partial z = 0$ for all t and x, which implies $\varphi_1(t,x) = 0$, so that all φ_{2n+1} identically vanish. As a consequence, ansatz (VI.48) with the recursion relation (VI.49) gives

$$\varphi(t,x,z) = \varphi_0(t,x) - \frac{z^2}{2} \frac{\partial^2 \varphi_0(t,x)}{\partial x^2} + \frac{z^4}{4!} \frac{\partial^4 \varphi_0(t,x)}{\partial x^4} + \dots .$$

Differentiating with respect to x or z yields the components of the velocity $\vec{\mathsf{v}} = -\vec{\nabla}\varphi$

$$\mathsf{v}_x(t,x,z) = -\frac{\partial \varphi_0(t,x)}{\partial x} + \frac{z^2}{2} \frac{\partial^3 \varphi_0(t,x)}{\partial x^3} - \frac{z^4}{4!} \frac{\partial^5 \varphi_0(t,x)}{\partial x^5} + \dots ,$$

$$\mathsf{v}_z(t,x,z) = z \frac{\partial^2 \varphi_0(t,x)}{\partial x^2} - \frac{z^3}{3!} \frac{\partial^4 \varphi_0(t,x)}{\partial x^4} + \dots .$$

Introducing the notation $\mathsf{u}(t,x) \equiv -\partial\varphi_0(t,x)/\partial x$ and anticipating that the maximal value of z relevant for the problem is of order δ, these components may be expressed as

$$\mathsf{v}_x(t,x,z) = \mathsf{u}(t,x) - \frac{z^2}{2} \frac{\partial^2 \mathsf{u}(t,x)}{\partial x^2} + o(\delta^3), \tag{VI.50a}$$

$$\mathsf{v}_z(t,x,z) = -z \frac{\partial \mathsf{u}(t,x)}{\partial x} + \frac{z^3}{3!} \frac{\partial^3 \mathsf{u}(t,x)}{\partial x^3} + o(\delta^3), \tag{VI.50b}$$

where the omitted terms are beyond $\mathcal{O}(\delta^3)$.

Linear waves rediscovered

If we momentarily set $\varepsilon = 0$—which amounts to linearizing the equations of motion and boundary conditions—, consistency requires that we consider equations up to order δ at most. That is, we keep only the first terms from Eqs. (VI.50): at the surface at $z \simeq \delta$, they become

$$\mathsf{v}_x(t,x,z=\delta) \simeq \mathsf{u}(t,x), \qquad \mathsf{v}_z(t,x,z=\delta) \simeq -\delta \frac{\partial \mathsf{u}(t,x)}{\partial x}, \tag{VI.51a}$$

while the boundary condition (VI.45e) simplifies to

$$\mathsf{v}_z(t,x,z=\delta) = \frac{\partial \delta h(t,x)}{\partial t} = \delta \frac{\partial \phi(t,x)}{\partial t}, \tag{VI.51b}$$

where we have introduced $\phi(t,x) \equiv \delta h(t,x)/\delta$. Meanwhile, Eq. (VI.46) with $\varepsilon = 0$ reads

$$\frac{\partial \mathsf{v}_x(t,x)}{\partial t} + \frac{\delta}{\mathrm{Fr}^2} \frac{\partial \phi(t,x)}{\partial x} = 0. \tag{VI.51c}$$

Together, Eqs. (VI.51a)–(VI.51c) yield after some straightforward manipulations the equation

$$\frac{\partial^2 \mathsf{u}(t,x)}{\partial t^2} - \frac{\delta}{\mathrm{Fr}^2} \frac{\partial^2 \mathsf{u}(t,x)}{\partial x^2} = 0, \tag{VI.52}$$

i.e. a linear equation describing waves with the dimensionless phase velocity $\sqrt{\delta}/\mathrm{Fr} = \sqrt{gh_0}/(L_c/t_c)$. Since the scaling factor of x resp. t is L_c resp. t_c, the corresponding dimensionful phase velocity is $c_\varphi = \sqrt{gh_0}$, as was already found in § VI.3.1 c for waves on shallow sea.

Until now, the scaling factor t_c was independent from L_c. Choosing $t_c \equiv L_c/\sqrt{gh_0}$, i.e. the unit in which times are measured, the factor δ/Fr^2 equals 1, leading to the simpler-looking equation

$$\frac{\partial \mathsf{v}_x(t,x,z)}{\partial t} + \varepsilon\left[\mathsf{v}_x(t,x,z)\frac{\partial \mathsf{v}_x(t,x,z)}{\partial x} + \mathsf{v}_z(t,x,z)\frac{\partial \mathsf{v}_z(t,x,z)}{\partial x}\right] + \frac{\partial \mathsf{\phi}(t,x)}{\partial x} = 0 \tag{VI.53}$$

instead of Eq. (VI.46).

VI.3.2 c Non-linear waves on shallow water

Taking now $\varepsilon \neq 0$ and investigating the equations up to order $\mathcal{O}(\delta^3)$, $\mathcal{O}(\delta\varepsilon)$, Eqs. (VI.50) at the free surface at $z = \delta(1+\varepsilon\mathsf{\phi})$ become

$$\mathsf{v}_x\big(t,x,z{=}\delta(1+\varepsilon\mathsf{\phi})\big) = \mathsf{u}(t,x) - \frac{\delta^2}{2}\frac{\partial^2 \mathsf{u}(t,x)}{\partial x^2}, \tag{VI.54a}$$

$$\mathsf{v}_z\big(t,x,z{=}\delta(1+\varepsilon\mathsf{\phi})\big) = -\delta\big[1+\varepsilon\mathsf{\phi}(t,x)\big]\frac{\partial \mathsf{u}(t,x)}{\partial x} + \frac{\delta^3}{6}\frac{\partial^3 \mathsf{u}(t,x)}{\partial x^3}. \tag{VI.54b}$$

Inserting these velocity components in (VI.53) while retaining only the relevant orders yields

$$\frac{\partial \mathsf{u}(t,x)}{\partial t} - \frac{\delta^2}{2}\frac{\partial^3 \mathsf{u}(t,x)}{\partial t\,\partial x^2} + \varepsilon\,\mathsf{u}(t,x)\frac{\partial \mathsf{u}(t,x)}{\partial x} + \frac{\partial \mathsf{\phi}(t,x)}{\partial x} = 0. \tag{VI.55}$$

On the other hand, the velocity components are also related by the boundary condition (VI.45e), which reads

$$\mathsf{v}_z\big(t,x,z{=}\delta(1+\varepsilon\mathsf{\phi})\big) = \delta\frac{\partial \mathsf{\phi}(t,x)}{\partial t} + \delta\varepsilon\,\mathsf{v}_x\big(t,x,z{=}\delta(1+\varepsilon\mathsf{\phi})\big)\frac{\partial \mathsf{\phi}(t,x)}{\partial x}.$$

Substituting Eq. (VI.54a) resp. (VI.54b) in the right resp. left member yields

$$\frac{\partial \mathsf{\phi}(t,x)}{\partial t} + \varepsilon\,\mathsf{u}(t,x)\frac{\partial \mathsf{\phi}(t,x)}{\partial x} + \big[1+\varepsilon\mathsf{\phi}(t,x)\big]\frac{\partial \mathsf{u}(t,x)}{\partial x} - \frac{\delta^2}{6}\frac{\partial^3 \mathsf{u}(t,x)}{\partial x^3} = 0. \tag{VI.56}$$

To leading order in δ and ε, the system of nonlinear partial differential equations (VI.55)–(VI.56) simplifies to the linear system

$$\begin{cases} \dfrac{\partial \mathsf{u}(t,x)}{\partial t} + \dfrac{\partial \mathsf{\phi}(t,x)}{\partial x} = 0 \\ \dfrac{\partial \mathsf{\phi}(t,x)}{\partial t} + \dfrac{\partial \mathsf{u}(t,x)}{\partial x} = 0, \end{cases}$$

which admits the solution $\mathsf{u}(t,x) = \mathsf{\phi}(t,x)$ under the condition

$$\frac{\partial \mathsf{u}(t,x)}{\partial t} + \frac{\partial \mathsf{u}(t,x)}{\partial x} = 0, \tag{VI.57}$$

which describes a traveling wave with (dimensionless) velocity 1, $\mathsf{u}(t,x) = \mathsf{u}(x-t)$. We again recover the linear sea surface waves which we have already encountered twice.

Going to next-to-leading order $\mathcal{O}(\delta^2)$, $\mathcal{O}(\varepsilon)$, we look for solutions in the form

$$\mathsf{u}(t,x) = \mathsf{\phi}(t,x) + \varepsilon\,\mathsf{u}^{(\varepsilon)}(t,x) + \delta^2\mathsf{u}^{(\delta)}(t,x) \tag{VI.58}$$

with $\mathsf{\phi}$, $\mathsf{u}^{(\varepsilon)}$, $\mathsf{u}^{(\delta)}$ functions that obey condition (VI.57) up to terms of order ε or δ^2. Inserting this ansatz in Eqs. (VI.55)–(VI.56) yields the system

$$\begin{cases} \dfrac{\partial \mathsf{\phi}}{\partial t} + \dfrac{\partial \mathsf{\phi}}{\partial x} + \varepsilon\dfrac{\partial \mathsf{u}^{(\varepsilon)}}{\partial x} + \delta^2\dfrac{\partial \mathsf{u}^{(\delta)}}{\partial x} + 2\varepsilon\,\mathsf{\phi}\dfrac{\partial \mathsf{\phi}}{\partial x} - \dfrac{\delta^2}{6}\dfrac{\partial^3 \mathsf{\phi}}{\partial x^3} = 0 \\ \dfrac{\partial \mathsf{\phi}}{\partial t} + \dfrac{\partial \mathsf{\phi}}{\partial x} + \varepsilon\dfrac{\partial \mathsf{u}^{(\varepsilon)}}{\partial t} + \delta^2\dfrac{\partial \mathsf{u}^{(\delta)}}{\partial t} + \varepsilon\,\mathsf{\phi}\dfrac{\partial \mathsf{\phi}}{\partial x} - \dfrac{\delta^2}{2}\dfrac{\partial^3 \mathsf{\phi}}{\partial x^2\,\partial t} = 0, \end{cases}$$

where for the sake of brevity, the (t,x)-dependence of the functions was not written. Subtracting both equations and using condition (VI.57) to relate the time and space derivatives of ϕ, $\mathsf{u}^{(\varepsilon)}$, and $\mathsf{u}^{(\delta)}$, one finds

$$\varepsilon\left[\frac{\partial \mathsf{u}^{(\varepsilon)}(t,x)}{\partial x}+\frac{1}{2}\phi(t,x)\frac{\partial\phi(t,x)}{\partial x}\right]+\delta^2\left[\frac{\partial \mathsf{u}^{(\delta)}(t,x)}{\partial x}-\frac{1}{3}\frac{\partial^3\phi(t,x)}{\partial x^3}\right]=0.$$

Since the two small parameters ε and δ are independent, each term between square brackets in this identity must identically vanish. Straightforward integrations then yield

$$\mathsf{u}^{(\varepsilon)}(t,x)=-\frac{1}{4}\phi(t,x)+C^{(\varepsilon)}(t),\qquad \mathsf{u}^{(\delta)}(t,x)=\frac{1}{3}\frac{\partial^2\phi(t,x)}{\partial x^2}+C^{(\delta)}(t),$$

with $C^{(\varepsilon)}$, $C^{(\delta)}$ two functions of time only.

These functions can then be substituted in the ansatz (VI.58). Inserting the latter in Eq. (VI.56) yields an equation involving the unknown function ϕ only, namely

$$\frac{\partial\phi(t,x)}{\partial t}+\frac{\partial\phi(t,x)}{\partial x}+\frac{3}{2}\varepsilon\,\phi(t,x)\frac{\partial\phi(t,x)}{\partial x}+\frac{1}{6}\delta^2\frac{\partial^3\phi(t,x)}{\partial x^3}=0. \tag{VI.59}$$

The first two terms only are those of the linear-wave equation of motion (VI.57). Since the the nonlinear corrections in ε and δ also obey the same condition, it is fruitful to perform a change of variables from (t,x) to (τ,ξ) with $\tau\equiv t$, $\xi\equiv x-t$. Equation (VI.59) then becomes

$$\boxed{\frac{\partial\phi(\tau,\xi)}{\partial\tau}+\frac{3}{2}\varepsilon\,\phi(\tau,\xi)\frac{\partial\phi(\tau,\xi)}{\partial\xi}+\frac{1}{6}\delta^2\frac{\partial^3\phi(\tau,\xi)}{\partial\xi^3}=0,} \tag{VI.60}$$

which is the *Korteweg–de Vries equation*.[ao],[ap]

Remark: By rescaling the variables τ and ξ to a new set (τ,ξ), one can actually absorb the parameters ε, δ which were introduced in the derivation. Accordingly, the equation takes the form

$$\boxed{\frac{\partial\phi(\tau,\xi)}{\partial\tau}+6\,\phi(\tau,\xi)\frac{\partial\phi(\tau,\xi)}{\partial\xi}+\frac{\partial^3\phi(\tau,\xi)}{\partial\xi^3}=0,} \tag{VI.61}$$

which is the more standard form of the Korteweg–de Vries equation.

Solitary waves

The Korteweg–de Vries (KdV) equation admits many different solutions. Among those, there is the class of *solitary waves* or *solitons*, which describe signals that propagate without changing their shape.

A specific subclass of solitons of the KdV equation of special interest in fluid dynamics consists of those which at each given instant vanish at (spatial) infinity. As solutions of the normalized equation (VI.61), they read

$$\phi(\tau,\xi)=\frac{\phi_0}{\cosh^2\left[\sqrt{\phi_0/2}\,(\xi-2\phi_0\tau)\right]} \tag{VI.62a}$$

with ϕ_0 the amplitude of the wave. Note that ϕ_0 must be nonnegative, which means that these solutions describe bumps above the mean sea level—which is indeed the only instance of solitary wave observed experimentally on the surface of water.

Going back first to the variables (τ,ξ), then to the dimensionless variables (t^*,x^*), and eventually to the dimensionful variables (t,x) and field δh, the soliton solution reads

[ao] D. Korteweg, 1848–1941 [ap] G. de Vries, 1866–1934

$$\delta h(t,x) = \frac{\delta h_{\max}}{\cosh^2\left\{\frac{1}{2h_0}\sqrt{\frac{3\delta h_{\max}}{h_0}}\left[x - \sqrt{gh_0}\left(1+\frac{\delta h_{\max}}{2h_0}\right)t\right]\right\}}, \tag{VI.62b}$$

with $\delta h_{\max}$ the maximum amplitude of the solitary wave. This solution, represented in Fig. VI.2, has a few properties that can be read directly off its expression and differ from those of linear sea surface waves, namely

- the propagation velocity c_{soliton} of the soliton—which is the factor in front of t—is larger than for linear waves;
- the velocity c_{soliton} increases with the amplitude $\delta h_{\max}$ of the soliton;
- the width of the soliton decreases with its amplitude.

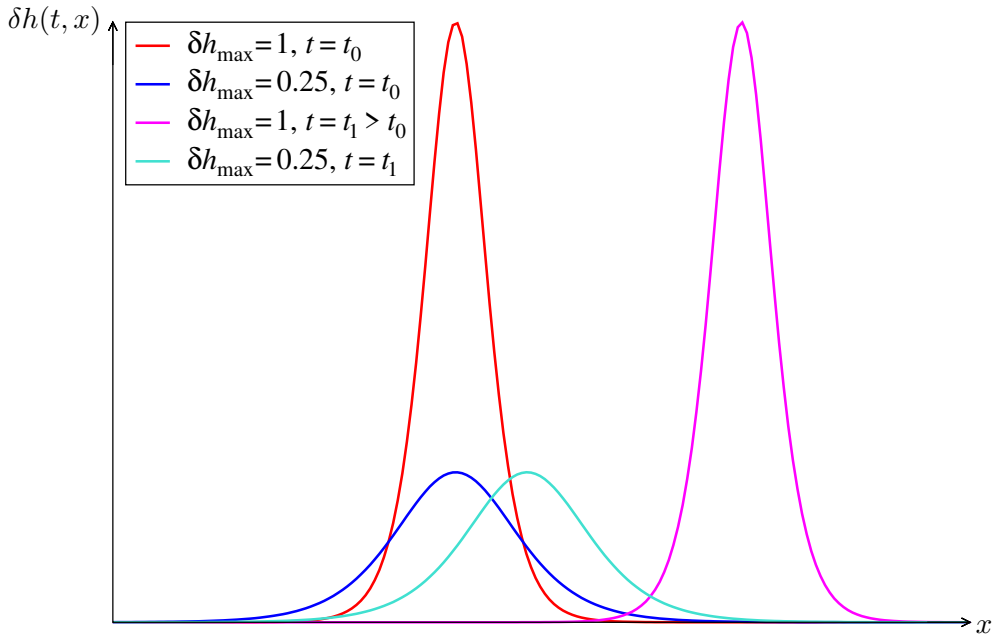

Figure VI.2 – Profile of the soliton solution (VI.62).

Bibliography for Chapter VI

- National Committee for Fluid Mechanics film & film notes on *Waves in Fluids*;
- Guyon *et al.* [2] Chapter 6.4;
- Landau–Lifshitz [4, 5] Chapters I § 12, VIII § 64–65, IX § 84–85, and X § 99;
- Sommerfeld [7, 8] Chapters III § 13, V § 23, 24 & 26 and VII § 37.

CHAPTER VII

Fluid instabilities

Small perturbation of fluid flow → dispersion relation. Can involve complex numbers (§ VI.1.4). Exponentially increasing solution? Signals instability of the fluid flow! (against linear perturbation)

VII.1 Gravitational instability in a perfect fluid at rest

VII.1.1 Sound waves in a fluid in a uniform gravity field

Consider a perfect fluid in a constant and uniform gravity field $\vec{g}$, which defines the z direction: $\vec{g} = -g\vec{e}_z$. As seen in § IV.1.2, in a domain over which the fluid mass density is assumed to be constant the Euler equation

$$\rho(t,\vec{r})\left\{\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathrm{v}}(t,\vec{r})\right\} = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \rho(t,\vec{r})\vec{g} \tag{VII.1}$$

and the continuity equation admit the static solution involving Pascal's law (IV.4)

$$\rho(t,\vec{r}) = \rho_0 \quad , \quad \mathcal{P}_0(t,\vec{r}) = \mathcal{P}^{(0)} - \rho_0 g z \quad , \quad \vec{\mathrm{v}}_0(t,\vec{r}) = \vec{0} \tag{VII.2}$$

with $\mathcal{P}^{(0)}$ the pressure at some reference altitude $z = 0$.

Let us consider a small adiabatic perturbation [cf. Eq. (VI.1)] of that reference state:

$$\rho(t,\vec{r}) = \rho_0 + \delta\rho(t,\vec{r}) \tag{VII.3a}$$

$$\mathcal{P}(t,\vec{r}) = \mathcal{P}_0(t,\vec{r}) + \delta\mathcal{P}(t,\vec{r}) = \mathcal{P}^{(0)} - \rho_0 g z + \delta\mathcal{P}(t,\vec{r}) \tag{VII.3b}$$

$$\vec{\mathrm{v}}(t,\vec{r}) = \vec{\mathrm{v}}_0(t,\vec{r}) + \delta\vec{\mathrm{v}}(t,\vec{r}) = \delta\vec{\mathrm{v}}(t,\vec{r}) \tag{VII.3c}$$

Inserting these fields into the continuity equation and Euler equation (VII.1) yields dynamical equations for the perturbations $\delta\rho$, $\delta\mathcal{P}$, $\delta\vec{\mathrm{v}}$. Linearizing these equations in the perturbations, one obtains

$$\frac{\partial\delta\rho(t,\vec{r})}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{\mathrm{v}}(t,\vec{r}) = 0 \tag{VII.4a}$$

as in Eq. (VI.4a) and

$$\rho_0\frac{\partial\delta\vec{\mathrm{v}}(t,\vec{r})}{\partial t} = -\vec{\nabla}\delta\mathcal{P}(t,\vec{r}) + \delta\rho(t,\vec{r})\vec{g}. \tag{VII.4b}$$

Using the adiabaticity condition as in § VI.1.1, the small variations of pressure and mass density are related via the speed of sound (VI.5) (estimated at ρ_0):

$$\delta\mathcal{P}(t,\vec{r}) = c_s^2 \delta\rho(t,\vec{r}). \tag{VII.4c}$$

Combining the time derivative of Eq. (VII.4a) and the divergence of Eq. (VII.4b) under consideration of relation (VII.4c), one finds

$$\frac{\partial^2 \delta\rho(t,\vec{r})}{\partial t} - c_s^2 \triangle \delta\rho(t,\vec{r}) + \vec{g}\cdot\vec{\nabla}\delta\rho(t,\vec{r}) = 0. \tag{VII.5}$$

This is a linear partial differential equation, which reduces to the classical wave equation in the absence of gravitational field.

Inserting in this evolution equation the Fourier ansatz

$$\delta\rho(t,\vec{r}) = \widetilde{\delta\rho}(\omega,\vec{k})\, \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)} \tag{VII.6}$$

leads at once to the dispersion relation

$$\omega^2 = c_s^2 \vec{k}^{\,2} + \mathrm{i}\vec{g}\cdot\vec{k} \tag{VII.7}$$

for small perturbations. Effect of gravitation ($\mathrm{i}\vec{g}\cdot\vec{k}$) only when wave vector $\vec{k}$ has a component along the direction or $\vec{g}$.

Sound waves in air on Earth: $c_s \approx 340\ \mathrm{m\cdot s^{-1}}$, $g \approx 9.8\ \mathrm{m\cdot s^{-2}}$. The second term on the right hand side of Eq. (VII.7) becomes comparable in absolute value to the first one when $|\vec{k}| \lesssim 10^{-4}\ \mathrm{m^{-1}}$. This corresponds to wavelengths of order 10 km or higher, i.e. to frequencies below 1 Hz. Effect of gravity on usual sound waves is thus negligible.

The discussion should be further expanded.

VII.1.2 Sound wave in a self-gravitating gas

Instead of the setup of the previous paragraph, in which the fluid is in an external gravity field, let us now consider the case of an isolated perfect fluid, whose mass distribution creates a Newtonian gravitational potential Φ according to the Poisson equation

$$\triangle\Phi(t,\vec{r}) = 4\pi G_\mathrm{N}\, \rho(t,\vec{r}) \tag{VII.8}$$

with Newton's gravitational constant $G_\mathrm{N} \simeq 6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$. The paramount example is that of an interstellar cloud, consisting mostly of hydrogen.

While Eq. (VII.8) describes the influence of mass density on the gravitational potential, they are further coupled together by the Euler equation

$$\rho(t,\vec{r})\left\{\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathrm{v}}(t,\vec{r})\right\} = -\vec{\nabla}\mathcal{P}(t,\vec{r}) - \rho(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}). \tag{VII.9}$$

In addition, mass density and velocity field obey the continuity equation. One could invoke a fourth equation, namely an equation of state, to relate the pressure to the mass density, but this will not be needed afterwards.

Instead, assume[(43)] that we take as starting point the stationary and uniform reference state

$$\rho_0(t,\vec{r}) = \rho_0 \quad , \quad \mathcal{P}_0(t,\vec{r}) = \mathcal{P}_0 \quad , \quad \vec{\mathrm{v}}_0(t,\vec{r}) = \vec{0} \quad , \quad \Phi_0(t,\vec{r}) = \Phi_0. \tag{VII.10}$$

As in § VII.1.1, we consider small adiabatic perturbations of the fields:

$$\rho(t,\vec{r}) = \rho_0 + \delta\rho(t,\vec{r}) \tag{VII.11a}$$
$$\mathcal{P}(t,\vec{r}) = \mathcal{P}_0 + \delta\mathcal{P}(t,\vec{r}) \tag{VII.11b}$$
$$\vec{\mathrm{v}}(t,\vec{r}) = \delta\vec{\mathrm{v}}(t,\vec{r}) \tag{VII.11c}$$
$$\Phi(t,\vec{r}) = \Phi_0 + \delta\Phi(t,\vec{r}) \tag{VII.11d}$$

[(43)] ... following Jeans' historical swindle! This assumption is *not* a valid solution of the equations, unless $\rho_0 = 0$.

with in addition

$$\delta\mathcal{P}(t,\vec{r}) = c_s^2 \delta\rho(t,\vec{r}). \tag{VII.12}$$

Invoking the continuity and Euler equations and the Poisson equation (VII.8), the perturbations satisfy the coupled partial differential equations

$$\frac{\partial\delta\rho(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\Big(\big[\rho_0 + \delta\rho(t,\vec{r})\big]\delta\vec{\mathrm{v}}(t,\vec{r})\Big) = 0 \tag{VII.13a}$$

$$\big[\rho_0+\delta\rho(t,\vec{r})\big]\left\{\frac{\partial\delta\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \big[\delta\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\delta\vec{\mathrm{v}}(t,\vec{r})\right\} = -\vec{\nabla}\delta\mathcal{P}(t,\vec{r}) - \big[\rho_0+\delta\rho(t,\vec{r})\big]\vec{\nabla}\delta\Phi(t,\vec{r}) \tag{VII.13b}$$

and

$$\triangle\delta\Phi(t,\vec{r}) = 4\pi G_{\mathrm{N}}\,\delta\rho(t,\vec{r}). \tag{VII.13c}$$

For small perturbations, we can linearize these equations, which thus simplify to the system

$$\frac{\partial\delta\rho(t,\vec{r})}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{\mathrm{v}}(t,\vec{r}) = 0 \tag{VII.14a}$$

$$\rho_0\frac{\partial\delta\vec{\mathrm{v}}(t,\vec{r})}{\partial t} = -\vec{\nabla}\delta\mathcal{P}(t,\vec{r}) - \rho_0\vec{\nabla}\delta\Phi(t,\vec{r}) \tag{VII.14b}$$

$$\triangle\delta\Phi(t,\vec{r}) = 4\pi G_{\mathrm{N}}\,\delta\rho(t,\vec{r}). \tag{VII.14c}$$

Combining the divergence of the linearized Euler equation (VII.14b), using Eq. (VII.14c) for the rightmost term, and the time derivative of Eq. (VII.14a), one obtains the partial differential equation

$$\frac{\partial^2\delta\rho(t,\vec{r})}{\partial t} - c_s^2\triangle\delta\rho(t,\vec{r}) - 4\pi G_{\mathrm{N}}\rho_0\,\delta\rho(t,\vec{r}) = 0 \tag{VII.15}$$

governing the evolution of the perturbation $\delta\rho(t,\vec{r})$. The Fourier ansatz $\delta\rho(t,\vec{r}) = \widetilde{\delta\rho}(\omega,\vec{k})\,\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)}$ leads to the dispersion relation

$$\omega^2 = c_s^2\vec{k}^{\,2} - 4\pi G_{\mathrm{N}}\rho_0 = c_s^2\big(k^2 - k_{\mathrm{J}}^2\big) \tag{VII.16}$$

between the angular frequency ω and the wave vector $\vec{k}$, where in the second identity we have denoted $k \equiv |\vec{k}|$ and introduced the wave number

$$k_{\mathrm{J}} \equiv \frac{\sqrt{4\pi G_{\mathrm{N}}\rho_0}}{c_s}. \tag{VII.17}$$

According to this dispersion relation, (linear) perturbations with wave number $k > k_J$ propagate with the phase velocity

$$c_\varphi = \frac{\omega}{k} = c_s\sqrt{1 - k_{\mathrm{J}}^2/k^2}, \tag{VII.18}$$

close to c_s for very small wavelengths $\lambda = 2\pi/k$ and vanishing for $k = k_{\mathrm{J}}$.

On the other hand, in the case of perturbations with a wavelength larger than the *Jeans*[(aq)] *length* $\lambda_{\mathrm{J}} \equiv 2\pi/k_{\mathrm{J}}$, the dispersion relation (VII.16) yields two modes with purely imaginary (angular) frequencies

$$\omega_\pm(k) = \pm\mathrm{i}c_s\sqrt{k_{\mathrm{J}}^2 - k^2}, \tag{VII.19}$$

which do not propagate since the real part of ω vanishes. While the amplitude of the mode with $\omega = \omega_-$ decreases exponentially, that of the mode with $\omega = \omega_+$ will increase.[(44)] Since there is no physical argument to discard these wildly growing modes, their existence signals the instability of the reference state (VII.10) with respect to large-wavelength perturbations.

For air ($\rho_0 \simeq 1.3\,\mathrm{kg\cdot m^{-3}}$) at 300 K, resulting in a speed of sound $c_s \simeq 350\,\mathrm{m\cdot s^{-1}}$, the Jeans length is of order 6.7×10^4 km, three orders of magnitude larger than the thickness of the Earth atmosphere — in which perturbations are thus stable.

[(44)] At least, within the limit of validity of the linear analysis performed here.

[(aq)] J. Jeans, 1877–1946

VII.2 Couette–Taylor instability

could be added at some point

Bibliography for Chapter VII

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Chapter VIII

Turbulence in non-relativistic fluids

All examples of flows considered until now in these notes, either of perfect or Newtonian fluids (Chapters IV–VI), shared a common property, namely they were all assumed to be laminar. This assumption—which at once translates into a relative simplicity of the flow velocity profile—is however not the generic case in real flows, which are most often turbulent to a more or less large extent. The purpose of this Chapter is to provide an introduction to the problematic of turbulence in non-relativistic fluid motions.

A number of experiments, in particular those conducted by O. Reynolds, have hinted at the possibility that turbulence occurs when the Reynolds number (V.12) is large enough in the flow, i.e. when convective effects predominate over the shear viscous ones in the mean fluid motion over which the instabilities develop. This distinction between mean flow and turbulent fluctuations can be modeled directly by splitting the dynamical fields into two parts, and one recovers with the help of dimensional arguments the role of the Reynolds number in separating two regimes, one in which viscous effects dominate the mean flow, and one in which turbulence takes over (Sec. VIII.1).

Despite its appeal, the decomposition into a mean flow and a turbulent motion has the drawback that it leads to a system of equations of motion which is not closed. A possibility to remedy this issue is to invoke the notion of a turbulent viscosity, for which various models have been proposed (Sec. VIII.2).

Even when the system of equations of motion is closed, it still involves averages—with an a priori unknown underlying probability distribution. That is, the description of turbulent part of the motion necessitates the introduction of a few concepts characterizing the statistics of the velocity field (Sec. VIII.3).

For the sake of simplicity, we shall mostly consider turbulence in the three-dimensional incompressible motion of Newtonian fluids with constant and homogeneous properties (mass density, viscosity…), in the absence of relevant external bulk forces, and neglecting possible temperature gradients—and thereby convective heat transport.

VIII.1 Generalities on turbulence in fluids

In this Section, a few experimental facts on turbulence in fluid flows are presented, and the first steps towards a modeling of the phenomenon are introduced.

VIII.1.1 Phenomenology of turbulence

VIII.1.1 a Historical example: Hagen–Poiseuille flow

The idealized Hagen–Poiseuille flow of a Newtonian fluid in a cylindrical tube was already partly discussed in § V.1.4. There, it was found that in the stationary *laminar* regime in which the velocity field $\vec{\mathsf{v}}$ is everywhere parallel to the walls of the tube, the mass flow rate Q across the cylinder cross section is given by the Hagen–Poiseuille law

$$Q = -\frac{\pi\rho a^4}{8\eta}\frac{\delta\mathcal{P}}{L}, \tag{V.9}$$

with a the tube radius, $\delta\mathcal{P}/L$ the pressure drop per unit length, and ρ, η the fluid mass density and shear viscosity, respectively.

Due to the viscous friction forces, part of the kinetic energy of the fluid motion is transformed into heat. To compensate for these "losses" and keep the flow in the stationary regime, energy has to be provided to the fluid, namely in the form of the mechanical work of the pressure forces driving the flow. Thus, the rate of energy dissipation per unit fluid mass is[45]

$$\dot{\mathcal{E}}_{\text{diss.}} = -\frac{1}{\rho}\frac{\delta\mathcal{P}}{L}\langle\mathsf{v}\rangle = \frac{8\nu\langle\mathsf{v}\rangle^2}{a^2} \tag{VIII.1}$$

with $\langle\mathsf{v}\rangle$ the average flow velocity across the tube cross section,

$$\langle\mathsf{v}\rangle = \frac{Q}{\pi a^2\rho} = -\frac{a^2}{8\eta}\frac{\delta\mathcal{P}}{L}.$$

In the laminar regime the rate $\dot{\mathcal{E}}_{\text{diss.}}$ is thus proportional to the kinematic viscosity $\nu \equiv \eta/\rho$ and to the square of the average velocity.

According to the Hagen–Poiseuille law (V.9), at fixed pressure gradient the average velocity $\langle\mathsf{v}\rangle$ grows quadratically with the tube radius. In practice, the rise is actually slower, reflecting a higher rate of energy loss in the flow than the laminar prediction (VIII.1). Thus, the mean rate of energy dissipation is no longer proportional to $\langle\mathsf{v}\rangle^2$, but rather to a higher power of $\langle\mathsf{v}\rangle$. Besides, the flow velocity profile across the tube cross section is no longer parabolic, but (in average) flatter around the cylinder axis, with a faster decrease at the tube walls.

VIII.1.1 b Transition to a turbulent regime

Consider a given geometry—say for instance that of the Hagen–Poiseuille flow or the motion of a fluid in a tube with fixed rectangular cross section. In the low-velocity regime, the flow in that geometry is laminar, and the corresponding state[46] is stable against small perturbations, which are damped by viscosity (see § VI.1.4).

However, when the average flow velocity exceeds some critical value, while all other characteristics of the flow, in particular the fluid properties, remain fixed, the motion cannot remain laminar. Small perturbations are no longer damped, but can grow by extracting kinetic energy from the "main", regular part of the fluid motion. As a consequence, instead of simple pathlines, the fluid particles now follow more twisted ones: the flow becomes *turbulent*.

[45] In this Chapter, we shall only discuss incompressible flows at constant mass density ρ, and thus always consider energies per unit mass.

[46] This term really refers to a macroscopic "state" of the system in the statistical-physical sense. In contrast to the global equilibrium states usually considered in thermostatics, it is here a non-equilibrium steady state, in which local equilibrium holds at every point.

In that case, the velocity gradients involved in the fluid motion are on average much larger than in a laminar flow. The amount of viscous friction per unit volume or unit mass of the fluid is thus increased, and a larger fraction of the kinetic energy is dissipated as heat.

The role of a critical parameter in the onset of turbulence was discovered by Reynolds in the case of the Hagen–Poiseuille flow of water, in which he injected some colored water on the axis of the tube, repeating the experiment for increasing flow velocities [21]. In the laminar regime found at small velocities, the streakline formed by the colored water forms a thin band along the tube axis, which does not mix with the surrounding water. Above some flow velocity, the streakline remains straight along some distance in the tube, then suddenly becomes unstable and fills the whole cross section of the tube.

As Reynolds understood himself by performing his experiments with tubes of various diameters, the important parameter is not the velocity itself, but rather the Reynolds number Re (V.12), which is proportional to the velocity. Thus, the transition to turbulence in flows with shear occurs at a "critical value" Re_c, which however depends on the geometry of the flow. For instance Re_c is of order 2000 for the Hagen–Poiseuille flow, but becomes of order 1000 for the plane Poiseuille flow investigated in § V.1.3, while $\mathrm{Re}_c \simeq 370$ for the plane Couette flow (§ V.1.2).

> The notion of a critical Reynolds number separating the laminar and turbulent regimes is actually a simplification. In theoretical studies of the stability of the laminar regime against *linear* perturbations, such a critical value Re_c can be computed for some very simple geometries, yielding e.g. $\mathrm{Re}_c = 5772$ for the plane Poiseuille flow. Yet the stability sometimes also depends on the size of the perturbation: the larger it is, the smaller the associated critical Re_c is, which hints at the role of nonlinear instabilities.

In the following, we shall leave aside the problem of the temporal onset of turbulence—and thereby of the (in)stability of laminar flows—, and focus on flows in which turbulence is already established.

VIII.1.2 Reynolds decomposition of the fluid dynamical fields

Experiment as well as reasoning hint at the existence of an underlying "simple", laminar flow over which turbulence develops. Accordingly, a reasonable ansatz for the description of the turbulent motion of a fluid is to split the relevant dynamical fields into two components: a first one that varies slowly both in time t and position $\vec{r}$, and a rapidly fluctuating component, which will be denoted with primed quantities. In the case of the flow velocity field $\vec{\mathrm{v}}(t,\vec{r})$, this *Reynolds decomposition*[lxii] reads [29]

$$\boxed{\vec{\mathrm{v}}(t,\vec{r}) = \overline{\vec{\mathrm{v}}(t,\vec{r})} + \vec{\mathrm{v}}'(t,\vec{r}),} \tag{VIII.2}$$

with $\overline{\vec{\mathrm{v}}}$ resp. $\vec{\mathrm{v}}'$ the "slow" resp. "fast" component. For the pressure, one similarly writes

$$\mathcal{P}(t,\vec{r}) = \overline{\mathcal{P}(t,\vec{r})} + \mathcal{P}'(t,\vec{r}). \tag{VIII.3}$$

The fluid motion with velocity $\overline{\vec{\mathrm{v}}}$ and pressure $\overline{\mathcal{P}}$ is then referred to as "mean flow", that with the rapidly varying quantities as "fluctuating motion".

Remarks:

∗ The "fast" and "short wavelength" degrees of freedom that constitute the turbulent motion should still be "slow enough" to be fluid dynamical, i.e. the corresponding scales are still "macroscopic" in the sense introduced in Sec. I.1.

[lxii] *Reynolds-Zerlegung*

∗ Let us already emphasize that the mean flow is *not* a valid solution of the usual fluid-dynamical equations of motion, in particular of the Navier–Stokes equation. Accordingly, the Reynolds decompositions (VIII.2)–(VIII.3) differ from the decomposition into a "background flow" and a "perturbation" introduced in the study of sound waves (Sec. VI.1).

As hinted at by the notation, $\overline{\vec{\mathsf{v}}(t,\vec{r})}$ represents an average, with some underlying probability distribution.

> Theoretically, the Reynolds average $\overline{\cdot}$ should be an ensemble average, obtained from an infinitely large number of realizations, namely experiments or computer simulations; in practice, however, there is only a finite number N of realizations $\vec{\mathsf{v}}^{(n)}(t,\vec{r})$, where $n = 1, 2, \ldots, N$ labels the realization. If the turbulent flow is statistically stationary, one may invoke the *ergodic assumption* and replace the ensemble average by a time average:
>
> $$\overline{\vec{\mathsf{v}}}(\vec{r}) \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \vec{\mathsf{v}}^{(n)}(t,\vec{r}) \approx \frac{1}{\mathcal{T}} \int_{t-\mathcal{T}/2}^{t+\mathcal{T}/2} \vec{\mathsf{v}}(t',\vec{r})\,\mathrm{d}t',$$
>
> where the duration $\mathcal{T}$ should be much larger than the autocorrelation time of the turbulent velocity $\vec{\mathsf{v}}'(t,\vec{r})$. If the flow is not statistically stationary, so that $\overline{\vec{\mathsf{v}}}(t,\vec{r})$ also depends on time, then $\mathcal{T}$ must also be much smaller than the typical time scale of the variations of the mean flow.

Using the same averaging procedure, the fluctuating velocity must obey the condition

$$\overline{\vec{\mathsf{v}}'(t,\vec{r})} = \vec{0}. \tag{VIII.4}$$

That is, the random variable $\vec{\mathsf{v}}'(t,\vec{r})$ is centered for every t and $\vec{r}$.

Despite this fact, the turbulent velocity $\vec{\mathsf{v}}'(t,\vec{r})$ still plays a role in the dynamics, in particular that of the mean flow, because its two-point, three-point and higher (auto)correlation functions are in general non-zero. For instance, one can write—assuming that the mass density ρ is constant and uniform throughout the fluid

$$\rho\,\overline{\mathsf{v}^i(t,\vec{r})\,\mathsf{v}^j(t,\vec{r})} = \rho\,\overline{\mathsf{v}^i(t,\vec{r})}\;\overline{\mathsf{v}^j(t,\vec{r})} + \rho\,\overline{\mathsf{v}'^i(t,\vec{r})\,\mathsf{v}'^j(t,\vec{r})}.$$

The first term of the right member corresponds to the convective part of the momentum-flux density of the mean flow, while the second one

$$\mathsf{T}_{\mathrm{R}}^{ij}(t,\vec{r}) \equiv \rho\,\overline{\mathsf{v}'^i(t,\vec{r})\,\mathsf{v}'^j(t,\vec{r})}, \tag{VIII.5}$$

which is the ij-component of the rank 2 tensor

$$\boxed{\mathsf{T}_{\mathrm{R}}(t,\vec{r}) \equiv \rho\,\overline{\vec{\mathsf{v}}'(t,\vec{r}) \otimes \vec{\mathsf{v}}'(t,\vec{r})},} \tag{VIII.6}$$

is due to the rapidly fluctuating motion. T_{R} is called *turbulent stress* or *Reynolds stress.*[(lxiii)]

VIII.1.3 Dynamics of the mean flow

For the sake of simplicity, the fluid motion will from now on be assumed to be incompressible. Thanks to the linearity of the averaging process, the kinematic condition $\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}) = 0$ leads to the two relations

$$\vec{\nabla}\cdot\overline{\vec{\mathsf{v}}}(t,\vec{r}) = 0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{\mathsf{v}}'(t,\vec{r}) = 0. \tag{VIII.7}$$

That is, both the mean flow and the turbulent motion are themselves incompressible.

The total flow velocity $\vec{\mathsf{v}}$ obeys the usual incompressible Navier–Stokes equation [cf. Eq. (III.33)]

$$\rho\left(\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r})\right) = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \eta\triangle\vec{\mathsf{v}}(t,\vec{r}), \tag{VIII.8}$$

[(lxiii)] *Reynolds-Spannung*

from which the equations governing the mean and turbulent flows can be derived. For the sake of brevity, the arguments $(t, \vec{r})$ of the various fields will be omitted in the following.

VIII.1.3 a Equations for the mean flow

Inserting the Reynolds decompositions (VIII.2)–(VIII.3) into the Navier–Stokes equation (VIII.8) and averaging with the Reynolds average $\overline{\cdot}$ leads to the so-called *Reynolds equation*

$$\boxed{\rho\left[\frac{\partial\overline{\vec{\mathrm{v}}}}{\partial t} + \big(\overline{\vec{\mathrm{v}}}\cdot\vec{\nabla}\big)\overline{\vec{\mathrm{v}}}\right] = -\vec{\nabla}\overline{\mathcal{P}} + \eta\triangle\overline{\vec{\mathrm{v}}} - \rho\overline{\big(\vec{\mathrm{v}}'\cdot\vec{\nabla}\big)\vec{\mathrm{v}}'}.} \tag{VIII.9a}$$

To avoid confusion, this equation is also sometimes referred to as *Reynolds-averaged Navier–Stokes equation*. In terms of components in a given system of coordinates, this becomes, after dividing by the mass density ρ and accounting for incompressibility (see below)

$$\frac{\partial\overline{\mathrm{v}^i}}{\partial t} + \big(\overline{\vec{\mathrm{v}}}\cdot\vec{\nabla}\big)\overline{\mathrm{v}^i} = -\frac{1}{\rho}\frac{\mathrm{d}\overline{\mathcal{P}}}{\mathrm{d}x_i} - \sum_{j=1}^{3}\frac{\mathrm{d}\overline{\mathrm{v}'^i\mathrm{v}'^j}}{\mathrm{d}x^j} + \nu\triangle\overline{\mathrm{v}^i} \qquad \forall i = 1,2,3. \tag{VIII.9b}$$

Using the incompressibility of the fluctuating motion, the rightmost term in Eq. (VIII.9a) can be rewritten as

$$-\rho\overline{\big(\vec{\mathrm{v}}'\cdot\vec{\nabla}\big)\vec{\mathrm{v}}'} = -\rho\vec{\nabla}\cdot\big(\overline{\vec{\mathrm{v}}'\otimes\vec{\mathrm{v}}'}\big) = -\vec{\nabla}\cdot\mathbf{T}_{\mathrm{R}}.$$

The Reynolds equation can thus be recast in the equivalent form [cf. Eq. (III.25b)]

$$\frac{\partial}{\partial t}\big(\rho\overline{\vec{\mathrm{v}}}\big) + \vec{\nabla}\cdot\overline{\mathbf{T}} = -\vec{\nabla}\cdot\mathbf{T}_{\mathrm{R}}, \tag{VIII.10}$$

with $\overline{\mathbf{T}}$ the momentum-flux density of the mean flow, given by [cf. Eqs. (III.27b), (III.27e)]

$$\overline{\mathbf{T}} \equiv \overline{\mathcal{P}}\,\mathbf{g}^{-1} + \rho\overline{\vec{\mathrm{v}}}\otimes\overline{\vec{\mathrm{v}}} - 2\eta\overline{\mathbf{S}} \tag{VIII.11a}$$

i.e., in terms of components,

$$\overline{\mathbf{T}^{ij}} \equiv \overline{\mathcal{P}}\,g^{ij} + \rho\overline{\mathrm{v}^i}\,\overline{\mathrm{v}^j} - 2\eta\overline{\mathbf{S}^{ij}}, \tag{VIII.11b}$$

where $\overline{\mathbf{S}}$ denotes the rate-of-shear tensor [Eq. (II.17b)] for the mean flow, whose components are given by [cf. Eq. (II.17d)]

$$\overline{\mathbf{S}^{ij}} \equiv \frac{1}{2}\left(\frac{\mathrm{d}\overline{\mathrm{v}^i}}{\mathrm{d}x_j} + \frac{\mathrm{d}\overline{\mathrm{v}^j}}{\mathrm{d}x_i} - \frac{2}{3}g^{ij}\vec{\nabla}\cdot\overline{\vec{\mathrm{v}}}\right). \tag{VIII.11c}$$

Note that the third term within the brackets actually vanishes due to the incompressibility of the mean flow, Eq. (VIII.7).

The form (VIII.10) of the Reynolds equation emphasizes perfectly the role of the Reynolds stress, i.e. the turbulent component of the flow, as "external" source driving the mean flow. In particular, the off-diagonal terms of the Reynolds stress describe sources of shear stresses, which will lead to the appearance of *eddies* in the flow.

Remark: The two equations (VIII.9a)–(VIII.9b) involve the material derivative "following the mean flow"

$$\frac{\overline{\mathrm{D}}}{\overline{\mathrm{D}t}} \equiv \frac{\partial}{\partial t} + \overline{\vec{\mathrm{v}}}\cdot\vec{\nabla}, \tag{VIII.12}$$

which we shall further use in the remainder of this chapter.

Starting from the Reynolds equation (VIII.9), one can derive the equation governing the evolution of the kinetic energy $\frac{1}{2}\rho\big(\overline{\vec{\mathrm{v}}}\big)^2$ associated with the mean flow, namely

$$\frac{\overline{\mathrm{D}}}{\overline{\mathrm{D}t}}\left(\frac{\rho\overline{\vec{\mathrm{v}}}^2}{2}\right) = -\vec{\nabla}\cdot\big[\overline{\mathcal{P}}\,\overline{\vec{\mathrm{v}}} + \big(\mathbf{T}_{\mathrm{R}} - 2\eta\overline{\mathbf{S}}\big)\cdot\overline{\vec{\mathrm{v}}}\big] + \big(\mathbf{T}_{\mathrm{R}} - 2\eta\overline{\mathbf{S}}\big) : \overline{\mathbf{S}}. \tag{VIII.13}$$

Traditionally, this equation is rather written in terms of the kinetic energy per unit mass $\overline{k} \equiv \frac{1}{2}\big(\overline{\vec{\mathsf{v}}}\big)^2$, in which case it reads

$$\frac{\overline{\mathrm{D}}\overline{k}}{\overline{\mathrm{D}}t} = -\vec{\nabla}\cdot\left[\frac{1}{\rho}\overline{\mathcal{P}}\,\overline{\vec{\mathsf{v}}} + \big(\overline{\vec{\mathsf{v}}'\otimes\vec{\mathsf{v}}'} - 2\nu\overline{\mathbf{S}}\big)\cdot\overline{\vec{\mathsf{v}}}\right] + \big(\overline{\vec{\mathsf{v}}'\otimes\vec{\mathsf{v}}'} - 2\nu\overline{\mathbf{S}}\big) : \overline{\mathbf{S}}, \tag{VIII.14a}$$

or in terms of components

$$\frac{\overline{\mathrm{D}}\overline{k}}{\overline{\mathrm{D}}t} = -\sum_{j=1}^{3}\frac{\mathrm{d}}{\mathrm{d}x^j}\left[\frac{1}{\rho}\overline{\mathcal{P}}\,\overline{\mathsf{v}^j} + \sum_{i=1}^{3}\Big(\overline{\mathsf{v}'^i\mathsf{v}'^j} - 2\nu\overline{\mathbf{S}^{ij}}\Big)\overline{\mathsf{v}_i}\right] + \sum_{i,j=1}^{3}\Big(\overline{\mathsf{v}'^i\mathsf{v}'^j} - 2\nu\overline{\mathbf{S}^{ij}}\Big)\overline{\mathbf{S}_{ij}}. \tag{VIII.14b}$$

In either form, the physical meaning of each term is rather transparent: first comes the convective transport of energy in the mean flow, given by the divergence of the energy flux density, inclusive a term from the turbulent motion. The second term represents the energy which the mean flow "loses", namely either because it is dissipated by the viscous friction forces (term in $\nu\overline{\mathbf{S}} : \overline{\mathbf{S}}$), or because it is transferred to the turbulent part of the motion (term involving the Reynolds stress).

> To prove Eq. (VIII.13), one should first average the inner product with $\overline{\vec{\mathsf{v}}}$ of the Reynolds equation (VIII.9), and then rewrite $\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\overline{\mathcal{P}}$ and $\overline{\vec{\mathsf{v}}}\cdot\overline{(\vec{\mathsf{v}}'\cdot\vec{\nabla})\vec{\mathsf{v}}'}$ under consideration of the incompressibility condition (VIII.7).

Remark: While equations (VIII.9) or (VIII.14) describe the dynamics of the mean flow, they rely on the Reynolds stress, which is not yet specified by the equations.

VIII.1.3 b Description of the transition to the turbulent regime

Turbulence takes place when the effects of Reynolds stress $\mathbf{T}_{\mathrm{R}}$—which represents a turbulent transport of momentum—predominates over those of the viscous stress tensor $2\rho\nu\overline{\mathbf{S}}$ associated with the mean flow, i.e. when the latter can no longer dampen the fluctuations corresponding to the former.

Let v_c resp. L_c denote a characteristic velocity resp. length scale of the fluid motion. Assuming that averages—here, a simple average over the volume is meant—over the flow yield the typical orders of magnitude

$$\left\langle\sum_{i,j=1}^{3}\big|\overline{\mathsf{v}'^i\mathsf{v}'^j}\,\overline{\mathbf{S}_{ij}}\big|\right\rangle \sim \frac{\mathsf{v}_c^3}{L_c} \quad\text{and}\quad \left\langle\sum_{i,j=1}^{3}\big|\nu\overline{\mathbf{S}^{ij}}\,\overline{\mathbf{S}_{ij}}\big|^2\right\rangle \sim \frac{\nu\mathsf{v}_c^2}{L_c^2}, \tag{VIII.15}$$

then in the turbulent regime the first of these terms is significantly larger than the second, which corresponds to having a large value of the Reynolds number $\mathrm{Re} \equiv \mathsf{v}_c L_c/\nu$ [Eq. (V.12)], in agreement with the qualitative discussion in § VIII.1.1 b.

In that situation, the equation (VIII.14) describing the evolution of the kinetic energy of the mean flow becomes

$$\frac{\overline{\mathrm{D}}\overline{k}}{\overline{\mathrm{D}}t} = -\vec{\nabla}\cdot\left[\frac{1}{\rho}\overline{\mathcal{P}}\,\overline{\vec{\mathsf{v}}} + \big(\overline{\vec{\mathsf{v}}'\otimes\vec{\mathsf{v}}'}\big)\cdot\overline{\vec{\mathsf{v}}}\right] + \big(\overline{\vec{\mathsf{v}}'\otimes\vec{\mathsf{v}}'}\big) : \overline{\mathbf{S}}, \tag{VIII.16a}$$

or component-wise

$$\frac{\overline{\mathrm{D}}\overline{k}}{\overline{\mathrm{D}}t} = -\sum_{j=1}^{3}\frac{\mathrm{d}}{\mathrm{d}x^j}\left[\frac{1}{\rho}\overline{\mathcal{P}}\,\overline{\mathsf{v}^j} + \sum_{i=1}^{3}\big(\overline{\mathsf{v}'^i\mathsf{v}'^j}\big)\overline{\mathsf{v}_i}\right] + \sum_{i,j=1}^{3}\overline{\mathsf{v}'^i\mathsf{v}'^j}\,\overline{\mathbf{S}_{ij}}. \tag{VIII.16b}$$

That is, the viscosity is no longer a relevant parameter for the dynamics of the mean flow.

As already discussed above, the first term on the right hand side of Eq. (VIII.16) represents the convective transport of energy in the mean flow, while the second term describes the transfer of energy from the mean flow into the turbulent motion, and thus corresponds to the energy "dissipated" by the mean flow. Invoking the first relation in Eq. (VIII.15), the rate of energy dissipation per

unit mass in the mean flow is thus

$$\dot{\mathcal{E}}_{\text{diss.}} = \left\langle \sum_{i,j=1}^{3} \overline{\mathsf{v}'^i \mathsf{v}'^j} \overline{\mathbf{S}_{ij}} \right\rangle \sim \frac{\mathsf{v}_c^3}{L_c}. \tag{VIII.17}$$

This grows like the third power of the typical velocity, i.e. faster than v_c^2, as was mentioned at the end of § VIII.1.1 a for the turbulent regime of the Hagen–Poiseuille flow. In addition, this energy dissipation rate is actually independent of the properties (mass density, viscosity…) of the flowing fluid: turbulence is a characteristic of the motion, not of the fluid itself.

Eventually, the middle term in Eq. (VIII.17) must be negative, so that the energy really flows from the mean flow to the turbulent motion, not in the other direction!

Remark: Looking naively at the definition of the Reynolds number, the limit of an infinitely large Re corresponds to the case of a vanishing shear viscosity, that is, to the model of a perfect fluid. As was just discussed, this is clearly not the case: with growing Reynolds number, i.e. increasing influence of the turbulent motion, the number of eddies in the flow also increases, in which energy is dissipated into heat. In contrast, the kinetic energy is conserved in the flow of a perfect fluid. The solution to this apparent paradox is simply that with increasing Reynolds number, the velocity gradients in the flow also increase. In the incompressible Navier–Stokes equation, the growth of $\triangle\vec{\mathsf{v}}$ compensates the decrease of the viscosity ν, so that the corresponding term does not disappear and the Navier–Stokes equation does not simplify to the Euler equation.

VIII.1.4 Necessity of a statistical approach

As noted above, the evolution equation for the mean flow involves the Reynolds stress, for which no similar equation has been determined yet.

A first, natural solution is simply to write down the evolution equation for the turbulent velocity $\vec{\mathsf{v}}'(t,\vec{r})$, see Eq. (VIII.25) below. Invoking then the identity

$$\frac{\partial}{\partial t}\left[\rho\,\overline{\vec{\mathsf{v}}'(t,\vec{r})\otimes\vec{\mathsf{v}}'(t,\vec{r})}\right] = \rho\overline{\frac{\partial\vec{\mathsf{v}}'(t,\vec{r})}{\partial t}\otimes\vec{\mathsf{v}}'(t,\vec{r})} + \rho\,\overline{\vec{\mathsf{v}}'(t,\vec{r})\otimes\frac{\partial\vec{\mathsf{v}}'(t,\vec{r})}{\partial t}},$$

one can derive a dynamical equation for $\mathbf{T}_{\text{R}}$, the so-called *Reynolds-stress equation*[lxiv], which in component form reads

$$\begin{aligned}\frac{\overline{\mathrm{D}}\mathbf{T}_{\text{R}}^{ij}}{\overline{\mathrm{D}}t} = &-2\overline{\mathcal{P}'\mathbf{S}'^{ij}} + \sum_{k=1}^{3}\frac{\mathrm{d}}{\mathrm{d}x^k}\left(\overline{\mathcal{P}'\mathsf{v}'^i}g^{jk} + \overline{\mathcal{P}'\mathsf{v}'^j}g^{ik} + \rho\overline{\mathsf{v}'^i\mathsf{v}'^j\mathsf{v}'^k} - \nu\frac{\mathrm{d}\mathbf{T}_{\text{R}}^{ij}}{\mathrm{d}x_k}\right)\\ &- \sum_{k=1}^{3}\left(\mathbf{T}_{\text{R}}^{ik}\frac{\mathrm{d}\overline{\mathsf{v}^j}}{\mathrm{d}x^k} + \mathbf{T}_{\text{R}}^{jk}\frac{\mathrm{d}\overline{\mathsf{v}^i}}{\mathrm{d}x^k}\right) - 2\eta\sum_{k=1}^{3}\overline{\frac{\mathrm{d}\mathsf{v}'^i}{\mathrm{d}x^k}\frac{\mathrm{d}\mathsf{v}'^j}{\mathrm{d}x_k}}.\end{aligned} \tag{VIII.18}$$

Irrespective of the physical interpretation of each of the terms in this equation, an important issue is that the evolution of $\rho\overline{\mathsf{v}'^i\mathsf{v}'^j}$ involves a contribution from the components $\rho\overline{\mathsf{v}'^i\mathsf{v}'^j\mathsf{v}'^k}$ of a tensor of degree 3. In turn, the evolution of $\rho\overline{\mathsf{v}'^i\mathsf{v}'^j\mathsf{v}'^k}$ involves the tensor with components $\rho\overline{\mathsf{v}'^i\mathsf{v}'^j\mathsf{v}'^k\mathsf{v}'^l}$, and so on: at each step, the appearance of a tensor of higher degree simply reflects the nonlinearity of the Navier–Stokes equation.

All in all, the incompressible Navier–Stokes equation (VIII.8) is thus equivalent to an infinite hierarchy of equations relating the successive n-point autocorrelation functions of the fluctuations of the velocity field. Any subset of this hierarchy is not closed and involves more unknown fields than equations. A closure prescription, based on some physical assumption, is therefore necessary to obtain a description with a finite number of equations governing the (lower-order) autocorrelation functions. Such an approach is presented in Sec. VIII.2.

[lxiv] *Reynolds-Spannungsgleichung*

An alternative possibility is to assume directly some ansatz for the statistical behavior of the turbulent velocity, especially for its general two-point autocorrelation function, of which the equal-time and position correlator $\overline{\mathsf{v}'^i(t,\vec{r})\mathsf{v}'^j(t,\vec{r})}$ is only a special case. This avenue will be pursued in Sec. VIII.3.

VIII.2 Model of the turbulent viscosity

A first possibility to close the system of equations describing turbulence consists in using the phenomenological concept of *turbulent viscosity*, which is introduced in § VIII.2.1, and for which various models are quickly presented in § VIII.2.2–VIII.2.4.

VIII.2.1 Turbulent viscosity

The basic idea underlying the model is to consider that at the level of the mean flow, the effect of the "turbulent friction" is to redistribute momentum from the high mean-velocity regions towards those with a smaller mean velocity, in the form of a diffusive transport. Accordingly, the traceless part of the turbulent Reynolds stress is dealt with like the corresponding part of the viscous stress tensor (III.27e), and assumed to be proportional to the rate-of-shear tensor of the mean flow (*Boussinesq hypothesis*[ar]):

$$\mathbf{T}_{\mathrm{R}}(t,\vec{r}) - \mathrm{Tr}\big[\mathbf{T}_{\mathrm{R}}(t,\vec{r})\big]\,\mathbf{g}^{-1}(t,\vec{r}) \equiv -2\rho\nu_{\mathrm{turb.}}(t,\vec{r})\,\overline{\mathbf{S}(t,\vec{r})}, \tag{VIII.19a}$$

where the proportionality factor involves the (kinematic) *turbulent viscosity* or *eddy viscosity*[47] ν_{turb}, which a priori depends on time and position. In terms of components in a coordinate system, and replacing the Reynolds stress and its trace by their expressions in terms of the fluctuating velocity, this reads

$$\rho\,\overline{\mathsf{v}'^i(t,\vec{r})\,\mathsf{v}'^j(t,\vec{r})} - \frac{1}{3}\rho\,\overline{[\vec{\mathsf{v}}'(t,\vec{r})]^2}\,g^{ij}(t,\vec{r}) \equiv -2\rho\nu_{\mathrm{turb}}(t,\vec{r})\,\overline{\mathbf{S}^{ij}(t,\vec{r})}. \tag{VIII.19b}$$

Using the ansatz (VIII.19) and invoking the incompressibility of the mean flow, which results in the identity $\vec{\nabla}\cdot\overline{\mathbf{S}} = \frac{1}{2}\triangle\overline{\vec{\mathsf{v}}}$, the Reynolds equation (VIII.9) can be rewritten as

$$\frac{\partial\overline{\vec{\mathsf{v}}}(t,\vec{r})}{\partial t} + \big[\overline{\vec{\mathsf{v}}}(t,\vec{r})\cdot\vec{\nabla}\big]\overline{\vec{\mathsf{v}}}(t,\vec{r}) = -\vec{\nabla}\left\{\frac{\overline{\mathcal{P}}(t,\vec{r})}{\rho} + \frac{\overline{[\vec{\mathsf{v}}'(t,\vec{r})]^2}}{3}\right\} + 2\nu_{\mathrm{eff}}(t,\vec{r})\triangle\overline{\vec{\mathsf{v}}}(t,\vec{r}), \tag{VIII.20}$$

with the effective viscosity

$$\nu_{\mathrm{eff}}(t,\vec{r}) = \nu + \nu_{\mathrm{turb}}(t,\vec{r}), \tag{VIII.21}$$

while the term in curly brackets may be seen as the ratio of an effective pressure over the mass density.

Even if the intrinsic fluid properties, in particular its kinematic viscosity ν, are assumed to be constant and uniform throughout the fluid, this does not hold for the turbulent and effective viscosities ν_{turb}, ν_{eff}, because they model not only the fluid, but also its flow—which is time and position dependent.

Either starting from Eq. (VIII.20) multiplied by $\overline{\vec{\mathsf{v}}}$, or substituting the Reynolds stress with the ansatz (VIII.19) in Eq. (VIII.14), one can derive the equation governing the evolution of the kinetic energy of the mean flow. In particular, one finds that the dissipative term now reads

$$\dot{\mathcal{E}}_{\mathrm{diss}} = 2\nu_{\mathrm{eff}}\,\overline{\mathbf{S}} : \overline{\mathbf{S}} = 2\nu_{\mathrm{eff}}\sum_{i,j=1}^{3}\overline{\mathbf{S}^{ij}}\,\overline{\mathbf{S}_{ij}}.$$

[47] *turbulente Viskosität, Wirbelviskosität*

[ar] J. Boussinesq, 1842–1929

Comparing with the rightmost term in Eq. (VIII.14) gives for the effective viscosity

$$\nu_{\text{eff}} = \frac{-\sum_{i,j} \overline{\mathsf{v}'^i \mathsf{v}'^j} \overline{\mathbf{S}_{ij}}}{2\sum_{i,j} \overline{\mathbf{S}^{ij}} \overline{\mathbf{S}_{ij}}} \gg \frac{2\nu \sum_{i,j} \overline{\mathbf{S}^{ij}} \overline{\mathbf{S}_{ij}}}{2\sum_{i,j} \overline{\mathbf{S}^{ij}} \overline{\mathbf{S}_{ij}}} = \nu,$$

where the inequality holds in the turbulent regime. There thus follows $\nu_{\text{eff}} \approx \nu_{\text{turb}} \gg \nu$.

> It has been argued that in plasmas the turbulent viscosity ν_{turb} could in some regimes be negative—and of the same magnitude as ν—, leading to a so-called "anomalous" effective viscosity ν_{eff} much smaller than ν [30, 31].

Remark: To emphasize the distinction with the turbulent viscosity, ν is sometimes referred to as "molecular" viscosity.

While the ansatz (VIII.19) allows the rewriting of the Reynolds equation in the seemingly simpler form (VIII.20)—in which the two terms contributing to the effective pressure are to be seen as constituting a single field—, it still involves an unknown, flow-dependent quantity, namely the effective viscosity ν_{eff}, which still needs to be specified, which will be the topic of the next three paragraphs.

VIII.2.2 Mixing-length model

A first phenomenological hypothesis for the turbulent viscosity is that implied in the *mixing-length model*[(lxv)] of Prandtl, which postulates the existence of a *mixing length*[(lxvi)] ℓ_m, that represents the typical scale over which momentum is transported by turbulence.

> The ansatz was motivated by an analogy with the kinetic theory of gases, in which the kinematic viscosity ν is proportional to the mean free path and to the typical velocity of particles.

In practice, ℓ_m is determined empirically by the geometry of the flow.

Under this assumption, the turbulent viscosity is given by

$$\nu_{\text{turb}}(t,\vec{r}) = \ell_m(t,\vec{r})^2 \left| \frac{\partial \overline{\mathsf{v}_x(t,\vec{r})}}{\partial y} \right|, \tag{VIII.22}$$

in the case of a two-dimensional flow like the plane Couette flow (§ V.1.2), or for a more general motion

$$\nu_{\text{turb}}(t,\vec{r}) = \ell_m^2(t,\vec{r}) \left| \overline{\mathbf{S}(t,\vec{r})} \right|,$$

with $|\overline{\mathbf{S}}|$ a typical value of the rate-of-shear tensor of the mean flow. In any case, the turbulent viscosity is determined by *local* quantities.

The latter point is actually a weakness of the model. For instance, it implies that the turbulent viscosity (VIII.22) vanishes at an extremum of the mean flow velocity—for instance, on the tube axis in the Hagen–Poiseuille flow—, which is not realistic. In addition, turbulence can be transported from a region into another one, which also not described by the ansatz.

Eventually, the mixing-length model actually merely displaces the arbitrariness from the choice of the turbulent viscosity ν_{turb} to that of the mixing length ℓ_m, i.e. it is just a change of unknown parameter.

VIII.2.3 k-model

In order to describe the possible transport of turbulence within the mean flow, the so-called *k-model* was introduced.

[(lxv)] *Mischungswegansatz* [(lxvi)] *Mischungsweglänge*

Denoting by $\overline{k'} \equiv \frac{1}{2}\overline{\vec{v}'^2}$ the average kinetic energy per unit mass of the fluctuating flow, the turbulent viscosity is postulated to be

$$\nu_{\text{turb}}(t,\vec{r}) = \ell_m(t,\vec{r})\,\overline{k'(t,\vec{r})}^{1/2}. \tag{VIII.23}$$

An additional relation is needed to describe the transport of $\overline{k'}$, to close the system of equations. For simplicity, the actual relation [see Eq (VIII.26) below] is replaced by a similar-looking equation, in which the material derivative following the main flow of the average turbulent kinetic energy equals the sum of a transport term—minus the gradient of a flux density, taken to be proportional to the gradient of $\overline{k'}$—, a production term—namely the energy extracted from the mean flow—, and a dissipation term that describes the rate of energy release as heat, and whose form

$$\overline{\dot{\mathcal{E}}_{\text{diss.}}} = C\overline{k'}^{3/2}/\ell_m$$

is motivated by dimensional arguments, with C a constant. Due to the introduction of this extra phenomenological transport equation for $\overline{k'}$, which was not present in the mixing-length model, the k-model is referred to as a *one-equation model*.[(lxvii)]

The k-model allows by construction the transport of turbulence. However, the mixing length ℓ_m remains an empirical parameter, while two extra ones are now introduced in the transport equation for the average turbulent kinetic energy.

VIII.2.4 (k-ε)-model

In the k-model, the dissipation term $\overline{\dot{\mathcal{E}}_{\text{diss.}}}$ which stands for the ultimate transformation of turbulent kinetic energy into heat under the influence of viscous friction, and should thus be proportional to the viscosity ν, is determined by a dimensional argument.

Another possibility is to consider the energy dissipation rate $\overline{\dot{\mathcal{E}}_{\text{diss}}(t,\vec{r})}$—which is usually rather denoted as $\bar{\varepsilon}$—as a dynamical variable, whose evolution is governed by a transport equation of its own. This approach yields a *two-equation model*,[(lxviii)] the so-called (k-ε)-model.
A dimensional argument then gives $\ell_m \sim \overline{k'}^{3/2}/\overline{\dot{\mathcal{E}}_{\text{diss.}}}$, and thus

$$\nu_{\text{turb}}(t,\vec{r}) = C\frac{\overline{k'(t,\vec{r})}^2}{\overline{\dot{\mathcal{E}}_{\text{diss.}}(t,\vec{r})}}, \tag{VIII.24}$$

with C an empirical constant.

In this model—or rather, this class of models—, the mixing length is totally fixed by the dynamical variables, so that it is no longer arbitrary. On the other hand, the two transport equations introduced for the average turbulent kinetic energy and the dissipation rate involve a handful of parameters, which have to be determined empirically for each flow.

In addition, the (k-ε)-model, like all descriptions involving a turbulent viscosity, relies on the assumption that the typical scale of variations of the mean flow velocity is clearly separated from the turbulent mixing length. This hypothesis is often not satisfied, in that many flows involve turbulent motion over many length scales, in particular with a larger scale comparable to that of the gradients of the mean flow. In such flows, the notion of turbulent viscosity is not really meaningful.

[(lxvii)] *Eingleichungsmodell* [(lxviii)] *Zweigleichungsmodell*

VIII.3 Statistical description of turbulence

Instead of handling the turbulent part of the motion like a source of momentum or a sink of kinetic energy for the mean flow, another approach consists in considering its dynamics more carefully (§ VIII.3.1). As already discussed in § VIII.1.4, this automatically involves higher-order autocorrelation functions of the fluctuating velocity, which hints at the interest of looking at the general autocorrelation functions, rather than just their values at equal times and equal positions. This more general approach allows on the one hand to determine length scales of relevance for turbulence (§ VIII.3.2), and on the other hand to motivate a statistical theory of (isotropic) turbulence (§ VIII.3.3).

VIII.3.1 Dynamics of the turbulent motion

Starting from the incompressible Navier–Stokes equation (VIII.8) for the "total" flow velocity $\vec{\mathsf{v}}$ and subtracting from it the Reynolds-averaged equation (VIII.9) for the mean flow, one finds the dynamical equation governing the evolution of the turbulent velocity $\vec{\mathsf{v}}'$, namely [for brevity, the $(t,\vec{r})$-dependence of the fields is omitted]

$$\rho\left[\frac{\partial\vec{\mathsf{v}}'}{\partial t}+\big(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\big)\vec{\mathsf{v}}'\right]=-\vec{\nabla}\mathcal{P}'+\eta\triangle\vec{\mathsf{v}}'-\rho\big(\vec{\mathsf{v}}'\cdot\vec{\nabla}\big)\overline{\vec{\mathsf{v}}}-\vec{\nabla}\cdot\big(\rho\vec{\mathsf{v}}'\otimes\vec{\mathsf{v}}'-\mathbf{T}_{\mathrm{R}}\big). \tag{VIII.25a}$$

Equivalently, after dividing by ρ and projecting along the x^i-axis of a coordinate system, one may write

$$\frac{\partial\mathsf{v}'^i}{\partial t}+\big(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\big)\mathsf{v}'^i=-\frac{1}{\rho}\frac{\mathrm{d}\mathcal{P}'}{\mathrm{d}x_i}+\nu\triangle\mathsf{v}'^i-\big(\vec{\mathsf{v}}'\cdot\vec{\nabla}\big)\overline{\mathsf{v}^i}-\frac{\mathrm{d}}{\mathrm{d}x^j}\Big(\mathsf{v}'^i\mathsf{v}'^j-\overline{\mathsf{v}'^i\mathsf{v}'^j}\Big). \tag{VIII.25b}$$

One recognizes on the left hand side of those equations the material derivative of the fluctuating velocity following the mean flow, $\overline{\mathrm{D}}\vec{\mathsf{v}}'/\overline{\mathrm{D}}t$.

From the turbulent Navier–Stokes equation (VIII.25), one finds for the average kinetic energy per unit mass of the fluctuating motion $\overline{k'}\equiv\frac{1}{2}\overline{\vec{\mathsf{v}}'^2}$

$$\frac{\overline{\mathrm{D}}\,\overline{k'}}{\overline{\mathrm{D}}t}=-\sum_{j=1}^{3}\frac{\mathrm{d}}{\mathrm{d}x^j}\left[\frac{1}{\rho}\overline{\mathcal{P}'\mathsf{v}'^j}+\sum_{i=1}^{3}\left(\frac{1}{2}\overline{\mathsf{v}'_i\mathsf{v}'^i\mathsf{v}'^j}-2\nu\overline{\mathsf{v}'_i\mathbf{S}'^{ij}}\right)\right]-\sum_{i,j=1}^{3}\overline{\mathsf{v}'^i\mathsf{v}'^j}\,\overline{\mathbf{S}}_{ij}-2\nu\sum_{i,j=1}^{3}\overline{\mathbf{S}'^{ij}\mathbf{S}'_{ij}} \tag{VIII.26}$$

with $\mathbf{S}'^{ij}\equiv\dfrac{1}{2}\left(\dfrac{\mathrm{d}\mathsf{v}'^i}{\mathrm{d}x_j}+\dfrac{\mathrm{d}\mathsf{v}'^j}{\mathrm{d}x_i}-\dfrac{2}{3}g^{ij}\vec{\nabla}\cdot\vec{\mathsf{v}}'\right)$ the components of the fluctuating rate-of-shear tensor.

- The first term on the right hand side describes a turbulent yet conservative transport—due to pressure, convective transport by the fluctuating flow itself, or diffusive transport due the viscous friction—, mixing the various length scales: the kinetic energy is transported without loss from the large scales, comparable to that of the variations of the mean flow, to the smaller ones. This process is referred to as *energy cascade*.

- The second term describes the "creation" of turbulent kinetic energy, which is actually extracted from the mean flow: it is precisely—up to the sign!—the loss term in Eq. (VIII.16) describing the transport of kinetic energy in the mean flow.

- Eventually, the rightmost term in Eq. (VIII.26) represents the average energy dissipated as heat by the viscous friction forces, and will hereafter be denoted as $\overline{\dot{\mathcal{E}}_{\mathrm{diss.}}}$.

In a statistically homogeneous and stationary turbulent flow, the amount of energy dissipated by viscous friction equals that extracted by turbulence from the mean flow, i.e.

$$-\sum_{i,j=1}^{3}\overline{\mathsf{v}'^i\mathsf{v}'^j}\,\overline{\mathbf{S}}_{ij}=2\nu\sum_{i,j=1}^{3}\overline{\mathbf{S}'^{ij}\mathbf{S}'_{ij}}. \tag{VIII.27}$$

VIII.3.2 Characteristic length scales of turbulence

VIII.3.2 a Two-point autocorrelation function of the turbulent velocity fluctuations

The fluctuations of the turbulent velocity $\vec{\mathsf{v}}'$ are governed by an unknown probability distribution. Instead of knowing the latter, it is equivalent to rely on the (*auto*)*correlation functions*

$$\kappa^{(n)}_{i_1 i_2 \dots i_n}(t_1,\vec{r}_1;t_2,\vec{r}_2;\dots;t_n,\vec{r}_n) \equiv \overline{\mathsf{v}'_{i_1}(t_1,\vec{r}_1)\,\mathsf{v}'_{i_2}(t_2,\vec{r}_2)\cdots\mathsf{v}'_{i_n}(t_n,\vec{r}_n)}\,,$$

in which the components of fluctuations at different instants and positions are correlated with each other. Remember that the 1-point averages vanish, Eq. (VIII.4).

The knowledge of *all* n-point autocorrelation functions is equivalent to that of the probability distribution. Yet the simplest—both from the experimental point of view as well as in numerical simulations—of these functions are the two-point autocorrelation functions [32]

$$\kappa^{(2)}_{ij}(t,\vec{r};t',\vec{r}') \equiv \overline{\mathsf{v}'_i(t,\vec{r})\,\mathsf{v}'_j(t',\vec{r}')}\,, \tag{VIII.28}$$

which will hereafter be considered only at equal times $t' = t$.

In the case of a statistically stationary turbulent flow,[48] the 2-point autocorrelation functions $\kappa^{(2)}_{ij}(t,\vec{r};t',\vec{r}')$ only depend on the time difference $t'-t$, which vanishes if both instants are equal, yielding a function of $\vec{r}$, $\vec{r}'$ only. If the turbulence is in addition statistically homogeneous[48]—which necessitates that one considers the flow far from any wall or obstacle, although this does not yet constitute a sufficient condition—, then the 2-point autocorrelation function only depends on the separation $\vec{X} \equiv \vec{r}' - \vec{r}$ of the two positions:

$$\kappa_{ij}(\vec{X}) = \overline{\mathsf{v}'_i(t,\vec{r})\,\mathsf{v}'_j(t,\vec{r}+\vec{X})}\,. \tag{VIII.29}$$

If the turbulence is statistically locally isotropic,[48] the tensor κ_{ij} only depends on the distance $X \equiv |\vec{X}|$ between the two points. Such a statistical local isotropy often represents a good assumption for the structure of the turbulent motion on small scales—again, far from the boundaries of the flow—and will be assumed hereafter.

Consider two points at $\vec{r}$ and $\vec{r}+\vec{X}$. Let $\vec{\mathrm{e}}_\parallel$ denote a unit vector along $\vec{X}$, $\vec{\mathrm{e}}_\perp$ a unit vector in a direction orthogonal to $\vec{\mathrm{e}}_\parallel$, and $\vec{\mathrm{e}}'_\perp$ perpendicular to both $\vec{\mathrm{e}}_\parallel$ and $\vec{\mathrm{e}}_\perp$. The component $\mathsf{v}'_\parallel$ of the turbulent velocity—at $\vec{r}$ or $\vec{r}+\vec{X}$—along $\vec{\mathrm{e}}_\parallel$ is referred to as "longitudinal", those along $\vec{\mathrm{e}}_\perp$ or $\vec{\mathrm{e}}'_\perp$ ($\mathsf{v}'_\perp$, $\mathsf{v}'_{\perp'}$) as "lateral".

The autocorrelation function (VIII.29) can be expressed with the help of the two-point functions $\kappa_\parallel(X) \equiv \overline{\mathsf{v}'_\parallel(t,\vec{r})\,\mathsf{v}'_\parallel(t,\vec{r}+\vec{X})}$, $\kappa_\perp(X) \equiv \overline{\mathsf{v}'_\perp(t,\vec{r})\,\mathsf{v}'_\perp(t,\vec{r}+\vec{X})}$, and $\kappa'_\perp(X) \equiv \overline{\mathsf{v}'_\perp(t,\vec{r})\,\mathsf{v}'_{\perp'}(t,\vec{r}+\vec{X})}$ as

$$\kappa_{ij}(X) = \frac{X_i X_j}{\vec{X}^2}\big[\kappa_\parallel(X) - \kappa_\perp(X)\big] + \kappa_\perp(X)\,\delta_{ij} + \kappa'_\perp(X)\sum_{k=1}^{3}\frac{\epsilon^{ijk}X_k}{X},$$

with $\{X_i\}$ the Cartesian components of $\vec{X}$, where the last term vanishes for statistically space-parity invariant turbulence,[49] which is assumed to be the case from now on.[50]
Multiplying the incompressibility condition $\vec{\nabla}\cdot\vec{\mathsf{v}}' = 0$ with v'_j and averaging yields

$$\sum_{i=1}^{3}\frac{\partial\kappa_{ij}(X)}{\partial X_i} = 0,$$

resulting in the identity

$$\kappa_\perp(X) = \kappa_\parallel(X) + \frac{X}{2}\frac{\mathrm{d}\kappa_\parallel(X)}{\mathrm{d}X}.$$

[48] This means that the probability distribution of the velocity fluctuations $\vec{\mathsf{v}}'$ is stationary (time-independent) resp. homogeneous (position-independent) resp. locally isotropic (the same for all Cartesian components of $\vec{\mathsf{v}}'$).
[49] Invariance under the space-parity operation is sometimes considered to be part of the isotropy, sometimes not...
[50] In presence of a magnetic field—i.e. in the realm of magnetohydrodynamics—, this last term is indeed present.

This result means that all κ_{ij} can be expressed in terms of the autocorrelation function $\kappa_\parallel$ only, so that we only discuss the latter from now on.

VIII.3.2 b Microscopic and macroscopic length scales of turbulence

The assumed statistical isotropy gives $\kappa_\parallel(0) = \overline{[\mathsf{v}_\parallel(t,\vec{r})]^2} = \frac{1}{3}\overline{[\vec{\mathsf{v}}'(t,\vec{r})]^2}$. Let $f(X)$ be the function such that

$$\kappa_\parallel(X) \equiv \frac{1}{3}\overline{[\vec{\mathsf{v}}'(t,\vec{r})]^2}\, f(X).$$

and that

- $f(0) = 1$;
- the fluctuations of the velocity at points separated by a large distance X are not correlated with another, so that $\kappa_\parallel(X)$ must vanish: $\lim\limits_{X\to\infty} f(X) = 0$.
- In addition, f is assumed to be integrable over $\mathbb{R}_+$, and such that its integral from 0 to $+\infty$ is convergent.

The dimensionless function f then defines a typical macroscopic length scale, namely that over which f resp. $\kappa_\parallel$ decreases,[(51)] the *integral scale* or *external scale*[(lxix)]

$$L_I \equiv \int_0^\infty f(X)\,\mathrm{d}X. \tag{VIII.30}$$

Empirically, this integral scale is found to be comparable to the scale of the variations of the mean flow velocity, i.e. characteristic for the production of turbulence in the flow. For example, in a flow past an obstacle, L_I is of the same order of magnitude as the size of the obstacle.

Assuming—as has been done till now—locally isotropic and space-parity invariant turbulence, the function $f(X)$ is even, so that its Taylor expansion around $X = 0$ defines a microscopic length scale:

$$f(X) \underset{X\to 0}{\simeq} 1 - \frac{1}{2}\left(\frac{X}{\ell_T}\right)^2 + \mathcal{O}(X^4) \quad \text{with} \quad \ell_T^2 \equiv -\frac{1}{f''(0)} > 0. \tag{VIII.31}$$

ℓ_T is the *Taylor microscale*.[(lxx)(52)]
Let $x_\parallel$ denote the coordinate along $\vec{X}$. One finds

$$\ell_T^2 = \overline{\left[\mathsf{v}'_\parallel(t,\vec{r})\right]^2} \Big/ \overline{\left[\mathrm{d}\mathsf{v}'_\parallel(t,\vec{r})/\mathrm{d}x_\parallel\right]^2}, \tag{VIII.32}$$

i.e. ℓ_T is the typical length scale of the gradients of the velocity fluctuations.

> Using the definition of f, the Taylor expansion (VIII.31) can be rewritten as
>
> $$\frac{\overline{\mathsf{v}'_\parallel(t,\vec{r})\,\mathsf{v}'_\parallel(t,\vec{r}+\vec{X})}}{\overline{[\mathsf{v}'_\parallel(t,\vec{r})]^2}} \underset{X\to 0}{\simeq} 1 + \frac{1}{2}\frac{\overline{\mathsf{v}'_\parallel(t,\vec{r})\,\partial_\parallel^2\mathsf{v}'_\parallel(t,\vec{r})}}{\overline{[\mathsf{v}'_\parallel(t,\vec{r})]^2}}\,X^2,$$
>
> where $\partial_\parallel$ denotes the derivative with respect to $x_\parallel$. Invoking the statistical homogeneity of the turbulence, $\overline{[\mathsf{v}'_\parallel(t,\vec{r})]^2}$ is independent of position, thus of $x_\parallel$, which after differentiation leads successively to $\overline{\mathsf{v}'_\parallel(t,\vec{r})\,\partial_\parallel\mathsf{v}'_\parallel(t,\vec{r})} = 0$ and then $\overline{[\partial_\parallel\mathsf{v}'_\parallel(t,\vec{r})]^2} + \overline{\mathsf{v}'_\parallel\,\partial_\parallel^2\mathsf{v}'_\parallel(t,\vec{r})} = 0$, proving relation (VIII.32).

(51) The reader should think of the example $\kappa_\parallel(X) = \kappa_\parallel(0)\,\mathrm{e}^{-X/L_I}$, or at least $\kappa_\parallel(X) \gtrsim \mathrm{e}^{-X/L_I}$ for X large enough compared to a microscopic scale much smaller than L_I.

(52) ... named after the fluid dynamics practitioner G. I. Taylor, not after B. Taylor of the Taylor series.

(lxix) *Integralskala, äußere Skala* (lxx) *Taylor-Mikroskala*

Remark: Even if the Taylor microscale emerges naturally from the formalism, it does not represent the length scale of the smallest eddies in the flow, despite what one could expect.

To find another, physically more relevant microscopic scale, it is necessary to investigate the behavior of the *longitudinal increment*

$$\delta \mathsf{v}'_{\parallel}(X) \equiv \mathsf{v}'_{\parallel}(t, \vec{r} + \vec{X}) - \mathsf{v}'_{\parallel}(t, \vec{r}) \tag{VIII.33}$$

of the velocity fluctuations, which compares the values of the longitudinal component of the latter at different points. According to the definition of the derivative, $\mathrm{d}\mathsf{v}'_{\parallel}/\mathrm{d}x_{\parallel}$ is the limit when $X \to 0$ of the ratio $\delta \mathsf{v}'_{\parallel}(X)/X$. The microscopic *Kolmogorov*[(as)] *length scale* ℓ_K is then defined by

$$\frac{\overline{[\delta \mathsf{v}'_{\parallel}(\ell_K)]^2}}{\ell_K^2} \equiv \lim_{X \to 0} \frac{\overline{[\delta \mathsf{v}'_{\parallel}(X)]^2}}{X^2} = \overline{\left[\frac{\mathrm{d}\mathsf{v}'_{\parallel}(t, \vec{r})}{\mathrm{d}x_{\parallel}}\right]^2}. \tag{VIII.34}$$

The role of this length scale will be discussed in the following Section, yet it can already be mentioned that it is the typical scale of the smallest turbulent eddies, and thus the pendant to the integral scale L_I.

Remark: Squaring the longitudinal velocity increment (VIII.33) and averaging under consideration of the statistical homogeneity, one finds when invoking Eq. (VIII.31)

$$\frac{\overline{[\delta \mathsf{v}'_{\parallel}(X)]^2}}{2\overline{[\mathsf{v}'_{\parallel}(X)]^2}} \underset{X \to 0}{\sim} \frac{1}{2}\left(\frac{X}{\ell_T}\right)^2.$$

On the other hand, experiments or numerical simulations show that the term on the left hand side of this relation equals about 1 when X is larger than the integral scale L_I. That is, the latter and the Taylor microscale can also be recovered from the longitudinal velocity increment.

VIII.3.3 The Kolmogorov theory (K41) of isotropic turbulence

A first successful statistical theory of turbulence was proposed in 1941 by Kolmogorov for statistically locally isotropic turbulent motion, assuming further stationarity, homogeneity and space-parity invariance [33, 34]. This *K41-theory* describes the fluctuations of the velocity increments $\delta \mathsf{v}'_i(X)$, and relies on two assumptions—originally termed *similarity hypotheses* by Kolmogorov:

1st Kolmogorov hypothesis

> *The probability distributions of the turbulent-velocity increments* $\delta \mathsf{v}'_i(X)$, $i = 1, 2, 3$, *are universal on separation scales* X *small compared to the integral scale* L_I, *and are entirely determined by the kinematic viscosity* ν *of the fluid and by the average energy dissipation rate per unit mass* $\overline{\dot{\mathcal{E}}_{\mathrm{diss.}}}$. (K41-1)

Here "universality" refers to an independence from the precise process which triggers the turbulence.

Considering e.g. the longitudinal increment, this hypothesis gives for the second moment of the probability distribution

$$\overline{[\delta \mathsf{v}'_{\parallel}(X)]^2} = \sqrt{\nu \overline{\dot{\mathcal{E}}_{\mathrm{diss.}}}}\, \Phi^{(2)}\left(\frac{X}{\ell_K}\right) \quad \text{for } X \ll L_I \quad \text{with} \quad \ell_K = \left(\frac{\nu^3}{\overline{\dot{\mathcal{E}}_{\mathrm{diss.}}}}\right)^{1/4} \tag{VIII.35}$$

and $\Phi^{(2)}$ a universal function, irrespective of the flow under study. The factor $\sqrt{\nu \overline{\dot{\mathcal{E}}_{\mathrm{diss.}}}}$ and the form of ℓ_K follow from dimensional considerations—the n-point autocorrelation function involves another function $\Phi^{(n)}$ multiplying a factor $\left(\nu \overline{\dot{\mathcal{E}}_{\mathrm{diss.}}}\right)^{n/4}$.

[(as)] А. Н. Колмогоров = A. N. Kolmogorov, 1903–1987

The hypothesis (K41-1) amounts to assuming that the physics of the fluctuating motion, far from the scale at which turbulence is created, is fully governed by the available energy extracted from the mean flow—which in the stationary regime equals the average energy dissipated by viscous friction in the turbulent motion—and by the amount of friction.

2nd Kolmogorov hypothesis

The probability distribution of the turbulent-velocity increments $\delta \mathsf{v}'_i(X)$, $i=1,2,3$, is independent of the kinematic viscosity ν of the fluid on separation scales X large compared to the microscopic scale ℓ_K. (K41-2)

The idea here is that viscous friction only plays a role at the microscopic scale, while the rest of the turbulent energy cascade is conservative.

The assumption holds for the longitudinal increment (VIII.35) if and only if the universal function $\Phi^{(2)}$ scales like $\Phi^{(2)}(x) \underset{x\gg 1}{\sim} B^{(2)} x^{2/3}$ at large x, with $B^{(2)}$ a universal constant, i.e. if

$$\overline{[\delta \mathsf{v}'_\parallel(X)]^2} \sim B^{(2)} \big(\overline{\dot{\mathcal{E}}_{\text{diss.}}} X\big)^{2/3} \quad \text{for } \ell_K \ll X \ll L_I. \tag{VIII.36}$$

The Kolmogorov 2/3-*law* (VIII.36) does not involve any length scale: this reflects the length-scale "self-similarity" of the conservative energy-cascading process in the *inertial range*[(lxxi)] $\ell_K \ll X \ll L_I$, in which the only relevant parameter is the energy dissipation rate.

The increase of the autocorrelation function $\overline{[\delta \mathsf{v}'_\parallel(X)]^2}$ as $X^{2/3}$ is observed both experimentally and in numerical simulations.[(53)]

A further prediction of the K41-theory regards the energy spectrum of the turbulent motion. Let $\tilde{\vec{\mathsf{v}}}'(t,\vec{k})$ denote the spatial Fourier transform of the fluctuating velocity. Up to a factor involving the inverse of the (infinite) volume of the flow, the kinetic energy per unit mass of the turbulent motion component with wave vector equal to $\vec{k}$ up to $\mathrm{d}^3\vec{k}$ is $\frac{1}{2}[\tilde{\vec{\mathsf{v}}}'(t,\vec{k})]^2\,\mathrm{d}^3\vec{k}$. In the case of statistically isotropic turbulence, $\frac{1}{2}[\tilde{\vec{\mathsf{v}}}'(t,\vec{k})]^2\,\mathrm{d}^3\vec{k} = 2\pi k^2 [\tilde{\vec{\mathsf{v}}}'(t,\vec{k})]^2\,\mathrm{d}k \equiv S_E(k)\,\mathrm{d}k$ with $S_E(k)$ the kinetic-energy spectral density.

From the 2/3-law (VIII.36),[(54)] one can then derive the $-5/3$-*law* for the latter, namely

$$S_E(k) = C_K\, \overline{\dot{\mathcal{E}}_{\text{diss.}}}^{2/3} k^{-5/3} \quad \text{for } L_I^{-1} \ll k \ll \ell_K^{-1} = \left(\frac{\overline{\dot{\mathcal{E}}_{\text{diss.}}}}{\nu^3}\right)^{1/4}, \tag{VIII.37}$$

with C_K a universal constant, the *Kolmogorov constant*, independent from the fluid or the flow geometry, yet depending—like the $-5/3$-law itself—on the space dimensionality. Experimentally[(53)] one finds $C_K \approx 1.45$.

As already mentioned, the laws (VIII.36) and (VIII.37) provide a rather satisfactory description of the results of experiments or numerical simulations. The K41-theory also predicts that the higher-order moments of the probability distribution of the velocity increments should be universal as well—and the reader can easily determine their scaling behavior $\overline{[\delta \mathsf{v}'_\parallel(X)]^n} \sim B^{(n)} \big(\overline{\dot{\mathcal{E}}_{\text{diss.}}} X\big)^{n/3}$ in the inertial range using dimensional arguments—, yet this prediction is no longer supported by experiment: the moments do depend on X as power laws, yet not with the predicted exponents.

A deficiency of Kolmogorov's theory is that in his energy cascade, only eddies of similar size interact with each other to transfer the energy from large to small length scales, which is encoded

[(53)] Examples from experimental results are presented in Ref. [35, Chapter 5].

[(54)] ... and assuming that $S_E(k)$ behaves properly, i.e. decreases quickly enough, at large k.

[(lxxi)] *Trägheitsbereich*

in the self-similarity assumption. In that picture, the distribution of the eddy sizes is statistically stationary.

In contrast, turbulent motion itself tends to deform eddies, by stretching vortices into tubes of smaller cross section, until they become so small that shear viscosity becomes efficient to counteract this process. This behavior somewhat clashes with Kolmogorov's picture.

Bibliography for Chapter VIII

- Chandrasekhar [36].
- Feynman [10, 11] Chapter 41-4–41–6.
- Faber [1] Chapter 9.1, 9.2–9.6.
- Frisch [35].
- Guyon *et al.* [2] Chapter 12.
- Landau–Lifshitz [4, 5] Chapter III § 33–34.

CHAPTER IX

Convective heat transfer

The examples of dissipative flows we have seen until now were dominated either by viscous effects (Chap. V and § VI.1.4) or by convective motion (Chap. VIII). In either case, the energy-conservation equation (III.37), and in particular the term representing heat conduction, was never playing a dominant role, with the exception of a brief mention of heat transport in the study of static Newtonian fluids (§ V.1.1).

The purpose of this Chapter is to shift the focus, and to discuss motions of Newtonian fluids in which heat is transferred from one region of the fluid to another. A first such type of transfer is heat conduction, which was already encountered in the static case. Under the generic term "convection", or "convective heat transfer", one encompasses flows in which heat is also transported by the moving fluid, not only via conduction.

Heat transfer will be caused by differences in temperature in a fluid. Going back to the equations of motion, one can make a few assumptions so as to eliminate or at least suppress other effects, and emphasize the role of temperature gradients in moving fluids (Sec IX.1). A specific instance of fluid motion driven by a temperature difference, yet also controlled by the fluid viscosity, which allows for a richer phenomenology, is then presented in Sec. IX.2.

IX.1 Equations of convective heat transfer

The fundamental equations of the dynamics of Newtonian fluids introduced in Chap. III include heat conduction, in the form of a term involving the gradient of temperature, yet the change in time of temperature does not appear explicitly. To obtain an equation involving the time derivative of temperature, some rewriting of the basic equations is thus needed, which will be done together with a few simplifications (§ IX.1.1). Conduction in a static fluid is then recovered as a limiting case.

In many instances, the main effect of temperature differences is however rather to lead to variations of the mass density, which in turn trigger the fluid motion. To have a more adapted description of such phenomena, a few extra simplifying assumptions are made, leading to a new, closed set of coupled equations (§ IX.1.2).

IX.1.1 Basic equations of heat transfer

Consider a Newtonian fluid submitted to conservative volume forces $\vec{f}_V = -\rho\vec{\nabla}\Phi$. Its motion is governed by the laws established in Chap. III, namely by the continuity equation, the Navier–Stokes

equation, and the energy-conservation equation or equivalently the entropy-balance equation, which we now recall.

Expanding the divergence of the mass flux density, the continuity equation (III.9) becomes

$$\frac{\mathrm{D}\rho(t,\vec{r})}{\mathrm{D}t} = -\rho(t,\vec{r})\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r}). \tag{IX.1a}$$

In turn, the Navier–Stokes equation (III.31a) may be written in the form

$$\rho(t,\vec{r})\frac{\mathrm{D}\vec{\mathrm{v}}(t,\vec{r})}{\mathrm{D}t} = -\vec{\nabla}\mathcal{P}(t,\vec{r}) - \rho(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}) + 2\vec{\nabla}\cdot\big[\eta(t,\vec{r})\mathbf{S}(t,\vec{r})\big] + \vec{\nabla}\big[\zeta(t,\vec{r})\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]. \tag{IX.1b}$$

Eventually, straightforward algebra using the continuity equation allows one to rewrite the entropy balance equation (III.42b) as

$$\rho(t,\vec{r})\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{s(t,\vec{r})}{\rho(t,\vec{r})}\right] = \frac{1}{T(t,\vec{r})}\vec{\nabla}\cdot\big[\kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r})\big] + \frac{2\eta(t,\vec{r})}{T(t,\vec{r})}\mathbf{S}(t,\vec{r}):\mathbf{S}(t,\vec{r}) + \frac{\zeta(t,\vec{r})}{T(t,\vec{r})}\big[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]^2. \tag{IX.1c}$$

Since we wish to isolate effects directly related with the transfer of heat, or playing a role in it, we shall make a few assumptions, so as to simplify the above set of equations.

- The transport coefficients η, ζ, κ depend on the local thermodynamic state of the fluid, i.e. on its local mass density ρ and temperature T, and thereby indirectly on time and position. Nevertheless, they will be taken as constant and uniform throughout the fluid, and pulled out of the various derivatives in Eqs. (IX.1b)–(IX.1c). This is a reasonable assumption as long as only small variations of the fluid properties are considered, which is consistent with the next assumption.

  Somewhat abusively, we shall in fact even allow ourselves to consider η resp. κ as uniform in Eq. (IX.1b) resp. (IX.1c), later replace them by related (diffusion) coefficients $\nu = \eta/\rho$ resp. $\alpha = \kappa/\rho c_{\mathcal{P}}$, and then consider the latter as uniform constant quantities.
  The whole procedure is only "justified" in that one can check—by comparing calculations using this assumption with numerical computations performed without the simplifications—that it does not lead to omitting a physical phenomenon.

- The fluid motions under consideration will be assumed to be "slow", i.e. to involve a small flow velocity, in the following sense:
  - The incompressibility condition $\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r}) = 0$ will hold on the right hand sides of each of Eqs. (IX.1). Accordingly, Eq. (IX.1a) simplifies to $\mathrm{D}\rho(t,\vec{r})/\mathrm{D}t = 0$ while Eq. (IX.1b) becomes the incompressible Navier–Stokes equation
  $$\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r}) = -\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}) - \vec{\nabla}\Phi(t,\vec{r}) + \nu\triangle\vec{\mathrm{v}}(t,\vec{r}), \tag{IX.2}$$
  with a constant and uniform kinematic (shear) viscosity ν.
  - The rate of shear is small, so that its square can be neglected in Eq. (IX.1c). Accordingly, that equation simplifies to
  $$\rho(t,\vec{r})T(t,\vec{r})\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{s(t,\vec{r})}{\rho(t,\vec{r})}\right] = \kappa\triangle T(t,\vec{r}). \tag{IX.3}$$

The term on the left hand side of that equation can be further rewritten. As a matter of fact, one can show that the differential of the specific entropy in a fluid particle is related to the change in temperature by

$$T\,\mathrm{d}\left(\frac{s}{\rho}\right) = c_{\mathcal{P}}\,\mathrm{d}T, \tag{IX.4}$$

where $c_{\mathcal{P}}$ denotes the specific heat capacity at constant pressure of the fluid. In turn, this relation translates into an identity relating the material derivatives when the fluid particles are followed in their motion. The left member of Eq. (IX.3) may then be reformulated in terms of the material

derivative of the temperature. Introducing the *thermal diffusivity*[lxxii]

$$\alpha \equiv \frac{\kappa}{\rho c_{\mathcal{P}}}, \tag{IX.5}$$

which will from now on be assumed to be constant and uniform in the fluid, where $\rho c_{\mathcal{P}}$ is the volumetric heat capacity at constant pressure, one eventually obtains

$$\boxed{\frac{\mathrm{D}T(t,\vec{r})}{\mathrm{D}t} = \frac{\partial T(t,\vec{r})}{\partial t} + \left[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\right]T(t,\vec{r}) = \alpha\triangle T(t,\vec{r}),} \tag{IX.6}$$

which is sometimes referred to as (*convective*) *heat transfer equation*.

If the fluid is at rest or if its velocity is "small" enough to ensure that the convective term $\vec{\mathrm{v}}\cdot\vec{\nabla}T$ remains negligible, Eq. (IX.6) simplifies to the classical heat diffusion equation, with diffusion constant α.

The thermal diffusivity α thus measures the ability of a medium to transfer heat diffusively, just like the kinematic shear viscosity ν characterizes the diffusive transfer of momentum. Accordingly, both coefficients have the same dimension $\mathsf{L}^2\,\mathsf{T}^{-1}$, and can thus be compared meaningfully. Their relative strength is measured by the dimensionless *Prandtl number*

$$\boxed{\mathrm{Pr} \equiv \frac{\nu}{\alpha} = \frac{\eta c_{\mathcal{P}}}{\kappa}} \tag{IX.7}$$

which in contrast to the Mach, Reynolds, Froude, Ekman, Rossby… numbers encountered in the previous chapters is entirely determined by the fluid, independent of any flow characteristics.

IX.1.2 Boussinesq approximation

If there is a temperature gradient in a fluid, it will lead to a heat flux density, and thereby to the transfer of heat, thus influencing the fluid motion. However, heat exchanges by conduction are often slow—except in metals—, so that another effect due to temperature differences is often the first to play a significant role, namely thermal expansion (or contraction), which will lead to buoyancy (§ IV.1.1) when a fluid particle acquires a mass density different from that of its surroundings.

The simplest approach to account for this effect, due to Boussinesq,[55] consists in considering that even though the fluid mass density changes, nevertheless the motion can to a very good approximation be viewed as incompressible—which is what was assumed in § IX.1.1:

$$\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r}) \simeq 0, \tag{IX.8}$$

where $\simeq$ is used to allow for small relative variations in the mass density, which are directly related to the expansion rate through Eq. (IX.1a).

Denoting by T_0 a typical temperature in the fluid and ρ_0 the corresponding mass density (strictly speaking, at a given pressure), the effect of thermal expansion on the latter reads

$$\rho(\Theta) = \rho_0(1 - \alpha_{(\mathcal{V})}\Theta), \tag{IX.9}$$

with

$$\Theta \equiv T - T_0 \tag{IX.10}$$

the temperature difference measured with respect to the reference value, and

$$\alpha_{(\mathcal{V})} \equiv -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{\mathcal{P},N} \tag{IX.11}$$

the *thermal expansion coefficient* for volume, where the derivative is taken at the thermodynamic

[55] Hence its denomination *Boussinesq approximation* (for buoyancy).

[lxxii] *Temperaturleitfähigkeit*

point corresponding to the reference value ρ_0. Strictly speaking, the linear regime (IX.9)[(56)] only holds when $|\alpha_{(\mathcal{V})}\Theta| \ll 1$, as will be assumed hereafter.

Consistent with relation (IX.9), the pressure term in the incompressible Navier–Stokes equation can be approximated as

$$-\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}) \simeq -\frac{\vec{\nabla}\mathcal{P}(t,\vec{r})}{\rho_0}\big[1+\alpha_{(\mathcal{V})}\Theta(t,\vec{r})\big].$$

Introducing an effective pressure $\mathcal{P}_{\text{eff}.}$ that accounts for the leading effect of the potential Φ from which the volume forces derive,

$$\mathcal{P}_{\text{eff}.}(t,\vec{r}) \equiv \mathcal{P}(t,\vec{r}) + \rho_0\Phi(t,\vec{r}),$$

one finds

$$-\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}) - \vec{\nabla}\Phi(t,\vec{r}) \simeq -\frac{\vec{\nabla}\mathcal{P}_{\text{eff}.}(t,\vec{r})}{\rho_0} + \alpha_{(\mathcal{V})}\Theta(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}),$$

where a term of subleading order $\alpha_{(\mathcal{V})}\Theta\vec{\nabla}\mathcal{P}_{\text{eff}.}$ has been dropped. To this level of approximation, the incompressible Navier–Stokes equation (IX.2) becomes

$$\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r}) = -\frac{\vec{\nabla}\mathcal{P}_{\text{eff}.}(t,\vec{r})}{\rho_0} + \alpha_{(\mathcal{V})}\Theta(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}) + \nu\triangle\vec{\mathrm{v}}(t,\vec{r}). \tag{IX.12}$$

This form of the Navier–Stokes equation emphasizes the role of a finite temperature difference Θ in providing an extra force density which contributes to the buoyancy, supplementing the effective pressure term.

Eventually, definition (IX.10) together with the convective heat transfer equation (IX.6) lead at once to

$$\frac{\partial\Theta(t,\vec{r})}{\partial t} + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\Theta(t,\vec{r}) = \alpha\triangle\Theta(t,\vec{r}). \tag{IX.13}$$

The (*Oberbeck*[(at)]–)*Boussinesq equations* (IX.8), (IX.12), and (IX.13) represent a closed system of five coupled scalar equations for the dynamical fields $\vec{\mathrm{v}}$, Θ—which in turn determines the variation of the mass density—and $\mathcal{P}_{\text{eff}.}$.

IX.2 Rayleigh–Bénard convection

A relatively simple example of flow in which thermal effects play a major role is that of a fluid with a positive[(57)] thermal expansion coefficient $\alpha_{(\mathcal{V})}$ between two horizontal plates at constant but different temperatures, the lower plate being at the higher temperature, in a uniform gravitational potential $-\vec{\nabla}\Phi(t,\vec{r}) = \vec{g}$, in the absence of horizontal pressure gradient.

The distance between the two plates will be denoted by d, and the temperature difference between them by ΔT, with $\Delta T > 0$ when the lower plate is warmer. When needed, a system of Cartesian coordinates will be used, with the (x,y)-plane midway between the plates and a vertical z-axis, with the acceleration of gravity pointing towards negative values of z, i.e. $\Phi(t,\vec{r}) = gz$.

IX.2.1 Phenomenology of the Rayleigh–Bénard convection

IX.2.1 a Experimental findings

If both plates are at the same temperature or if the upper one is warmer ($\Delta T < 0$), the fluid between them can simply be at rest, with a stationary linear temperature profile.

[(56)] ... which is in fact the beginning of a Taylor expansion in Θ.

[(57)] This is the "usual" behavior, but there famously exist fluids with a density anomaly, like liquid water below 4°C.

[(at)] A. Oberbeck, 1849–1900

As a matter of fact, denoting by T_0 resp. $\mathcal{P}_0$ the temperature resp. pressure at a point at $z = 0$ and ρ_0 the corresponding mass density, one easily checks that equations (IX.8), (IX.12), (IX.13) admit the static solution

$$\vec{v}_{\text{st.}}(t,\vec{r}) = \vec{0}, \quad \Theta_{\text{st.}}(t,\vec{r}) = -\frac{z}{d}\Delta T, \quad \mathcal{P}_{\text{eff.,st.}}(t,\vec{r}) = \mathcal{P}_0 - \rho_0 g\frac{z^2}{2d}\alpha_{(\mathcal{V})}\Delta T, \tag{IX.14}$$

with the pressure given by $\mathcal{P}_{\text{st.}}(t,\vec{r}) = \mathcal{P}_{\text{eff.,st.}}(t,\vec{r}) - \rho_0 g z$. Since $|z/d| < \frac{1}{2}$ and $|\alpha_{(\mathcal{V})}\Delta T| \ll 1$, one sees that the main part of the pressure variation due to gravity is already absorbed in the definition of the effective pressure.

If $\Delta T = 0$, one recognizes the usual linear pressure profile of a static fluid at constant temperature in a uniform gravity field.

One can check that the fluid state defined by the profile (IX.14) is stable against small perturbations of any of the dynamical fields. To account for that property, that state (for a given temperature difference ΔT) will be referred to as "equilibrium state".

Increasing now the temperature of the lower plate with respect to that of the upper plate, for small positive temperature differences ΔT nothing happens, and the static solution (IX.14) still holds—and is still stable.

When ΔT reaches a critical value ΔT_c, the fluid starts developing a pattern of somewhat regular cylindrical domains rotating around their longitudinal, horizontal axes, two neighboring regions rotating in opposite directions. These domains in which warmer and thus less dense fluid rises on the one side while colder, denser fluid descends on the other side, are called *Bénard cells*.[au]

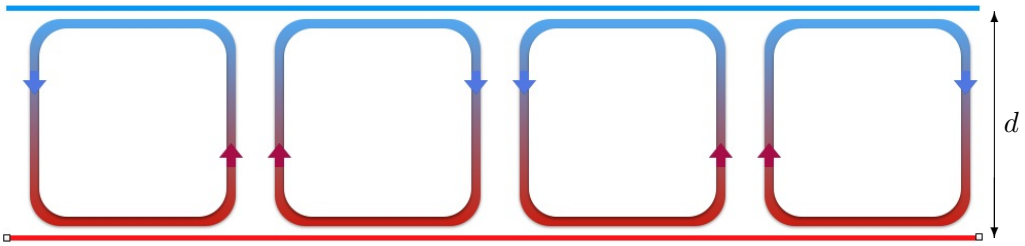

Figure IX.1 – Schematic representation of Bénard cells between two horizontal plates.

The transition from a situation in which the static fluid is a stable state, to that in which motion develops—i.e. the static case is no longer stable—, is referred to as (onset of the) *Rayleigh–Bénard instability*. Since the motion of the fluid appears spontaneously, without the need to impose any external pressure gradient, it is an instance of *free convection* or *natural convection*—in opposition to *forced convection*).

Remarks:

∗ Such convection cells are omnipresent in Nature, as e.g. in the Earth mantle, in the Earth atmosphere, or in the Sun convective zone.

∗ When ΔT further increases, the structure of the convection pattern becomes more complicated, eventually becoming chaotic.

> In a series of experiments with liquid helium or mercury, A. Libchaber[av] and his collaborators observed the following features [37, 38, 39]: Shortly above ΔT_c, the stable fluid state involve cylindrical convective cells with a constant profile. Above a second threshold, "oscillatory convection" develops: that is, undulatory waves start to propagate along the "surface" of the convective cells, at first at a unique (angular) frequency ω_1, then—as ΔT further increases—also at higher harmonics $n_1\omega_1$, $n_1 \in \mathbb{N}$. As the temperature difference ΔT reaches a third threshold, a second undulation frequency ω_2 appears, incommensurate with ω_1, later accompanied by the combina-

[au] H. Bénard, 1874–1939. [av] A. Libchaber, born 1934

tions $n_1\omega_1 + n_2\omega_2$, with $n_1, n_2 \in \mathbb{N}$. At higher ΔT, the oscillator with frequency ω_2 experiences a shift from its proper frequency to a neighboring submultiple of ω_1—e.g., $\omega_1/2$ in the experiments with He—, illustrating the phenomenon of *frequency locking*. For even higher ΔT, submultiples of ω_1 appear ("frequency demultiplication"), then a low-frequency continuum, and eventually chaos.

Each appearance of a new frequency may be seen as a *bifurcation*. The ratios of the experimentally measured lengths of consecutive intervals between successive bifurcations provide an estimate of the (first) *Feigenbaum constant*[(aw)] in agreement with its theoretical value—thereby providing the first empirical confirmation of Feigenbaum's theory.

IX.2.1 b Qualitative discussion

Consider the fluid in its "equilibrium" state of rest, in the presence of a positive temperature difference ΔT, so that the lower layers of the fluid are warmer than the upper ones.

If a fluid particle at altitude z acquires, for some reason, a temperature that differs from the equilibrium temperature—measured with respect to some reference value—$\Theta(z)$, then its mass density given by Eq. (IX.9) will differ from that of its environment. As a result, the Archimedes force acting on it no longer exactly balances its weight, so that it will experience a buoyancy force. For instance, if the fluid particle is warmer that its surroundings, it will be less dense and experience a force directed upwards. Consequently, the fluid particle should start to move in that direction, in which case it encounters fluid which is even colder and denser, resulting in an increased buoyancy and a continued motion. According to that reasoning, *any* vertical temperature gradient should result in a convective motion.

There are however two effects that counteract the action of buoyancy, and explain why the Rayleigh–Bénard instability necessitates a temperature difference larger than a given threshold. First, the rising particle fluid will also experience a viscous friction force from the other fluid regions it passes through, which slows its motion. Secondly, if the rise of the particle is too slow, heat has time to diffuse—by heat conduction—through its surface: this tends to equilibrate the temperature of the fluid particle with that of its surroundings, thereby suppressing the buoyancy.

Accordingly, we can expect to find that the Rayleigh–Bénard instability will be facilitated when $\alpha_{(\mathcal{V})}\Delta T g$—i.e. the buoyancy per unit mass—increases, as well as when the thermal diffusivity α and the shear viscosity ν decrease.

Translating the previous argumentation in formulas, let us consider a spherical fluid particle with radius R, and assume that it has some vertically directed velocity v, while its temperature initially equals that of its surroundings.

With the fluid particle surface area, proportional to R^2, and the thermal diffusivity α, one can estimate the characteristic time for heat exchanges between the particle and the neighboring fluid, namely

$$\tau_Q = C\frac{R^2}{\alpha}$$

with C a geometrical factor. If the fluid particle moves with constant velocity v during that duration τ_Q, while staying at almost constant temperature since heat exchanges remain limited, the temperature difference $\delta\Theta$ it acquires with respect to the neighboring fluid is

$$\delta\Theta = \frac{\partial\Theta}{\partial z}\delta z = \frac{\partial\Theta}{\partial z} v\tau_Q = C\frac{\Delta T}{d}\frac{R^2}{\alpha}v,$$

where $\Delta T/d$ is the vertical temperature gradient in the fluid imposed by the two plates. This temperature difference gives rise to a mass density difference

$$\delta\rho = -\rho_0\alpha_{(\mathcal{V})}\delta\Theta = -C\rho_0 v\frac{R^2}{\alpha}\frac{\alpha_{(\mathcal{V})}\Delta T}{d},$$

[(aw)] M. Feigenbaum, 1944–2019

between the particle and its surroundings. As a result, the fluid particle experiences an upwards directed buoyancy

$$-\frac{4\pi}{3}R^3\delta\rho\, g = \frac{4\pi C}{3}\rho_0 g v \frac{R^5}{\alpha}\frac{\alpha_{(\mathcal{V})}\Delta T}{d}. \tag{IX.15}$$

On the other hand, the fluid particle is slowed in its vertical motion by the downwards oriented Stokes friction force acting on it, namely, in projection on the z-axis

$$F_{\text{Stokes}} = -6\pi R\eta v. \tag{IX.16}$$

> Note that assuming that the velocity v remains constant, with a counteracting Stokes force that is automatically the "good" one, relies on the picture that viscous effects adapt instantaneously, i.e. that momentum diffusion is fast. That is, the above reasoning actually assumes that the Prandtl number (IX.7) is much larger than 1; yet its result is independent from that assumption.

Comparing Eqs. (IX.15) and (IX.16), buoyancy will overcome friction, and thus the Rayleigh–Bénard instability take place, when

$$\frac{4\pi C}{3}\rho_0 g v \frac{R^5}{\alpha}\frac{\alpha_{(\mathcal{V})}\Delta T}{d} > 6\pi R\rho_0\nu v \quad \Leftrightarrow \quad \frac{\alpha_{(\mathcal{V})}\Delta T g R^4}{\alpha\nu d} > \frac{9}{2C}.$$

Note that the velocity v that was invoked in the reasoning actually drops out from this condition. Taking for instance $R = d/2$—which maximizes the left member of the inequality—, this becomes

$$\mathrm{Ra} \equiv \frac{\alpha_{(\mathcal{V})}\Delta T g\, d^3}{\nu\alpha} > \frac{72}{C} = \mathrm{Ra}_c.$$

Ra is the so-called *Rayleigh number* and Ra_c its critical value, above which the static-fluid state is unstable against perturbation and convection takes place. The "value" $72/C$ found with the above simple reasoning on force equilibrium is totally irrelevant—both careful experiments and theoretical calculations agree with $\mathrm{Ra}_c = 1708$ for a fluid between two very large plates—, the important lesson is the existence of a threshold.

IX.2.2 Toy model for the Rayleigh–Bénard instability

A more refined—although still crude—toy model of the transition to convection consists in considering small perturbations $\vec{\mathsf{v}}$, $\delta\Theta$, $\delta\mathcal{P}_{\text{eff.}}$ around a static state $\vec{\mathsf{v}}_{\text{st.}} = \vec{0}$, $\Theta_{\text{st.}}$, $\mathcal{P}_{\text{eff.,st.}}$, and to linearize the Boussinesq equations to first order in these perturbations. As shown by Eq. (IX.14), the effective pressure $\mathcal{P}_{\text{eff.,st.}}$ actually already includes a small correction, due to $\alpha_{(\mathcal{V})}\Delta T$ being much smaller than 1, so that we may from the start neglect $\delta\mathcal{P}_{\text{eff.}}$.

To first order in the perturbations, Eqs. (IX.12), projected on the z-axis, and (IX.13) give, after subtraction of the contributions from the static solution

$$\frac{\partial \mathsf{v}_z}{\partial t} = \nu\triangle\mathsf{v}_z + \alpha_{(\mathcal{V})}\delta\Theta\, g, \tag{IX.17a}$$

$$\frac{\partial\delta\Theta}{\partial t} - \frac{\Delta T}{d}\mathsf{v}_z = \alpha\triangle\delta\Theta. \tag{IX.17b}$$

Moving the second term of the latter equation to the right hand side increases the parallelism of this set of coupled equations. In addition, there is also the projection of Eq. (IX.12) along the x-axis, and the velocity field must obey the incompressibility condition (IX.8).

The proper approach would now be to specify the boundary conditions, namely: the vanishing of v_z at both plates—impermeability condition—, the vanishing of v_x at both plates—no-slip condition—, and the identity of the fluid temperature at each plate with that of the corresponding plate; that is, all in all, 6 conditions. By manipulating the set of equations, it can be turned into a 6th-order linear partial differential equation for $\delta\Theta$, on which the boundary conditions can be imposed.

Instead of following this long road,[58] we refrain from trying to really solve the equations, but rather make a simple ansatz, namely $\mathsf{v}_z(t,\vec{r}) = \mathsf{v}_0\,\mathrm{e}^{\gamma t}\cos(kx)$—which automatically fulfills the incompressibility equation, but clearly violates the impermeability conditions—, and a similar one for $\delta\Theta$, with γ a constant. Substituting these forms in Eqs. (IX.17) yields the linear system

$$\begin{aligned}\gamma\mathsf{v}_0 = -k^2\nu\mathsf{v}_0 + \alpha_{(\mathcal{V})}\delta\Theta_0 g \quad &\Leftrightarrow \quad \big(\gamma+\nu k^2\big)\mathsf{v}_0 - g\alpha_{(\mathcal{V})}\delta\Theta_0 = 0,\\ \gamma\delta\Theta_0 = -k^2\alpha\delta\Theta_0 + \frac{\Delta T}{d}\mathsf{v}_0 \quad &\Leftrightarrow \quad \frac{\Delta T}{d}\mathsf{v}_0 - \big(\gamma+\alpha k^2\big)\delta\Theta_0 = 0\end{aligned}$$

for the amplitudes v_0, $\delta\Theta_0$. This admits a non-trivial solution only if

$$\big(\gamma+\nu k^2\big)\big(\gamma+\alpha k^2\big) - \frac{\alpha_{(\mathcal{V})}\Delta T}{d}g = 0. \tag{IX.18}$$

This is a straightforward quadratic equation for γ. It always has two real solutions, one of which is negative—corresponding to a dampened perturbation—since their sum is $-(\alpha+\nu)k^2 < 0$; the other solution may change sign since their product

$$\alpha\nu k^4 - \frac{\alpha_{(\mathcal{V})}\Delta T}{d}g$$

is positive for $\Delta T = 0$, yielding a second negative solution, yet changes sign as ΔT increases. The vanishing of this product thus signals the onset of instability. Taking for instance $k = \pi/d$ to fix ideas, this occurs at a critical Rayleigh number

$$\mathrm{Ra}_c = \frac{\alpha_{(\mathcal{V})}\Delta T\, g\, d^3}{\alpha\nu} = \pi^4,$$

where the precise value (here π^4) is irrelevant.

From Eq. (IX.18) also follows that the growth rate of the instability is given in the neighborhood of the threshold by

$$\gamma = \frac{\mathrm{Ra}-\mathrm{Ra}_c}{\mathrm{Ra}_c}\frac{\alpha\nu}{\alpha+\nu}k^2,$$

i.e. it is infinitely slow at Ra_c. This is reminiscent of a similar behavior in the vicinity of the critical point associated with a thermodynamic phase transition.

> By performing a more rigorous calculation including non-linear effects, one can show that the velocity amplitude at a given point behaves like
>
> $$\mathsf{v} \propto \left(\frac{\mathrm{Ra}-\mathrm{Ra}_c}{\mathrm{Ra}_c}\right)^{\beta} \quad \text{with} \quad \beta = \frac{1}{2} \tag{IX.19}$$
>
> in the vicinity of the critical value, and this prediction is borne out by experiments [41]. Such a power law behavior is again reminiscent of the thermodynamics of phase transitions, more specifically here—since v vanishes below Ra_c and is finite above—of the behavior of the *order parameter* in the vicinity of a critical point. Accordingly the notation β used for the exponent in relation (IX.19) is the traditional choice for the critical exponent associated with the order parameter of phase transitions.

Eventually, a last analogy with phase transitions regards the breaking of a symmetry at the threshold for the Rayleigh–Bénard instability. Below Ra_c, the system is invariant under translations parallel to the plates, while above Ra_c that symmetry is spontaneously broken.

[58] The reader may find details in Ref. [40, Chap. II].

Bibliography for Chapter IX

- A nice introduction to the topic is to be found in Ref. [42], which is a popular science account of part of Ref. [43].
- Faber [1] Chapter 8.5–8.7 & 9.2.
- Guyon *et al.* [2] Chapter 11.2.
- Landau–Lifshitz [4, 5] Chapter V § 49–53 & 56–57.

CHAPTER X

Fundamental equations of relativistic fluid dynamics

Under a number of "extreme conditions"—for instance inside compact astrophysical objects, in the early Universe, or in high-energy collisions of heavy nuclei[(59)]—the "atoms" constituting a fluid can acquire very high kinetic energies, that become comparable to their (rest) mass energy. A non-relativistic description of the medium is then no longer appropriate, and must be replaced by a relativistic model. Some introductory elements of such a description are presented in this Chapter—in which the basic laws governing the dynamics of relativistic fluids are formulated and discussed—, and the following one—which will deal with a few simple analytically tractable solutions of the equations of motions.

As in the non-relativistic case, the basic equations governing the motion of a fluid in the relativistic regime are nothing but formulations of the most fundamental laws of physics, namely conservation laws for "particle number"—in fact, for the various conserved quantum numbers carried by the fluid atoms—, and for energy and momentum (Sec. X.1).

Precisely because the equations simply express general conservation laws, they are not very specific, and contain at first too many degrees of freedom to be tractable. To make progress, it is necessary to introduce models for the fluid under consideration, leading for instance to distinguishing between perfect and dissipative fluids. A convenient way to specify the constitutive equations characteristic of such models is to do so in terms of a fluid four-velocity, which generalizes the non-relativistic flow velocity, yet in a non-unique way (Sec. X.2).

[(59)] ... as performed at the CERN Large Hadron Collider (LHC) or at the dedicated Relativistic Heavy Ion Collider (RHIC) at the Brookhaven National Laboratory.

Such a fluid four-velocity also automatically singles out a particular reference frame, the local rest frame, in which the conserved currents describing the physics of the fluid take a simpler form, whose physical interpretation is clearer. The perfect fluids are thus those whose properties at each point are spatially isotropic in the corresponding local rest frame, from which there follows that the conserved currents can only depend on the flow four-velocity, not on its derivatives (Sec. X.3). Conversely, when the conserved currents involve (spatial) gradients of the fluid four-velocity, these derivatives signal real fluids with dissipative effects (Sec. X.4).

Two topics that lie beyond the main stream of this Chapter are given in appendices, namely the expression of the conserved currents of relativistic fluid dynamics in terms of underlying microscopic quantities (Sec. X.A) and a discussion of relativistic kinematics (Sec. X.B).

Throughout this Chapter and the following one, the fluids occupy domains of the four-dimensional Minkowski[(ax)] space-time $\mathscr{M}_4$ of Special Relativity. The position of a generic point of $\mathscr{M}_4$ will be designated by a 4-vector x. Given a reference frame $\mathcal{R}$ and a system of coordinates, those of x will be denoted by $\{x^\mu\} \equiv (x^0, x^1, x^2, x^3)$—where in the case of Minkowski coordinates[(60)] $x^0 = ct$ with t the time measured by an observer at rest in $\mathcal{R}$.

For the metric tensor $\mathbf{g}$ on $\mathscr{M}_4$, we use the "mostly plus" convention, with signature $(-,+,+,+)$, i.e. in the case of Minkowski coordinates $x_0 = -x^0$ while $x_i = x^i$ for $i = 1, 2, 3$. Thus, time-like resp. space-like 4-vectors have a negative resp. positive semi-norm.

X.1 Conservation laws

As stated in the introduction, the equations governing the dynamics of fluids in the relativistic regime, just as in the non-relativistic case, embody conservation principles—or more generally, balance equations. More precisely, the usual fundamental equations of relativistic fluid dynamics are differential formulations of these laws. Instead of proceeding as in Chap. III, in which the local formulations were derived from integral ones, we shall hereafter postulate the differential laws, and check or argue that they lead to the expected macroscopic behavior.

> Starting from the local level is more natural here, since one of the tenets underlying relativistic theories, as e.g. quantum field theory, is precisely locality—the absence of action at distance—besides causality. Thus, both conservation equations (X.2) and (X.7) actually emerge as those expressing the invariance of microscopic theories under specific transformations, involving associated Noether currents.

We first discuss the conservation of so-called "charges" (§ X.1.1), then that of energy and momentum, which in a relativistic context are inseparable (§ X.1.2).

X.1.1 Charge conservation

The conservation law that was discussed first in the Chapter III introducing the equations of non-relativistic hydrodynamics was that of mass, which, in the case of a single-component fluid, is fully equivalent to the conservation of particle number. In a relativistic system, the number of particles is strictly speaking not conserved, even if the system is closed. Indeed, thanks to the high kinetic energies available, particle–antiparticle pairs can continuously either be created, or annihilate.

[(60)] We shall call *Minkowski coordinates* the analog on the space-time $\mathscr{M}_4$ of the Cartesian coordinates on Euclidean space $\mathscr{E}_3$, i.e. those corresponding to a set of four mutually orthogonal 4-vectors $(\mathsf{e}_0, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3)$ such that the metric tensor has components $g_{\mu\nu} = \mathsf{e}_\mu \cdot \mathsf{e}_\nu = \mathrm{diag}(-1, +1, +1, +1)$ for $\mu, \nu = 0, 1, 2, 3$. They are also alternatively referred to as *Lorentz*[(ay)] *coordinates*.

[(ax)] H. Minkowski, 1864–1909 [(ay)] H. A. Lorentz, 1853–1926

If the particles carry some conserved additive quantum number—as e.g. electric charge or baryon number—, then the value of the quantum number carried by an antiparticle is the negative of that of the corresponding particle. Accordingly, this quantum number is (by definition!) conserved in both creation and annihilation processes.

Throughout this Chapter and the following, such a conserved additive quantum number will be called a "charge", where the reader should beware that this does not necessarily mean the electric charge. Similarly, we shall use the designations "charge (number) density" or "charge flux density". Since there may be several different conserved charges in a given fluid, the corresponding densities will possibly be labeled with an index[(61)] a—which should not be confused with a coordinate index—and accordingly called "charge of type a".

X.1.1 a Local formulation of charge conservation

By definition, the *local charge (number) density* $n_a(t,\vec{r})$ in a fluid is such that the product $n_a(t,\vec{r})\,\mathrm{d}^3\vec{r}$ represents the net amount of charge of type a in the infinitesimal spatial volume $\mathrm{d}^3\vec{r}$ about position $\vec{r}$ at time t.

Since the volume element $\mathrm{d}^3\vec{r}$ depends on the reference frame in which it is measured—remember that in special relativity there is the length contraction phenomenon—, this is also the case of the charge density $n_a(t,\vec{r})$, to ensure that the total charge of type a in $\mathrm{d}^3\vec{r}$ remains independent of the reference frame. Hereafter, $n_a(t,\vec{r})$ will also be denoted by $n_a(\mathsf{x})$.

The *charge flux density* (or *current*) $\vec{\jmath}_{N_a}(t,\vec{r})$ is defined in a similar way, as the amount of charge of type a that crosses a unit area per unit time interval. Since both "unit area" and "unit time interval" in this definition are reference frame-dependent concepts, $\vec{\jmath}_{N_a}(t,\vec{r})$ also depends on the reference frame in which it is measured.

Together, $n_a(\mathsf{x})$ and $\vec{\jmath}_{N_a}(\mathsf{x})$ build up a *charge (number) four-current*[(lxxiii)] $\mathsf{N}_a(\mathsf{x})$, whose contravariant Minkowski components at every point x are $N_a^0(\mathsf{x}) = cn_a(\mathsf{x})$, $N_a^i(\mathsf{x}) = j_{N_a}^i(\mathsf{x})$ for $i = 1,2,3$. This is conveniently summarized in the form

$$\mathsf{N}_a(\mathsf{x}) = \begin{pmatrix} cn_a(\mathsf{x}) \\ \vec{\jmath}_{N_a}(\mathsf{x}) \end{pmatrix} \tag{X.1a}$$

or, somewhat improperly,

$$N_a^\mu(\mathsf{x}) = \begin{pmatrix} cn_a(\mathsf{x}) \\ \vec{\jmath}_{N_a}(\mathsf{x}) \end{pmatrix}. \tag{X.1b}$$

With the help of the charge four-current, the local formulation of the conservation of charge of type a in the motion of the system reads, using coordinates

$$\boxed{\mathrm{d}_\mu N_a^\mu(\mathsf{x}) = 0,} \tag{X.2a}$$

where $\mathrm{d}_\mu \equiv \mathrm{d}/\mathrm{d}x^\mu$ denote the components of the 4-gradient. Denoting the latter, which is a one-form, by d, one may write the even shorter "geometric" (i.e. coordinate-invariant) equation

$$\mathsf{d}\cdot\mathsf{N}_a(\mathsf{x}) = 0, \tag{X.2b}$$

with $\mathsf{d}\cdot$ the four-divergence.

Remarks:

$*$ Whether $\mathsf{N}_a(\mathsf{x})$ defined by Eq. (X.1) is a 4-vector—that is, whether it behaves as it should under Lorentz transformations—is at first far from clear. That $n_a(\mathsf{x})\,\mathrm{d}^3\vec{r}$ need be a number—i.e. a Lorentz scalar, like $\mathrm{d}^4\mathsf{x} = \mathrm{d}x^0\,\mathrm{d}^3\vec{r}$—suggests that $n_a(\mathsf{x})$ should transform like the time-like component of a

(61) mostly in subscript

(lxxiii) (*Ladungs-*) *Viererstrom*

4-vector. Yet it is admittedly not clear that the associated spatial part should be the particle flux density.
We shall see in § X.3.3 that assuming that there exists a 4-vector field obeying the conservation equation (X.2) leads in the non-relativistic limit to the above interpretation of its time-like and space-like parts, which may be viewed as a justification.

> A better argument is to introduce the particle number 4-current from a microscopic definition, see App. X.A.1. Yet strictly speaking, this goes beyond the fluid-dynamical framework.

∗ If Minkowski coordinates $\{x^\mu\}$ are used, the components of the 4-gradient d are simply the partial derivatives $\partial_\mu \equiv \partial/\partial x^\mu$, so that Eq. (X.2a) becomes $\partial_\mu N_a^\mu(\mathsf{x}) = 0$.

X.1.1 b Global formulation

Consider in $\mathscr{M}_4$ a space-like 3-dimensional hypersurface Σ—i.e. a hypersurface at every point of which the normal 4-vector is time-like—which extends far enough so that the whole fluid passes through it in its motion; that is, Σ intercepts the worldlines of all fluid particles. Since Σ is 3-dimensional, it may be parameterized (at least in the vicinity of each of its points) by parameters α, β, γ.

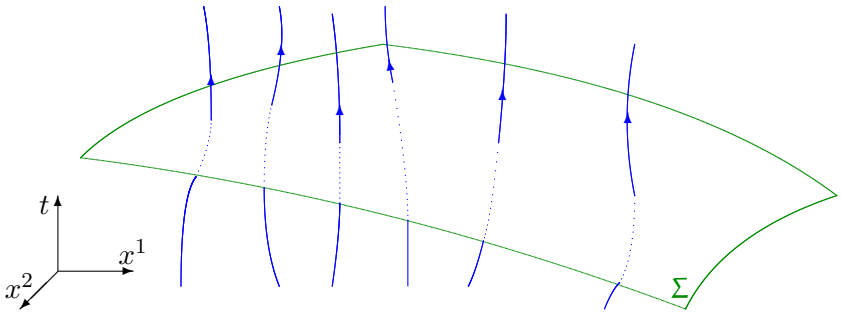

Figure X.1

The total amount N_a of charge of type a in the fluid is the flux of the charge number 4-current $\mathsf{N}_a(\mathsf{x})$ across Σ

$$N_a = \int_\Sigma N_a^\mu(\mathsf{x})\, \mathrm{d}^3\sigma_\mu = \int_\Sigma \mathsf{N}_a(\mathsf{x}) \cdot \mathrm{d}^3\boldsymbol{\sigma}, \tag{X.3}$$

where $\mathrm{d}^3\sigma_\mu$ denotes the components of the 3-hypersurface element

$$\mathrm{d}^3\sigma_\mu \equiv \frac{1}{3!}\sqrt{-\det \mathbf{g}}\, \epsilon_{\mu\nu\rho\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\alpha}\frac{\mathrm{d}x^\rho}{\mathrm{d}\beta}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\gamma}\, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma \tag{X.4}$$

with $\epsilon_{\mu\nu\rho\lambda}$ the four-dimensional Levi-Civita symbol, with the convention $\epsilon_{0123} = +1$.[(62)]

Let Ω denote a 4-volume in $\mathscr{M}_4$, and $\partial\Omega$ its 3-surface. Applying the Gauss theorem, the flux of the charge number 4-current across $\partial\Omega$ is the integral of the 4-divergence of $\mathsf{N}_a(\mathsf{x})$ over Ω

$$\oint_{\partial\Omega} \mathsf{N}_a(\mathsf{x}) \cdot \mathrm{d}^3\boldsymbol{\sigma} = \int_\Omega \mathsf{d} \cdot \mathsf{N}_a(\mathsf{x})\, \mathrm{d}^4\mathsf{x}, \tag{X.5}$$

where the right member vanishes thanks to the local expression (X.2) of charge conservation. Splitting $\partial\Omega$ into two space-like parts through which charges enter resp. leave Ω in their motion—the technical criterion is the sign of $\mathsf{N}_a(\mathsf{x}) \cdot \mathrm{d}^3\boldsymbol{\sigma}$—, one finds that there are as many charge carriers that leave as those that enter, which expresses charge conservation globally.

(62) This choice is not universal: the alternative convention $\epsilon^{0123} = +1$ results in $\epsilon_{0123} < 0$ due to the odd number of minus signs in the signature of the metric tensor.

X.1.2 Energy-momentum conservation

In a relativistic theory, energy and momentum constitute the temporal and spatial components of a four-vector, the four-momentum. To express the local conservation—in the absence of external forces—of the latter, the densities and flux densities of energy and momentum at each space-time point x must be combined into a four-tensor of degree 2, the *energy-momentum tensor*[(lxxiv)]—also called *stress-energy tensor*—$\mathbf{T}(\mathsf{x})$, of type $\binom{2}{0}$.

This energy-momentum tensor[(63)] may be defined by the physical content of its 16 Minkowski components $T^{\mu\nu}(\mathsf{x})$ in a given reference frame $\mathcal{R}$:

- $T^{00}(\mathsf{x})$ is the energy density;
- $cT^{0j}(\mathsf{x})$ is the j-th component of the energy flux density, with $j = 1, 2, 3$;
- $\frac{1}{c}T^{i0}(\mathsf{x})$ is the density of the i-th component of momentum, with $i = 1, 2, 3$;
- $T^{ij}(\mathsf{x})$ for $i, j = 1, 2, 3$ is the momentum flux-density tensor.

(X.6)

All physical quantities are to be measured with respect to the reference frame $\mathcal{R}$.

Remarks:

∗ The similarity of the notations $\mathbf{T}$ resp. $\mathbf{T}$ for the energy-momentum four-tensor resp. the three-dimensional momentum flux-density tensor is not accidental! The former is *the* natural generalization to the 4-dimensional relativistic framework of the latter, just like four-momentum p, with components p^μ is the four-vector associated to the three-dimensional momentum $\vec{p}$. That is, the 3-tensor $\mathbf{T}$ is the spatial part of the 4-tensor $\mathbf{T}$, just like momentum $\vec{p}$ is the spatial part of four-momentum p.

∗ Starting from a microscopic description of the fluid, one can show that the energy-momentum tensor is symmetric, i.e. $T^{\mu\nu}(\mathsf{x}) = T^{\nu\mu}(\mathsf{x})$ for all $\mu, \nu = 0, 1, 2, 3$. Accordingly, only 10 of the 16 entries of $\mathbf{T}$ are independent.

In the absence of external force acting on the fluid, the local conservation of the energy-momentum tensor reads component-wise

$$\boxed{\mathrm{d}_\mu T^{\mu\nu}(\mathsf{x}) = 0 \quad \forall \nu = 0, 1, 2, 3,} \tag{X.7a}$$

which represents four equations: the equation with $\nu = 0$ is the conservation of energy, while the equations $\mathrm{d}_\mu T^{\mu j}(\mathsf{x}) = 0$ for $j = 1, 2, 3$ are the components of the momentum conservation equation.

In geometric formulation, Eq. (X.7a) becomes

$$\mathrm{d} \cdot \mathbf{T}(\mathsf{x}) = 0. \tag{X.7b}$$

This is exactly the same form as Eq. (X.2b), just like Eqs. (X.2a) and (X.7a) are similar, up to the difference in the tensorial degree of the conserved quantity.

As in § X.1.1 b, one associates to the energy-momentum tensor $\mathbf{T}(\mathsf{x})$ a 4-vector P by

$$\mathsf{P} \equiv \int_\Sigma \mathbf{T}(\mathsf{x}) \cdot \mathrm{d}^3\sigma \quad \Leftrightarrow \quad P^\mu = \int_\Sigma T^{\mu\nu}(\mathsf{x})\, \mathrm{d}^3\sigma_\nu, \tag{X.8}$$

with Σ a space-like 3-hypersurface. P represents the total 4-momentum crossing Σ, and invoking the Gauss theorem, Eq. (X.7) implies that it is a conserved quantity.

X.1.3 Fundamental equations in the presence of sources

This may come at some point.

(63) As in the case of the charge number 4-current, the argument showing that $\mathbf{T}(\mathsf{x})$ is a Lorentz tensor is to define it microscopically as a tensor—see App. X.A.2—and to later interpret the physical meaning of the components.

(lxxiv) *Energieimpulstensor*

X.2 Four-velocity of a fluid flow. Local rest frame

The four-velocity of a flow is a field, defined at each point x of a space-time domain $\mathcal{D}$, of time-like 4-vectors $\mathsf{u}(\mathsf{x})$ with constant magnitude c, i.e. such that

$$[\mathsf{u}(\mathsf{x})]^2 = u_\mu(\mathsf{x})u^\mu(\mathsf{x}) = -c^2 \quad \forall \mathsf{x}, \tag{X.9}$$

with $u^\mu(\mathsf{x})$ the (contravariant) components of $\mathsf{u}(\mathsf{x})$.

At each point x of the fluid, one can define a proper reference frame, the so-called *local rest frame*,[(lxxv)] hereafter abbreviated as LR(x), in which the space-like Minkowski components of the local flow 4-velocity vanish:

$$u^\mu(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = (c, 0, 0, 0). \tag{X.10}$$

Let $\vec{\mathsf{v}}(\mathsf{x})$ denote the instantaneous velocity of (an observer at rest in) the local rest frame LR(x) with respect to a fixed reference frame $\mathcal{R}$, which hereafter will be referred to as the "laboratory frame". In the latter, the components of the flow four-velocity are

$$u^\mu(\mathsf{x})\big|_{\mathcal{R}} = \begin{pmatrix} \gamma(\mathsf{x})c \\ \gamma(\mathsf{x})\vec{\mathsf{v}}(\mathsf{x}) \end{pmatrix}, \tag{X.11}$$

with $\gamma(\mathsf{x}) = 1/\sqrt{1 - \vec{\mathsf{v}}(\mathsf{x})^2/c^2}$ the corresponding Lorentz factor.

In a situation where the system is locally close to thermodynamic equilibrium, the local rest frame represents the reference frame in which the local thermodynamic variables—charge densities $n_a(\mathsf{x})$ and energy density $\epsilon(\mathsf{x})$—are defined in their usual sense:

$$n_a(\mathsf{x}) \equiv n_a(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})}, \qquad \epsilon(\mathsf{x}) \equiv T^{00}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})}. \tag{X.12}$$

For the remaining local thermodynamic variables, it is assumed that they are related to $n_a(\mathsf{x})$ and $\epsilon(\mathsf{x})$ in the same way as when the fluid is at thermodynamic equilibrium. Thus, the pressure $\mathcal{P}(\mathsf{x})$ is given by the mechanical equation of state

$$\mathcal{P}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = \mathcal{P}(\epsilon(\mathsf{x}), \{n_a(\mathsf{x})\}); \tag{X.13}$$

the temperature $T(\mathsf{x})$ is given by the thermal equation of state; the entropy density $s(\mathsf{x})$ is defined by the Gibbs fundamental relation, and so on.

Remarks:

∗ A slightly more formal approach to define the 4-velocity and the local rest frame is to turn the reasoning round. Namely, one first introduces the latter as a reference frame LR(x) in which "physics at point x is easy", which in particular means that the fluid should be locally "motionless".[(64)] Introducing then an instantaneous inertial reference frame that momentarily coincides with LR(x), one considers an observer $\mathcal{O}$ at rest in that inertial frame. The fluid four-velocity $\mathsf{u}(\mathsf{x})$ with respect to the laboratory frame $\mathcal{R}$ is then the four-velocity of $\mathcal{O}$ (assumed to be pointlike) in $\mathcal{R}$—defined as the derivative of $\mathcal{O}$'s space-time trajectory with respect to $\mathcal{O}$'s proper time.

∗ The relativistic energy density ϵ differs from its at first sight obvious non-relativistic counterpart, the internal energy density e. The reason is that ϵ also contains the contribution from the mass energy of the particles and antiparticles—mc^2 per (anti)particle—, which is conventionally not taken into account in the non-relativistic internal energy density.

(64) As we shall discuss in Ref. X.4.2, this requirement may not define a unique reference frame.

(lxxv) *lokales Ruhesystem*

∗ To distinguish between the frame-dependent quantities, like charge densities $n_a(\mathsf{x})$ or energy density $T^{00}(\mathsf{x})$, and the corresponding quantities measured in the local rest frame, namely $\mathfrak{n}_a(\mathsf{x})$ or $\epsilon(\mathsf{x})$, the latter are referred to as *comoving*.

The comoving quantities can actually be computed easily within any reference frame and coordinate system. Let us thus write

$$\mathfrak{n}_a(\mathsf{x}) \equiv n_a(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = \frac{1}{c}N^0_a(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = \left.\frac{N^0_a(\mathsf{x})u^0(\mathsf{x})}{[u^0(\mathsf{x})]^2}\right|_{\mathrm{LR}(\mathsf{x})} = \left.\frac{N^0_a(\mathsf{x})u_0(\mathsf{x})}{g_{00}(\mathsf{x})[u^0(\mathsf{x})]^2}\right|_{\mathrm{LR}(\mathsf{x})} = \left.\frac{N^\mu_a(\mathsf{x})u_\mu(\mathsf{x})}{u^\nu(\mathsf{x})u_\nu(\mathsf{x})}\right|_{\mathrm{LR}(\mathsf{x})},$$

where we used the fact that $u_0(\mathsf{x}) = g_{00}(\mathsf{x})u^0(\mathsf{x})$ in the local rest frame. The rightmost term of the above equation is the ratio of two Lorentz-invariant scalars, and thus is itself a Lorentz scalar field, independent of the reference frame in which it is computed:

$$\mathfrak{n}_a(\mathsf{x}) = \frac{N^\mu_a(\mathsf{x})u_\mu(\mathsf{x})}{u^\nu(\mathsf{x})u_\nu(\mathsf{x})} = \frac{\mathsf{N}_a(\mathsf{x})\cdot\mathsf{u}(\mathsf{x})}{[\mathsf{u}(\mathsf{x})]^2}. \tag{X.14}$$

Similarly one finds

$$\epsilon(\mathsf{x}) \equiv T^{00}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = c^2\left.\frac{u_\mu(\mathsf{x})T^{\mu\nu}(\mathsf{x})u_\nu(\mathsf{x})}{[u^\rho(\mathsf{x})u_\rho(\mathsf{x})]^2}\right|_{\mathrm{LR}(\mathsf{x})} = \frac{1}{c^2}u_\mu(\mathsf{x})T^{\mu\nu}(\mathsf{x})u_\nu(\mathsf{x}) = \frac{1}{c^2}\mathsf{u}(\mathsf{x})\cdot\mathsf{T}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x}), \tag{X.15}$$

where the normalization of the 4-velocity was used.

In the following Sections, we introduce fluid models, defined by the relations between the conserved currents—charge 4-currents $\mathsf{N}_a(\mathsf{x})$ and energy-momentum tensor $\mathsf{T}(\mathsf{x})$—and the fluid 4-velocity $\mathsf{u}(\mathsf{x})$ and comoving thermodynamic quantities.

X.3 Perfect relativistic fluid

By definition, a fluid is perfect when there is no dissipative current in it, see definition (III.16a). As a consequence, one can at each point x of the fluid find a reference frame in which the local properties in the neighborhood of x are spatially *isotropic* [cf. definition (III.24)]. This reference frame represents the natural choice for the local rest frame at point x, LR(x).

The expressions of the particle-number 4-current and the energy-momentum tensor of a perfect fluid are first introduced in § X.3.1. It is then shown that the postulated absence of dissipative current automatically leads to the conservation of entropy in the motion (§ X.3.2). Eventually, the low-velocity limit of the dynamical equations is investigated in § X.3.3.

X.3.1 Charge four-current and energy-momentum tensor of a perfect fluid

To express the defining feature of the local rest frame LR(x), namely the spatial isotropy of the local fluid properties, it is convenient to adopt a Cartesian coordinate system for the space-like directions in LR(x): since the fluid characteristics are the same in all spatial directions, this in particular holds along the three mutually perpendicular axes defining Cartesian coordinates.

Adopting momentarily such a system—and accordingly Minkowski coordinates on space-time—the local-rest-frame values of the charge flux density $\vec{\jmath}_a(\mathsf{x})$, the j-th component $cT^{0j}(\mathsf{x})$ of the energy flux density, and the density $c^{-1}T^{i0}(\mathsf{x})$ of the i-th component of momentum should all vanish. In addition, the momentum flux-density 3-tensor $\mathsf{T}(\mathsf{x})$ in LR(x) should be diagonal, and even proportional to the three-dimensional identity. All in all, one thus necessarily has

$$N^0_a(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = c\mathfrak{n}_a(\mathsf{x}), \qquad \vec{\jmath}_a(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = \vec{0}, \tag{X.16a}$$

and

$$\begin{aligned} T^{00}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} &= \epsilon(\mathsf{x}), \\ T^{ij}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} &= \mathcal{P}(\mathsf{x})\delta^{ij}, \qquad \forall i,j=1,2,3 \\ T^{i0}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} &= T^{0j}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = 0, \qquad \forall i,j=1,2,3 \end{aligned} \tag{X.16b}$$

where the definitions (X.12) were taken into account, while $\mathcal{P}(\mathsf{x})$ denotes the pressure. Representing the energy-momentum tensor as a matrix in which $T^{\mu\nu}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})}$ is the entry in the $\mu+1$-th line and $\nu+1$-th column, Eq. (X.16b) yields

$$T^{\mu\nu}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} \cong \begin{pmatrix} \epsilon(\mathsf{x}) & 0 & 0 & 0 \\ 0 & \mathcal{P}(\mathsf{x}) & 0 & 0 \\ 0 & 0 & \mathcal{P}(\mathsf{x}) & 0 \\ 0 & 0 & 0 & \mathcal{P}(\mathsf{x}) \end{pmatrix}. \tag{X.16c}$$

Remark: The identification of the diagonal spatial components with a "pressure" term is warranted by the physical interpretation of $T^{ii}(\mathsf{x})$. Referring to it as "the" pressure anticipates the fact that it behaves as the thermodynamic quantity that is related to energy density and particle number by the mechanical equation of state of the fluid.

In an arbitrary reference frame $\mathcal{R}$ and allowing for the possible use of curvilinear coordinates, the components of the charge 4-currents and the energy-momentum tensor of a perfect fluid are

$$\boxed{N^\mu_a(\mathsf{x}) = n_a(\mathsf{x})u^\mu(\mathsf{x})} \tag{X.17a}$$

and

$$\boxed{T^{\mu\nu}(\mathsf{x}) = \mathcal{P}(\mathsf{x})g^{\mu\nu}(\mathsf{x}) + \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\frac{u^\mu(\mathsf{x})\,u^\nu(\mathsf{x})}{c^2}} \tag{X.17b}$$

respectively, with $u^\mu(\mathsf{x})$ the components of the fluid 4-velocity with respect to $\mathcal{R}$.

> Relation (X.17a) resp. (X.17b) is an identity between the components of two 4-vectors resp. two 4-tensors, which transform identically under Lorentz transformations—i.e. changes of reference frame—and coordinate basis changes. Since the components of these 4-vectors resp. 4-tensors are equal in a given reference frame—the local rest frame—and a given basis—that of Minkowski coordinates—, they remain equal in any coordinate system in any reference frame. □

In geometric formulation, the particle number 4-current and energy-momentum tensor respectively read

$$\boxed{\mathsf{N}_a(\mathsf{x}) = n_a(\mathsf{x})\mathsf{u}(\mathsf{x})} \tag{X.18a}$$

and

$$\boxed{\mathbf{T}(\mathsf{x}) = \mathcal{P}(\mathsf{x})\mathbf{g}^{-1}(\mathsf{x}) + \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\frac{\mathsf{u}(\mathsf{x})\otimes\mathsf{u}(\mathsf{x})}{c^2}.} \tag{X.18b}$$

The latter is very reminiscent of the 3-dimensional non-relativistic momentum flux density (III.22); similarly, the reader may also compare their component-wise expressions (III.21b) and (X.17a).

Remarks:

∗ The energy-momentum tensor is obviously symmetric—which is a non-trivial physical statement. For instance, the identity $T^{i0} = T^{0i}$ means that ($1/c$ times) the energy flux density in direction i equals (c times) the density of the i-th component of momentum—where one may rightly argue that the factors of c are historical accidents in the choice of units. This is possible in a relativistic theory only because the energy density also contains the mass energy.

∗ In Eq. (X.17b) or (X.18b), the sum $\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})$ is equivalently the *enthalpy density* $w(\mathsf{x})$.

∗ Equation (X.17b), (X.18b) or (X.19a) below represents the most general symmetric $\binom{2}{0}$-tensor that can be constructed using only the metric tensor and the 4-velocity.

The component form (X.17b) of the energy-momentum tensor can trivially be recast as

$$T^{\mu\nu}(\mathsf{x}) = \epsilon(\mathsf{x})\frac{u^\mu(\mathsf{x})\,u^\nu(\mathsf{x})}{c^2} + \mathcal{P}(\mathsf{x})\Delta^{\mu\nu}(\mathsf{x}) \tag{X.19a}$$

with

$$\Delta^{\mu\nu}(\mathsf{x}) \equiv g^{\mu\nu}(\mathsf{x}) + \frac{u^\mu(\mathsf{x})\,u^\nu(\mathsf{x})}{c^2} \tag{X.19b}$$

the components of a tensor $\mathbf{\Delta}$ which—in its $\binom{1}{1}$-form—is actually a projector on the 3-dimensional space orthogonal to the 4-velocity $\mathsf{u}(\mathsf{x})$, while $u^\mu(\mathsf{x})\,u^\nu(\mathsf{x})/c^2$ projects on the time-like direction of the 4-velocity.

One easily checks the identities $\Delta^\mu{}_\nu(\mathsf{x})\Delta^\nu{}_\rho(\mathsf{x}) = \Delta^\mu{}_\rho(\mathsf{x})$ and $\Delta^\mu{}_\nu(\mathsf{x})u^\nu(\mathsf{x}) = 0$.

From Eq. (X.19a) follows at once that the comoving pressure $\mathcal{P}(\mathsf{x})$ can be found in any reference frame as

$$\mathcal{P}(\mathsf{x}) = \frac{1}{3}\Delta_{\mu\nu}(\mathsf{x})T^{\mu\nu}(\mathsf{x}). \tag{X.20}$$

which complements relations (X.14) and (X.15) for the charge density and energy density.

Remark: Contracting the energy-momentum tensor $\mathbf{T}$ with the metric tensor twice yields a scalar, the so-called *trace* of $\mathbf{T}$

$$\mathbf{T}(\mathsf{x}) : \mathbf{g}(\mathsf{x}) = T^{\mu\nu}(\mathsf{x})g_{\mu\nu}(\mathsf{x}) = T^\mu{}_\mu(\mathsf{x}) = 3\mathcal{P}(\mathsf{x}) - \epsilon(\mathsf{x}). \tag{X.21}$$

X.3.2 Entropy in a perfect fluid

Let $s(\mathsf{x})$ denote the (comoving) entropy density of the fluid, as defined in the local rest frame $\mathrm{LR}(\mathsf{x})$ at point x.

X.3.2 a Entropy conservation

For a perfect fluid, the fundamental equations of motion (X.2) and (X.7) lead automatically to the *local conservation of entropy*

$$\boxed{\mathrm{d}_\mu\big[s(\mathsf{x})u^\mu(\mathsf{x})\big] = 0} \tag{X.22}$$

with $s(\mathsf{x})u^\mu(\mathsf{x})$ the *entropy four-current*.

Proof: The relation $U = TS - \mathcal{P}\mathcal{V} + \sum \mu_a N_a$ with U resp. μ_a the internal energy resp. the chemical potential associated to the conserved charge of type a, gives for the local thermodynamic densities $\epsilon = Ts - \mathcal{P} + \sum \mu_a \mathfrak{n}_a$. Inserting this expression of the energy density in Eq. (X.17b) yields (dropping the x variable for the sake of brevity):

$$T^{\mu\nu} = \mathcal{P}g^{\mu\nu} + \big(Ts + \sum \mu_a \mathfrak{n}_a\big)\frac{u^\mu u^\nu}{c^2} = \mathcal{P}g^{\mu\nu} + \big[T(su^\mu) + \sum \mu_a(\mathfrak{n}_a u^\mu)\big]\frac{u^\nu}{c^2}.$$

Letting the 4-gradient d_μ act on both sides of this identity gives

$$\mathrm{d}_\mu T^{\mu\nu} = \mathrm{d}^\nu\mathcal{P} + \big[T(su^\mu) + \sum \mu_a(\mathfrak{n}_a u^\mu)\big]\frac{\mathrm{d}_\mu u^\nu}{c^2} + \big[s\,\mathrm{d}_\mu T + \sum \mathfrak{n}_a\,\mathrm{d}_\mu\mu_a\big]\frac{u^\mu u^\nu}{c^2} + \big[T\,\mathrm{d}_\mu(su^\mu) + \sum \mu_a \mathrm{d}_\mu(\mathfrak{n}_a u^\mu)\big]\frac{u^\nu}{c^2}.$$

Invoking the energy-momentum conservation equation (X.7), the left member of this identity vanishes. The second term between square brackets on the right hand side can be rewritten with the help of the Gibbs–Duhem relation as $s\,\mathrm{d}_\mu T + \sum \mathfrak{n}_a\,\mathrm{d}_\mu\mu_a = \mathrm{d}_\mu\mathcal{P}$. Eventually, the

charge conservation equation (X.7) can be used in the rightmost term. Multiplying everything by u_ν and summing over ν yields

$$0 = u_\nu \,\mathrm{d}^\nu \mathcal{P} + \big[T(su^\mu) + \sum \mu_a (n_a u^\mu)\big] \frac{u_\nu \,\mathrm{d}_\mu u^\nu}{c^2} + (\mathrm{d}_\mu \mathcal{P}) \frac{u^\mu u^\nu u_\nu}{c^2} + \big[T \,\mathrm{d}_\mu (su^\mu)\big] \frac{u_\nu u^\nu}{c^2}.$$

The constant normalization $u_\nu u^\nu = -c^2$ of the 4-velocity implies $u_\nu \,\mathrm{d}_\mu u^\nu = 0$ for $\mu = 0, \ldots, 3$, so that the equation becomes

$$0 = u_\nu \,\mathrm{d}^\nu \mathcal{P} - (\mathrm{d}_\mu \mathcal{P}) u^\mu - T \,\mathrm{d}_\mu (su^\mu),$$

leading to $\mathrm{d}_\mu (su^\mu) = 0$ since the first two terms cancel each other. □

X.3.2 b Isentropic distribution

Let $n(\mathsf{x})$ denote the (local) comoving density of one of the conserved charges. The local conservation of entropy (X.22) implies the conservation of the ratio $s(\mathsf{x})/n(\mathsf{x})$, conveniently referred to as the entropy per particle, along the motion.

Proof: The total time derivative of the entropy per particle reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{s}{n}\right) = \frac{\partial}{\partial t}\left(\frac{s}{n}\right) + \vec{\mathsf{v}} \cdot \vec{\nabla}\left(\frac{s}{n}\right) = \frac{1}{\gamma} \mathsf{u} \cdot \mathsf{d}\left(\frac{s}{n}\right),$$

where the second identity makes use of Eq. (X.11), with γ the Lorentz factor. The rightmost term is then

$$\mathsf{u} \cdot \mathsf{d}\left(\frac{s}{n}\right) = \frac{1}{n} \mathsf{u} \cdot \mathsf{d}s - \frac{s}{n^2} \mathsf{u} \cdot \mathsf{d}n = \frac{1}{n}\left(\mathsf{u} \cdot \mathsf{d}s - \frac{s}{n} \mathsf{u} \cdot \mathsf{d}n\right).$$

The continuity equation $\mathsf{d} \cdot (n\mathsf{u}) = 0$ gives $\mathsf{u} \cdot \mathsf{d}n = -n \,\mathsf{d} \cdot \mathsf{u}$, implying

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{s}{n}\right) = \frac{1}{\gamma} \mathsf{u} \cdot \mathsf{d}\left(\frac{s}{n}\right) = \frac{1}{\gamma n}\big(\mathsf{u} \cdot \mathsf{d}s + s \,\mathsf{d} \cdot \mathsf{u}\big) = \frac{1}{\gamma n} \mathsf{d} \cdot (s\mathsf{u}) = 0,$$

where the last identity expresses the conservation of entropy (X.22). □

X.3.2 c Special case of a "single-component fluid"

Assume that the fluid components carry only one conserved charge, whose comoving density is again denoted $n(\mathsf{x})$, without index. Let N be the corresponding number of charges in a given volume $\mathcal{V}$. From the macroscopic fundamental thermodynamic relations in integral and differential forms, one deduces the relation

$$T \,\mathrm{d}\left(\frac{S}{N}\right) = \mathrm{d}\left(\frac{U}{N}\right) + \mathcal{P} \,\mathrm{d}\left(\frac{\mathcal{V}}{N}\right) \tag{X.23}$$

between the changes in the entropy "per particle" to the changes in internal energy per particle and in density. Introducing the local densities, this identity reads

$$T \,\mathrm{d}\left(\frac{s}{n}\right) = \mathrm{d}\left(\frac{\epsilon}{n}\right) + \mathcal{P} \,\mathrm{d}\left(\frac{1}{n}\right). \tag{X.24}$$

Note that these relations are purely thermodynamic.

Proof of Eq. (X.23): Writing $U = N(U/N)$, $S = N(S/N)$ and $\mathcal{V} = N(\mathcal{V}/N)$ in the fundamental relation $\mathrm{d}U = T \,\mathrm{d}S - \mathcal{P} \,\mathrm{d}\mathcal{V} + \mu \,\mathrm{d}N$ yields

$$N \,\mathrm{d}\left(\frac{U}{N}\right) + \frac{U}{N} \,\mathrm{d}N = TN \,\mathrm{d}\left(\frac{S}{N}\right) + \frac{TS}{N} \,\mathrm{d}N - N\mathcal{P} \,\mathrm{d}\left(\frac{\mathcal{V}}{N}\right) - \frac{\mathcal{P}\mathcal{V}}{N} \,\mathrm{d}N + \mu \,\mathrm{d}N,$$

which after dividing by N can be recast as

$$T \,\mathrm{d}\left(\frac{S}{N}\right) = \mathrm{d}\left(\frac{U}{N}\right) + \mathcal{P} \,\mathrm{d}\left(\frac{\mathcal{V}}{N}\right) + \frac{U - TS + \mathcal{P}\mathcal{V} - \mu N}{N} \,\mathrm{d}N.$$

Thanks to the relation $U = TS - \mathcal{P}\mathcal{V} + \mu N$ the numerator of the rightmost term vanishes, yielding the result. □

According to the previous paragraph, the entropy per particle is conserved in a perfect fluid, so that $\mathrm{d}(s/n)$ on the left-hand side of Eq. (X.24) vanishes. Expanding the differentials on the right-hand side with the product rule then yields

$$0 = \frac{\mathrm{d}\epsilon}{n} - \frac{\epsilon\,\mathrm{d}n}{n^2} - \frac{\mathcal{P}\,\mathrm{d}n}{n^2},$$

which after some rewriting gives

$$\mathcal{P} = n\frac{\mathrm{d}\epsilon}{\mathrm{d}n} - \epsilon. \tag{X.25}$$

That is, the pressure is the Legendre[(az)] transform of the internal energy density with respect to the charge number density.

X.3.3 Non-relativistic limit

We shall now consider the low-velocity limit $|\vec{\mathsf{v}}| \ll c$ of the relativistic equations of motion (X.2) and (X.7), in the case when the conserved currents are those of perfect fluids, namely given by constitutive relations (X.17a) and (X.17b). Anticipating on the result, we shall recover the equations governing the dynamics of non-relativistic perfect fluids presented in Chapter III, as could be expected for the sake of consistency.

In the small-velocity limit, the typical velocity of the "atoms" forming the fluid is also much smaller than the speed of light, which has two consequences. On the one hand, the available energies are too low to allow the creation of particle–antiparticle pairs—while their annihilation remains possible—, so that the fluid consists of either particles or antiparticles. Assuming that there is a single type of particles in the fluid, the various charges labeled by the index a are all redundant, and the charge density $n_a(\mathsf{x})$, which is proportional to the difference of the amounts of particles and antiparticles in a unit volume, actually coincides with the "true" particle number density, which will be more shortly denoted $n(\mathsf{x})$.

On the other hand, the relativistic energy density ϵ can now be expressed as the sum of the contribution from the masses of the particles and of a kinetic energy term. By definition, the latter is the local internal energy density $e(\mathsf{x})$ of the fluid, while the former is simply the number density of particles multiplied by their mass energy:

$$\epsilon(\mathsf{x}) = n(\mathsf{x})mc^2 + e(\mathsf{x}) = \rho(\mathsf{x})c^2 + e(\mathsf{x}), \tag{X.26}$$

with $\rho(\mathsf{x})$ the mass density of the fluid constituents. It is important to note that the internal energy density e is of order $\vec{\mathsf{v}}^2/c^2$ with respect to the mass-energy term. The same holds for the pressure $\mathcal{P}$, which is of the same order of magnitude as e.[(65)]

Eventually, a Taylor expansion of the Lorentz factor associated with the flow velocity yields

$$\gamma(\mathsf{x}) \underset{|\vec{\mathsf{v}}|\ll c}{\sim} 1 + \frac{1}{2}\frac{\vec{\mathsf{v}}(\mathsf{x})^2}{c^2} + \mathcal{O}\left(\frac{\vec{\mathsf{v}}(\mathsf{x})^4}{c^4}\right). \tag{X.27}$$

Accordingly, to leading order in $\vec{\mathsf{v}}^2/c^2$, the components (X.11) of the flow 4-velocity read

$$u^\mu(\mathsf{x}) \underset{|\vec{\mathsf{v}}|\ll c}{\sim} \begin{pmatrix} c \\ \vec{\mathsf{v}}(\mathsf{x}) \end{pmatrix}. \tag{X.28}$$

Throughout this subsection, we shall omit for the sake of brevity the variables x resp. $(t, \vec{r})$ of the various fields. In addition, we adopt for simplicity a system of Minkowski coordinates.

(65) This is exemplified for instance by the non-relativistic classical ideal gas, in which the internal energy density is $e = nc_\nu k_\mathrm{B}T$ with c_ν a number of order 1—this results e.g. from the *equipartition theorem*—while its pressure is $\mathcal{P} = nk_\mathrm{B}T$.

(az) A.-M. Legendre, 1752-1833

X.3.3 a Particle number conservation

The 4-velocity components (X.28) give for those of the particle number 4-current (X.17a)

$$N^\mu \underset{|\vec{\mathsf{v}}|\ll c}{\sim} \begin{pmatrix} nc \\ n\vec{\mathsf{v}} \end{pmatrix}.$$

Accordingly, the particle number conservation equation (X.2) becomes

$$0 = \partial_\mu N^\mu \approx \frac{1}{c}\frac{\partial(nc)}{\partial t} + \sum_{i=1}^{3}\frac{\partial(n\mathsf{v}^i)}{\partial x^i} = \frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{\mathsf{v}}). \tag{X.29}$$

That is, one recovers the non-relativistic continuity equation (III.10).

X.3.3 b Momentum and energy conservation

The components of the energy-momentum tensor of a perfect fluid are given by Eq. (X.17b). Performing a Taylor expansion including the leading and next-to-leading terms in $|\vec{\mathsf{v}}|/c$ yields, under consideration of relation (X.26)

$$T^{00} = -\mathcal{P} + \gamma^2(\rho c^2 + e + \mathcal{P}) \underset{|\vec{\mathsf{v}}|\ll c}{\sim} \rho c^2 + e + \rho\vec{\mathsf{v}}^2 + \mathcal{O}\!\left(\frac{\vec{\mathsf{v}}^2}{c^2}\right); \tag{X.30a}$$

$$T^{0j} = T^{j0} = \gamma^2(\rho c^2 + e + \mathcal{P})\frac{\mathsf{v}^j}{c} \underset{|\vec{\mathsf{v}}|\ll c}{\sim} \rho c\mathsf{v}^j + \left(e + \mathcal{P} + \rho\vec{\mathsf{v}}^2\right)\frac{\mathsf{v}^j}{c} + \mathcal{O}\!\left(\frac{|\vec{\mathsf{v}}|^3}{c^3}\right); \tag{X.30b}$$

$$T^{ij} = \mathcal{P}g^{ij} + \gamma^2(\rho c^2 + e + \mathcal{P})\frac{\mathsf{v}^i\mathsf{v}^j}{c^2} \underset{|\vec{\mathsf{v}}|\ll c}{\sim} \mathcal{P}g^{ij} + \rho\,\mathsf{v}^i\mathsf{v}^j + \mathcal{O}\!\left(\frac{\vec{\mathsf{v}}^2}{c^2}\right) = \mathbf{T}^{ij} + \mathcal{O}\!\left(\frac{\vec{\mathsf{v}}^2}{c^2}\right). \tag{X.30c}$$

In the last line, we have introduced the components $\mathbf{T}^{ij}$, defined in Eq. (III.21b), of the three-dimensional momentum flux-density tensor for a perfect non-relativistic fluid. As emphasized below Eq. (X.26), the internal energy density and pressure in the rightmost terms of the first or second equations are of the same order of magnitude as the term $\rho\vec{\mathsf{v}}^2$ with which they appear, i.e. they are always part of the next-to-leading order term.

Momentum conservation

Considering first the components (X.30b), (X.30c), the low-velocity limit of the relativistic momentum-conservation equation $\partial_\mu T^{\mu j} = 0$ for $j = 1, 2, 3$ reads

$$0 = \frac{1}{c}\frac{\partial(\rho c\mathsf{v}^j)}{\partial t} + \sum_{i=1}^{3}\frac{\partial\mathbf{T}^{ij}}{\partial x^i} + \mathcal{O}\!\left(\frac{\vec{\mathsf{v}}^2}{c^2}\right) = \frac{\partial(\rho\mathsf{v}^j)}{\partial t} + \sum_{i=1}^{3}\frac{\partial\mathbf{T}^{ij}}{\partial x^i} + \mathcal{O}\!\left(\frac{\vec{\mathsf{v}}^2}{c^2}\right). \tag{X.31}$$

This is precisely the conservation-equation formulation (III.25a) of the Euler equation in absence of external volume forces.

Energy conservation

Given the physical interpretation of the components T^{00}, T^{i0} with $i = 1, 2, 3$, the component $\nu = 0$ of the energy-momentum conservation equation (X.7), $\partial_\mu T^{\mu 0} = 0$, should represent the conservation of energy.

As was mentioned several times, the relativistic energy density and flux density actually also contain a term from the rest mass of the fluid constituents. Thus, the leading order contribution to $\partial_\mu T^{\mu 0} = 0$, coming from the first terms in the right members of Eqs. (X.30a) and (X.30b), is

$$0 = \frac{\partial(\rho c)}{\partial t} + \sum_{i=1}^{3}\frac{\partial(\rho c\mathsf{v}^i)}{\partial x^i} + \mathcal{O}\!\left(\frac{|\vec{\mathsf{v}}|}{c}\right),$$

that is, up to a factor c, exactly the continuity equation (III.9), which was already shown to be the low-velocity limit of the conservation of the particle-number 4-current.

To isolate the internal energy contribution, it is thus necessary to subtract that of mass energy. In the fluid local rest frame, relation (X.26) shows that one must subtract ρc^2 from ϵ. The former simply equals $\rho c u^0|_{\mathrm{LR}}$, while the latter is the component $\mu = 0$ of $T^{\mu 0}|_{\mathrm{LR}}$, whose space-like components vanish in the local rest frame. To fully subtract the mass energy contribution in any frame from both the energy density and flux density, one should thus consider the 4-vector $T^{\mu 0} - \rho c u^\mu$.

Accordingly, instead of simply using $\partial_\mu T^{\mu 0} = 0$, one should start from the equivalent—thanks to Eq. (X.2) and the relation $\rho = m\mathcal{n}$—equation $\partial_\mu (T^{\mu 0} - \rho c u^\mu) = 0$. With the approximations

$$\rho c u^0 = \gamma \rho c^2 = \rho c^2 + \frac{1}{2}\rho \vec{\mathsf{v}}^2 + \mathcal{O}\!\left(\frac{\vec{\mathsf{v}}^2}{c^2}\right)$$

and

$$\rho c u^j = \gamma \rho c \mathsf{v}^j = \rho c \mathsf{v}^j + \left(\frac{1}{2}\rho \vec{\mathsf{v}}^2\right)\frac{v^j}{c} + \mathcal{O}\!\left(\frac{|\vec{\mathsf{v}}|^5}{c^3}\right)$$

one finds

$$0 = \partial_\mu (T^{\mu 0} - \rho c u^\mu) = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho \vec{\mathsf{v}}^2 + e\right) + \sum_{j=1}^{3} \frac{\partial}{\partial x^j}\left[\left(\frac{1}{2}\rho \vec{\mathsf{v}}^2 + e + \mathcal{P}\right)\frac{\mathsf{v}^j}{c}\right] + \mathcal{O}\!\left(\frac{\vec{\mathsf{v}}^2}{c^2}\right),$$

that is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho \vec{\mathsf{v}}^2 + e\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho \vec{\mathsf{v}}^2 + e + \mathcal{P}\right)\vec{\mathsf{v}}\right] \approx 0. \tag{X.32}$$

This is the non-relativistic local formulation of energy conservation (III.35) for a perfect fluid in absence of external volume forces. Since that equation had been postulated in Section III.4.1, the above derivation may be seen as its belated proof.

X.3.3 c Entropy conservation

Using the approximate 4-velocity components (X.28), the entropy conservation equation (X.22) becomes in the low-velocity limit

$$0 = \partial_\mu (s u^\mu) \approx \frac{1}{c}\frac{\partial (sc)}{\partial t} + \sum_{i=1}^{3} \frac{\partial (s\mathsf{v}^i)}{\partial x^i} = \frac{\partial s}{\partial t} + \vec{\nabla}\cdot(s\,\vec{\mathsf{v}}), \tag{X.33}$$

i.e. gives the non-relativistic equation (III.36), expressing locally the conservation of entropy in a fluid without dissipative effects.

X.4 Dissipative relativistic fluids

In a dissipative relativistic fluid, the transport of conserved charges and 4-momentum is not only convective—i.e. caused by the fluid motion—, but may also be diffusive, due e.g. to spatial gradients of the flow velocity, the temperature, or the chemical potentials associated with the charges. The description of these new types of transport necessitates the introduction of additional contributions to the charge-number 4-currents and the energy-momentum tensor (§ X.4.1), that break the local spatial isotropy of the fluid. As a matter of fact, the local rest frame of the fluid is no longer uniquely determined, but there are in general different choices that lead to "simple" expressions for the dynamical quantities (§ X.4.2).

Adopting one of these particular reference frames as local rest frame, the general dynamical equations describing the motion of a dissipative relativistic fluid are derived in § X.4.3. Eventually, § X.4.4 and X.4.5**...**

For the sake of brevity, we adopt in this Section a "natural" system of units in which the speed of light c and the Boltzmann constant k_B equal 1.

X.4.1 Dissipative currents

To account for the additional types of transport present in dissipative fluids, extra terms are added to the charge-number 4-currents and the energy-momentum tensor. Denoting with a subscript perf. the quantities for a perfect fluid, their counterparts in the dissipative case thus read

$$\mathsf{N}_a(\mathsf{x}) = \mathsf{N}_{a,\text{perf.}}(\mathsf{x}) + \boldsymbol{\nu}_a(\mathsf{x}) \quad , \quad \mathbf{T}(\mathsf{x}) = \mathbf{T}_{\text{perf.}}(\mathsf{x}) + \boldsymbol{\tau}(\mathsf{x}) \tag{X.34a}$$

or equivalently, in terms of the components on a given coordinate system

$$N_a^\mu(\mathsf{x}) = N_{a,\text{perf.}}^\mu(\mathsf{x}) + \nu_a^\mu(\mathsf{x}) \quad , \quad T^{\mu\nu}(\mathsf{x}) = T_{\text{perf.}}^{\mu\nu}(\mathsf{x}) + \tau^{\mu\nu}(\mathsf{x}). \tag{X.34b}$$

In these equations, $\boldsymbol{\nu}_a(\mathsf{x})$ resp. $\boldsymbol{\tau}(\mathsf{x})$ is a 4-vector resp. a 4-tensor of degree 2, with components $\nu_a^\mu(\mathsf{x})$ resp. $\tau^{\mu\nu}(\mathsf{x})$, that represents a dissipative charge-number resp. energy-momentum flux density.

As in the perfect-fluid case, it is natural to introduce a 4-velocity $\mathsf{u}(\mathsf{x})$ in terms of which the quantities $N_{a,\text{perf.}}(\mathsf{x})$, $\mathbf{T}_{\text{perf.}}(\mathsf{x})$ have a simple, "isotropic" expression. Accordingly, let $\mathsf{u}(\mathsf{x})$ be an at first *arbitrary* time-like 4-vector field with constant magnitude -1,[(66)] with components $u^\mu(\mathsf{x})$, $\mu \in \{0,1,2,3\}$. The reference frame in which the spatial components of this 4-velocity vanish will constitute the local rest frame $\text{LR}(\mathsf{x})$ associated with $\mathsf{u}(\mathsf{x})$.

The projector $\boldsymbol{\Delta}$ on the 3-dimensional space orthogonal to the 4-velocity $\mathsf{u}(\mathsf{x})$ is defined as in Eq. (X.19b), i.e. its components are[(66)]

$$\Delta^{\mu\nu}(\mathsf{x}) \equiv g^{\mu\nu}(\mathsf{x}) + u^\mu(\mathsf{x})u^\nu(\mathsf{x}), \tag{X.35}$$

with $g^{\mu\nu}(\mathsf{x})$ the components of the inverse metric tensor $\mathbf{g}^{-1}(\mathsf{x})$. For the comprehension it is important to realize that $\boldsymbol{\Delta}$ plays the role of the identity in the 3-space orthogonal to $\mathsf{u}(\mathsf{x})$.

By analogy with Eqs. (X.17a), (X.18), and (X.19a), one thus writes

$$\boxed{N_a^\mu(\mathsf{x}) = n_a(\mathsf{x})u^\mu(\mathsf{x}) + \nu_a^\mu(\mathsf{x})} \tag{X.36a}$$

or equivalently

$$\boxed{\mathsf{N}_a(\mathsf{x}) = n_a(\mathsf{x})\mathsf{u}(\mathsf{x}) + \boldsymbol{\nu}_a(\mathsf{x})} \tag{X.36b}$$

and

$$\boxed{T^{\mu\nu}(\mathsf{x}) = \epsilon(\mathsf{x})u^\mu(\mathsf{x})u^\nu(\mathsf{x}) + \mathcal{P}(\mathsf{x})\Delta^{\mu\nu}(\mathsf{x}) + \tau^{\mu\nu}(\mathsf{x})} \tag{X.37a}$$

i.e., in component-free form

$$\boxed{\mathbf{T}(\mathsf{x}) = \epsilon(\mathsf{x})\mathsf{u}(\mathsf{x})\otimes\mathsf{u}(\mathsf{x}) + \mathcal{P}(\mathsf{x})\boldsymbol{\Delta}(\mathsf{x}) + \boldsymbol{\tau}(\mathsf{x}).} \tag{X.37b}$$

Remark: It should be kept in mind that the various fields $\epsilon(\mathsf{x})$, $\mathcal{P}(\mathsf{x})$, ... in Eqs. (X.36)–(X.37) depend on the still arbitrary four-velocity $\mathsf{u}(\mathsf{x})$,

The precise physical content and mathematical form of the additional terms can now be further specified.

Tensor algebra

In order for $n_a(\mathsf{x})$ to represent the comoving charge density, the dissipative 4-vector $\boldsymbol{\nu}_a(\mathsf{x})$ can have no timelike component in the local rest frame $\text{LR}(\mathsf{x})$ defined by the fluid 4-velocity, see definition (X.12). Accordingly, the condition

$$u_\mu(\mathsf{x})\nu_a^\mu(\mathsf{x})\big|_{\text{LR}(\mathsf{x})} = 0$$

must hold in the local rest frame. Since the term on the left hand side of this identity is a Lorentz scalar, its value remains the same in any reference frame or coordinate system, i.e.

[(66)] since $c^2 = 1$

$$u_\mu(\mathsf{x})\nu_a^\mu(\mathsf{x}) = \mathsf{u}(\mathsf{x})\cdot\boldsymbol{\nu}_a(\mathsf{x}) = 0. \tag{X.38a}$$

Equations (X.36) thus represent the decomposition of a 4-vector in a component parallel to the flow 4-velocity and a component orthogonal to it. Accordingly, one can write

$$\nu_a^\mu(\mathsf{x}) = \Delta^\mu{}_\nu(\mathsf{x})N_a^\nu(\mathsf{x}). \tag{X.38b}$$

Physically, the 4-vector $\boldsymbol{\nu}_a(\mathsf{x})$ represents a *diffusive charge-number 4-current* in the local rest frame, which describes the non-convective transport of the conserved charge of type a.

Similarly, the dissipative energy-momentum current $\boldsymbol{\tau}(\mathsf{x})$ can have no 00-component in the local rest frame, to ensure that $T^{00}(\mathsf{x})$ in that frame still defines the comoving energy density $\epsilon(\mathsf{x})$. This means that the components $\tau^{\mu\nu}(\mathsf{x})$ may not be proportional to the product $u^\mu(\mathsf{x})u^\nu(\mathsf{x})$. The most general symmetric tensor of degree 2 which satisfies that requirement is of the form

$$\tau^{\mu\nu}(\mathsf{x}) = q^\mu(\mathsf{x})u^\nu(\mathsf{x}) + q^\nu(\mathsf{x})u^\mu(\mathsf{x}) + \varpi^{\mu\nu}(\mathsf{x}), \tag{X.39a}$$

with $q^\mu(\mathsf{x})$ resp. $\varpi^{\mu\nu}(\mathsf{x})$ the components of a 4-vector $\mathsf{q}(\mathsf{x})$ resp. of a $\binom{2}{0}$-type tensor $\boldsymbol{\varpi}(\mathsf{x})$ such that

$$u_\mu(\mathsf{x})q^\mu(\mathsf{x}) = \mathsf{u}(\mathsf{x})\cdot\mathsf{q}(\mathsf{x}) = 0 \tag{X.39b}$$

and

$$u_\mu(\mathsf{x})\varpi^{\mu\nu}(\mathsf{x})u_\nu(\mathsf{x}) = \mathsf{u}(\mathsf{x})\cdot\boldsymbol{\varpi}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x}) = 0. \tag{X.39c}$$

Condition (X.39b) expresses the orthogonality of the 4-velocity $\mathsf{u}(\mathsf{x})$ and the 4-vector $\mathsf{q}(\mathsf{x})$, which physically represents the *heat current* or *energy flux density* in the local rest frame.

In turn, the symmetric tensor $\boldsymbol{\varpi}(\mathsf{x})$ can be decomposed into the sum of a traceless symmetric tensor $\boldsymbol{\varpi}(\mathsf{x})$ with components $\varpi^{\mu\nu}(\mathsf{x})$ and a tensor proportional to the projector (X.19b) orthogonal to the 4-velocity

$$\varpi^{\mu\nu}(\mathsf{x}) = \pi^{\mu\nu}(\mathsf{x}) + \Pi(\mathsf{x})\Delta^{\mu\nu}(\mathsf{x}). \tag{X.39d}$$

The tensor $\boldsymbol{\pi}(\mathsf{x})$ is the *shear stress tensor* in the local rest frame of the fluid, that describes the transport of momentum due to shear deformations. Eventually, $\Pi(\mathsf{x})$ represents a *dissipative pressure* term, since it behaves as the thermodynamic pressure $\mathcal{P}(\mathsf{x})$ as shown by Eq. (X.40) below.

All in all, the components of the energy-momentum tensor in a dissipative relativistic fluid may thus be written as

$$\boxed{T^{\mu\nu}(\mathsf{x}) = \epsilon(\mathsf{x})u^\mu(\mathsf{x})u^\nu(\mathsf{x}) + \big[\mathcal{P}(\mathsf{x}) + \Pi(\mathsf{x})\big]\Delta^{\mu\nu}(\mathsf{x}) + q^\mu(\mathsf{x})u^\nu(\mathsf{x}) + q^\nu(\mathsf{x})u^\mu(\mathsf{x}) + \pi^{\mu\nu}(\mathsf{x}),} \tag{X.40a}$$

which in geometric formulation reads

$$\boxed{\mathbf{T}(\mathsf{x}) = \epsilon(\mathsf{x})\mathsf{u}(\mathsf{x})\otimes\mathsf{u}(\mathsf{x}) + \big[\mathcal{P}(\mathsf{x}) + \Pi(\mathsf{x})\big]\boldsymbol{\Delta}(\mathsf{x}) + \mathsf{q}(\mathsf{x})\otimes\mathsf{u}(\mathsf{x}) + \mathsf{u}(\mathsf{x})\otimes\mathsf{q}(\mathsf{x}) + \boldsymbol{\pi}(\mathsf{x}).} \tag{X.40b}$$

One easily checks the identities

$$q^\mu(\mathsf{x}) = \Delta^{\mu\nu}(\mathsf{x})T_{\nu\rho}(\mathsf{x})u^\rho(\mathsf{x}); \tag{X.41a}$$

$$\pi^{\mu\nu}(\mathsf{x}) = \frac{1}{2}\left[\Delta^\mu{}_\rho(\mathsf{x})\Delta^\nu{}_\sigma(\mathsf{x}) + \Delta^\nu{}_\rho(\mathsf{x})\Delta^\mu{}_\sigma(\mathsf{x}) - \frac{2}{3}\Delta^{\mu\nu}(\mathsf{x})\Delta_{\rho\sigma}(\mathsf{x})\right]T^{\rho\sigma}(\mathsf{x}); \tag{X.41b}$$

$$\mathcal{P}(\mathsf{x}) + \Pi(\mathsf{x}) = \frac{1}{3}\Delta^{\mu\nu}(\mathsf{x})T_{\mu\nu}(\mathsf{x}), \tag{X.41c}$$

which together with Eq. (X.15)

$$\epsilon(\mathsf{x}) = u_\mu(\mathsf{x})T^{\mu\nu}(\mathsf{x})u_\nu(\mathsf{x}) = \mathsf{u}(\mathsf{x})\cdot\mathbf{T}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x}) \tag{X.41d}$$

allow one to recover the various fields in which the energy-momentum tensor has been decomposed.

Remarks:

∗ The energy-momentum tensor comprises 10 unknown independent fields, namely the components $T^{\mu\nu}(\mathsf{x})$ with $\nu \geq \mu$. In the decomposition (X.40), written in the local rest frame, $\epsilon(\mathsf{x})$, $\mathcal{P}(\mathsf{x}) + \Pi(\mathsf{x})$, the spatial components $q^i(\mathsf{x})$ and $\pi^{ij}(\mathsf{x})$ represent 1+1+3+5=10 equivalent independent fields—out of the 6 components $\pi^{ij}(\mathsf{x})$ with $j \geq i$, one of the diagonal ones is fixed by the condition on the trace. This in particular shows that the decomposition of the left hand side of Eq. (X.41c) into two terms is as yet premature—the splitting actually requires of an equation of state to properly identify $\mathcal{P}(\mathsf{x})$.

Similarly, the 4 unknown components N_a^μ of the 4-current associated with the conserved charge of type a are expressed in terms of $n_a(\mathsf{x})$ and the three spatial components $\nu_a^i(\mathsf{x})$, i.e. an equivalent number of independent fields.

∗ Let $a^{\mu\nu}$ denote the (contravariant) components of an arbitrary $\binom{2}{0}$-tensor. One encounters in the literature the various notations

$$a^{(\mu\nu)} \equiv \frac{1}{2}\big(a^{\mu\nu} + a^{\nu\mu}\big),$$

which represents the symmetric part of the tensor,

$$a^{[\mu\nu]} \equiv \frac{1}{2}\big(a^{\mu\nu} - a^{\nu\mu}\big)$$

for the antisymmetric part—so that $a^{\mu\nu} = a^{(\mu\nu)} + a^{[\mu\nu]}$—, and

$$a^{\langle\mu\nu\rangle} \equiv \left(\Delta_\rho^{(\mu}\Delta^{\nu)}_\sigma - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\rho\sigma}\right)a^{\rho\sigma},$$

which is the symmetrized traceless projection on the 3-space orthogonal to the 4-velocity. Using these notations, the dissipative stress tensor (X.39a) reads

$$\tau^{\mu\nu}(\mathsf{x}) = q^{(\mu}(\mathsf{x})u^{\nu)}(\mathsf{x}) + \pi^{\mu\nu}(\mathsf{x}) + \Pi(\mathsf{x})\Delta^{\mu\nu}(\mathsf{x}),$$

while Eq. (X.41b) becomes $\pi^{\mu\nu}(\mathsf{x}) = T^{\langle\mu\nu\rangle}(\mathsf{x})$.

X.4.2 Local rest frames

At a given point in a dissipative relativistic fluid, the conserved charge(s) and the energy may flow in different directions. This can happen in particular because particle–antiparticle pairs, which do not contribute to the net charge density, still transport energy. Another, not exclusive, possibility is that different conserved quantum numbers flow in different directions. In any case, one can in general not find a preferred reference frame in which the local properties of the fluid are isotropic.

As a consequence, there is also no unique "natural" choice for the 4-velocity $\mathsf{u}(\mathsf{x})$ of the fluid motion. On the contrary, several definitions of the flow 4-velocity are possible, which imply varying relations for the dissipative currents, although the physics that is being described remains the same.

X.4.2 a Eckart frame

A first natural possibility, proposed by Eckart[(ba)] [44], is to take the 4-velocity proportional to the 4-current $\mathsf{N}(\mathsf{x})$ for a given conserved charge, namely

$$u^\mu_{\text{Eckart}}(\mathsf{x}) \equiv \frac{N^\mu(\mathsf{x})}{\sqrt{-N_\nu(\mathsf{x})N^\nu(\mathsf{x})}}. \tag{X.42}$$

Accordingly, the dissipative charge-number flux $\nu(\mathsf{x})$ vanishes automatically, so that the expression of charge conservation is simpler with that choice.
The local rest frame associated with the flow 4-velocity (X.42) is then referred to as *Eckart frame*.

[(ba)] C. Eckart, 1902–1973

A drawback of that definition of the fluid 4-velocity is that the net charge number can possibly vanish in some regions of a given flow, so that $\mathsf{u}_{\text{Eckart}}(\mathsf{x})$ is not defined unambiguously in such domains.

X.4.2 b Landau frame

An alternative natural definition is that of Landau[(bb)] (and Lifshitz[(bc)]), according to whom the fluid 4-velocity is taken to be proportional to the energy flux density. The corresponding 4-velocity is defined by the implicit equation

$$u^{\mu}_{\text{Landau}}(\mathsf{x}) = \frac{T^{\mu}{}_{\nu}(\mathsf{x})\,u^{\nu}_{\text{Landau}}(\mathsf{x})}{\sqrt{-u^{\lambda}_{\text{Landau}}(\mathsf{x})T_{\lambda}{}^{\rho}(\mathsf{x})T_{\rho\sigma}(\mathsf{x})u^{\sigma}_{\text{Landau}}(\mathsf{x})}}. \tag{X.43}$$

With this choice, which in turn determines the *Landau frame*, the heat current $\mathsf{q}(\mathsf{x})$ vanishes, see Eq. (X.41a), so that the dissipative tensor $\boldsymbol{\tau}(\mathsf{x})$ satisfies the condition

$$u^{\mu}_{\text{Landau}}(\mathsf{x})\tau_{\mu\nu}(\mathsf{x}) = 0 \tag{X.44}$$

and reduces to its "viscous" part $\boldsymbol{\varpi}(\mathsf{x})$.

For a fluid without conserved quantum number, the Landau definition of the 4-velocity is the only natural one. However, in the presence of a conserved charge, heat conduction now enters the dissipative part of the associated current $\mathsf{n}(\mathsf{x})$, which conflicts with the intuition gained in the non-relativistic case. This implies that the Landau choice does not lead to a simple behavior in the limit of low velocities.

Remark: Relation (X.43) means that the Landau 4-velocity is at every point x an eigenvector of the energy momentum tensor, and more precisely an eigenvector with a negative value — the negative of the local energy density $\epsilon(\mathsf{x})$. This property allows the determination of $\mathsf{u}_{\text{Landau}}(\mathsf{x})$.

Eventually, one may of course choose to work with a general fluid 4-velocity $\mathsf{u}(\mathsf{x})$, and thus to keep both the diffusive charge current(s) and the heat flux density in the dynamical fields (X.36)–(X.40). We shall come back to that point at the end of § X.4.4.

X.4.3 General equations of motion

By substituting the decompositions (X.36), (X.40) into the generic conservation laws (X.2), (X.7), one can obtain model-independent equations of motion, that do not depend on any assumption on the various dissipative currents.

For that purpose, let us introduce the notation

$$\nabla^{\mu}(\mathsf{x}) \equiv \Delta^{\mu\nu}(\mathsf{x})\mathrm{d}_{\nu}, \tag{X.45a}$$

where d_{ν}, $\nu \in \{0,1,2,3\}$ denotes the components of the 4-gradient d—involving covariant derivatives in case a non-Minkowski system of coordinates is being used. In geometric formulation, this definition reads

$$\boldsymbol{\nabla}(\mathsf{x}) \equiv \boldsymbol{\Delta}(\mathsf{x}) \cdot \mathsf{d}. \tag{X.45b}$$

As is most obvious in the local rest frame at point x, in which the timelike component $\nabla^0(\mathsf{x})$ vanishes, $\boldsymbol{\nabla}(\mathsf{x})$ is the projection of the gradient on the space-like 3-space orthogonal to the 4-velocity. Using this 3-gradient, the 4-gradient can be written $\mathsf{d} = -\mathsf{u}(\mathsf{u}\cdot\mathsf{d}) + \boldsymbol{\nabla}$, i.e. in terms of components

$$\mathrm{d}_{\mu} = -u_{\mu}(\mathsf{u}\cdot\mathsf{d}) + \nabla_{\mu}. \tag{X.45c}$$

Recognizing that $\mathsf{u}\cdot\mathsf{d}$ is the derivative with respect to (proper) time in the local rest frame, this decomposition has a clear meaning.

[(bb)] Л. Д. Ландау = L. D. Landau, 1908–1968 [(bc)] Е. М. Лифшиц = E. M. Lifshitz, 1915–1985

The charge conservation equation (X.2) first yields

$$\mathrm{d}_\mu N_a^\mu(\mathsf{x}) = u^\mu(\mathsf{x})\mathrm{d}_\mu n_a(\mathsf{x}) + n_a(\mathsf{x})\mathrm{d}_\mu u^\mu(\mathsf{x}) + \mathrm{d}_\mu \nu_a^\mu(\mathsf{x}) = 0. \tag{X.46a}$$

In turn, the conservation (X.7) of the energy momentum tensor (X.37), projected perpendicular to resp. along the 4-velocity, gives

$$\Delta^\rho{}_\nu(\mathsf{x})\mathrm{d}_\mu T^{\mu\nu}(\mathsf{x}) = \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]u^\mu(\mathsf{x})\mathrm{d}_\mu u^\rho(\mathsf{x}) + \nabla^\rho(\mathsf{x})\mathcal{P}(\mathsf{x}) + \Delta^\rho{}_\nu(\mathsf{x})\mathrm{d}_\mu \tau^{\mu\nu}(\mathsf{x}) = 0 \tag{X.46b}$$

resp.

$$u_\nu(\mathsf{x})\mathrm{d}_\mu T^{\mu\nu}(\mathsf{x}) = -u^\mu(\mathsf{x})\mathrm{d}_\mu\epsilon(\mathsf{x}) - \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\mathrm{d}_\mu u^\mu(\mathsf{x}) + u_\nu(\mathsf{x})\mathrm{d}_\mu \tau^{\mu\nu}(\mathsf{x}) = 0.$$

In the latter equation, one can substitute the rightmost term by

$$u_\nu(\mathsf{x})\mathrm{d}_\mu\tau^{\mu\nu}(\mathsf{x}) = \mathrm{d}_\mu\big[u_\nu(\mathsf{x})\tau^{\mu\nu}(\mathsf{x})\big] - \tau^{\mu\nu}(\mathsf{x})\,\mathrm{d}_\mu u_\nu(\mathsf{x}) = -\mathrm{d}_\mu q^\mu(\mathsf{x}) - \tau^{\mu\nu}(\mathsf{x})\mathrm{d}_\mu u_\nu(\mathsf{x}),$$

where the second equality follows from conditions (X.39b) and (X.39c), which altogether yield $u_\nu(\mathsf{x})\tau^{\mu\nu}(\mathsf{x}) = -q^\mu(\mathsf{x})$. One thus obtains

$$u^\mu(\mathsf{x})\mathrm{d}_\mu\epsilon(\mathsf{x}) + \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\mathrm{d}_\mu u^\mu(\mathsf{x}) + \mathrm{d}_\mu q^\mu(\mathsf{x}) + \tau^{\mu\nu}(\mathsf{x})\mathrm{d}_\mu(\mathsf{x})u_\nu(\mathsf{x}) = 0. \tag{X.46c}$$

Equations (X.46a)–(X.46c) represent the relations governing the dynamics of a dissipative fluid in an arbitrary frame.

Remarks:

∗ The last term on the left hand side of Eq. (X.46c) can be further transformed. First, invoking identity (X.45c) and again $u_\mu(\mathsf{x})\tau^{\mu\nu}(\mathsf{x}) = -q^\nu(\mathsf{x})$, one finds

$$\tau^{\mu\nu}(\mathsf{x})\mathrm{d}_\mu u_\nu(\mathsf{x}) = q^\nu(\mathsf{x})\mathsf{u}^\rho(\mathsf{x})\mathrm{d}_\rho\mathsf{u}_\nu(\mathsf{x}) + \tau^{\mu\nu}(\mathsf{x})\nabla_\mu(\mathsf{x})\mathsf{u}_\nu(\mathsf{x}).$$

Then the terms $u^\mu(\mathsf{x})q^\nu(\mathsf{x})$ and $u^\nu(\mathsf{x})q^\mu(\mathsf{x})$ in $\tau^{\mu\nu}(\mathsf{x})$ do not contribute to the rightmost term: the former, because the "spatial gradient" $\boldsymbol{\nabla}(\mathsf{x})$ is orthogonal to the four-velocity; the latter, because $u^\nu(\mathsf{x})\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x}) = \frac{1}{2}\nabla_\mu(\mathsf{x})\big[u^\nu(\mathsf{x})u_\nu(\mathsf{x})\big]$ is the derivative of a constant. All in all, one has

$$\tau^{\mu\nu}(\mathsf{x})\,\mathrm{d}_\mu u_\nu(\mathsf{x}) = q^\nu(\mathsf{x})\mathsf{u}^\rho(\mathsf{x})\mathrm{d}_\rho\mathsf{u}_\nu(\mathsf{x}) + \varpi^{\mu\nu}(\mathsf{x})\nabla_\mu(\mathsf{x})\mathsf{u}_\nu(\mathsf{x}). \tag{X.47}$$

∗ From the general equations of motion (X.46) one can deduce the simpler forms they take when adopting Eckart's or Landau's choice of four-velocity. With Eckart's definition, the third term $\mathsf{d}\cdot\mathsf{v}_a(\mathsf{x})$ in Eq. (X.46a) drops out for the charge current used to define $\mathsf{u}(\mathsf{x})$, since $\mathsf{v}_a(\mathsf{x}) = \mathsf{0}$.

In turn, when using Landau's definition,[(67)] the heat 4-current $\mathsf{q}(\mathsf{x})$ vanishes while $\tau(\mathsf{x})$ reduces to $\varpi(\mathsf{x})$, which leads to simplifications in Eqs. (X.46b) and (X.46c).

Entropy law in a dissipative relativistic fluid

Inserting the thermodynamic relation $\epsilon + \mathcal{P} = Ts + \sum \mu_a n_a$ into the dynamical equation (X.46c) and using $\mathrm{d}\epsilon = T\,\mathrm{d}s + \sum \mu_a\,\mathrm{d}n_a$, one finds

$$T(\mathsf{x})\mathrm{d}_\mu\big[s(\mathsf{x})u^\mu(\mathsf{x})\big] + \sum \mu_a \mathrm{d}_\mu\big[n_a(\mathsf{x})u^\mu(\mathsf{x})\big] + \mathrm{d}_\mu q^\mu(\mathsf{x}) + \tau^{\mu\nu}(\mathsf{x})\mathrm{d}_\mu u_\nu(\mathsf{x}) = 0.$$

The second term can be transformed with the help of the charge-conservation equation (X.46a): Invoking in addition relation (X.47) for the fourth term yields

$$T(\mathsf{x})\mathrm{d}_\mu\big[s(\mathsf{x})u^\mu(\mathsf{x})\big] = \sum \mu_a(\mathsf{x})\mathrm{d}_\mu\nu_a^\mu(\mathsf{x}) - \mathrm{d}_\mu q^\mu(\mathsf{x}) - \varpi^{\mu\nu}(\mathsf{x})\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x}) - q^\nu(\mathsf{x})\mathsf{u}^\rho(\mathsf{x})\mathrm{d}_\rho\mathsf{u}_\nu(\mathsf{x}).$$

Next we can add two terms inside the square brackets on the left hand side, which partly cancel

(67) This choice of form for $\mathsf{u}(\mathsf{x})$ is often announced as "let us work in the Landau frame", which means that the local rest frame at each point of the fluid is the Landau frame.

the first two terms in the right member:

$$T(\mathsf{x})\mathrm{d}_\mu\left[s(\mathsf{x})u^\mu(\mathsf{x})-\sum\frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}\nu_a^\mu(\mathsf{x})+\frac{q^\mu(\mathsf{x})}{T(\mathsf{x})}\right]=-T(\mathsf{x})\sum\nu_a^\mu(\mathsf{x})\mathrm{d}_\mu\left[\frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}\right]-\frac{q^\mu(\mathsf{x})}{T(\mathsf{x})}\mathrm{d}_\mu T(\mathsf{x})$$
$$-\varpi^{\mu\nu}(\mathsf{x})\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x})-q^\nu(\mathsf{x})u^\rho(\mathsf{x})\mathrm{d}_\rho u_\nu(\mathsf{x}). \quad \text{(X.48)}$$

The first two terms on the right hand side can be further transformed: since the dissipative four-currents $\boldsymbol{\nu}_a(\mathsf{x})$ and $\mathsf{q}(\mathsf{x})$ are orthogonal to the four-velocity, one has the identities $\nu_a^\mu\mathrm{d}_\mu=\nu_a^\mu\nabla_\mu$ and $q^\mu\mathrm{d}_\mu=q^\mu\nabla_\mu$. Then, since the viscous tensor $\boldsymbol{\varpi}$ is symmetric, one has

$$\varpi^{\mu\nu}(\mathsf{x})\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x})=\frac{1}{2}\varpi^{\mu\nu}(\mathsf{x})\big[\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x})+\nabla_\nu(\mathsf{x})u_\mu(\mathsf{x})\big].$$

Using now the decomposition $\varpi^{\mu\nu}(\mathsf{x})=\pi^{\mu\nu}(\mathsf{x})+\Pi(\mathsf{x})\Delta^{\mu\nu}(\mathsf{x})$ [Eq. (X.39d)] and

$$\frac{1}{2}\big(\nabla_\mu u_\nu+\nabla_\nu u_\mu\big)=\frac{1}{2}\left[\nabla_\mu u_\nu+\nabla_\nu u_\mu-\frac{2}{3}\Delta_{\mu\nu}\big(\boldsymbol{\nabla}\cdot\mathsf{u}\big)\right]+\frac{1}{3}\Delta_{\mu\nu}\big(\boldsymbol{\nabla}\cdot\mathsf{u}\big)\equiv\mathsf{S}_{\mu\nu}+\frac{1}{3}\Delta_{\mu\nu}\big(\boldsymbol{\nabla}\cdot\mathsf{u}\big),$$

where the $\mathsf{S}_{\mu\nu}$ are the components of a traceless tensor[(68)]—comparing with Eq. (II.17d), this is the rate-of-shear tensor—, while $\boldsymbol{\nabla}\cdot\mathsf{u}$ is the (spatial) 3-divergence of the 4-velocity field, one finds that Eq. (X.48) becomes

$$\mathrm{d}_\mu S^\mu(\mathsf{x})=-\frac{\pi^{\mu\nu}(\mathsf{x})}{T(\mathsf{x})}\mathsf{S}_{\mu\nu}(\mathsf{x})-\frac{\Pi(\mathsf{x})}{T(\mathsf{x})}\nabla_\mu(\mathsf{x})u^\mu(\mathsf{x})-\sum\nu_a^\mu(\mathsf{x})\nabla_\mu\left[\frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}\right]$$
$$-\frac{q^\mu(\mathsf{x})}{T(\mathsf{x})}\left[\frac{\nabla_\mu(\mathsf{x})T(\mathsf{x})}{T(\mathsf{x})}+u^\rho(\mathsf{x})\mathrm{d}_\rho u_\mu(\mathsf{x})\right]. \quad \text{(X.49a)}$$

where we have introduced the definition

$$S^\mu(\mathsf{x})\equiv s(\mathsf{x})u^\mu(\mathsf{x})-\sum\frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}\nu_a^\mu(\mathsf{x})+\frac{q^\mu(\mathsf{x})}{T(\mathsf{x})}. \quad \text{(X.49b)}$$

These are the components of an entropy 4-current $\mathsf{S}(\mathsf{x})$, comprising on the one hand the convective transport of "thermodynamic" entropy—which is the only contribution present in the perfect-fluid case, see Eq. (X.22)—, and on the other hand contributions from the diffusive currents $\boldsymbol{\nu}_a(\mathsf{x})$ and $\mathsf{q}(\mathsf{x})$. The term on the left hand side of Eq. (X.49a) is the 4-divergence of this current, and thus describes its local change.

Let Ω be the 4-volume that represents the space-time trajectory of the fluid between an initial and a final times. Integrating Eq. (X.49b) over Ω while using the same reasoning as in § X.1.1 b, one sees that the left member will yield the change in the total entropy of the fluid between these two times. This entropy variation must be positive to ensure that the second law of thermodynamics holds. Accordingly, one requests that the integrand should be positive: $\mathrm{d}_\mu S^\mu(\mathsf{x})\geq 0$. This requirement can be used to build models for the dissipative currents.

Remark: Equation (X.49b) is actually only correct at first order.

X.4.4 First order dissipative relativistic fluid dynamics

The decompositions (X.36), (X.40) are purely algebraic and do not imply anything regarding the physics of the fluid. Any such assumption involves two distinct elements: an equation of state, relating the energy density ϵ to the thermodynamic pressure $\mathcal{P}$ and the conserved-charge densities n_a; and a set of *constitutive equations* that model the dissipative effects, i.e. the diffusive charge 4-currents $\boldsymbol{\nu}_a(\mathsf{x})$, the heat flux density $\mathsf{q}(\mathsf{x})$ and the dissipative stress tensor $\boldsymbol{\tau}(\mathsf{x})$.

[(68)] In the notation introduced in the remark at the end of § X.4.1, $\mathsf{S}_{\mu\nu}=\nabla_{\langle\mu}u_{\nu\rangle}$.

Several approaches are possible to construct such constitutive equations. A first one would be to compute the conserved-charge 4-currents and energy-momentum tensor starting from an underlying microscopic theory, in particular from a kinetic description of the fluid constituents. Alternatively, one can work directly at the "macroscopic" level, using the various constraints applying to such models.

A first constraint is that the tensorial structure of the various currents should be the correct one: using as building blocks the 4-velocity u, the 4-gradients of the temperature T, the chemical potential μ, and of u, as well as the projector $\boldsymbol{\Delta}$, one writes the possible expressions of the Lorentz-scalar Π, the 4-vectors v_a and q, and the tensor $\boldsymbol{\pi}$. Another condition is that the second law of thermodynamics should hold, i.e. that when inserting the dissipative currents in Eq. (X.49b), one obtains a 4-divergence of the entropy 4-current that is always positive.

Working in the Landau frame,[69] in which the heat flux density $\mathsf{q}(\mathsf{x})$ vanishes, and assuming a single conserved charge, the simplest—but not the most general—possibility that satisfies all constraints is to request

$$\boxed{\Pi(\mathsf{x}) = -\zeta(\mathsf{x})\nabla^\mu(\mathsf{x})u_\mu(\mathsf{x})} \tag{X.50a}$$

for the dissipative pressure,

$$\boxed{\pi^{\mu\nu}(\mathsf{x}) = -\eta(\mathsf{x})\left[\nabla^\mu(\mathsf{x})u^\nu(\mathsf{x}) + \nabla^\nu(\mathsf{x})u^\mu(\mathsf{x}) - \frac{2}{3}\Delta^{\mu\nu}(\mathsf{x})\left[\nabla^\rho(\mathsf{x})u_\rho(\mathsf{x})\right]\right] = -2\eta(\mathsf{x})\mathbf{S}^{\mu\nu}(\mathsf{x})} \tag{X.50b}$$

for the components of the shear stress tensor, and

$$\boxed{\nu^\mu(\mathsf{x}) = -\kappa(\mathsf{x})\left[\frac{n(\mathsf{x})T(\mathsf{x})}{\epsilon(\mathsf{x})+\mathcal{P}(\mathsf{x})}\right]^2\nabla^\mu(\mathsf{x})\left[\frac{\mu(\mathsf{x})}{T(\mathsf{x})}\right]} \tag{X.50c}$$

for the components of the dissipative conserved-charge 4-current. η, ζ, κ are three positive numbers which implicitly depend on the space-time position, inasmuch as they vary with temperature and chemical potential. The first two ones are obviously the shear and bulk viscosity coefficients, respectively, as hinted at by the similarity with the form (III.27f) of the viscous stress tensor of a non-relativistic Newtonian fluid. Accordingly, the equation of motion (X.46b) in which the dissipative stress tensor is substituted by $\varpi^{\mu\nu} = \pi^{\mu\nu} + \Pi\Delta^{\mu\nu}$ with the forms (X.50a), (X.50b) yields a relativistic version of the Navier–Stokes equation.

What is less obvious is that κ in Eq. (X.50c) does correspond to the heat conductivity—which explains why the coefficient in front of the gradient is written in a rather contrived way.

Inserting the dissipative currents (X.50) in the entropy law (X.49b), the latter becomes

$$\mathsf{d}\cdot\mathsf{S}(\mathsf{x}) = \frac{\boldsymbol{\pi}(\mathsf{x}):\boldsymbol{\pi}(\mathsf{x})}{2\eta(\mathsf{x})T(\mathsf{x})} + \frac{\Pi(\mathsf{x})^2}{\zeta(\mathsf{x})T(\mathsf{x})} + \left[\frac{\epsilon(\mathsf{x})+\mathcal{P}(\mathsf{x})}{n(\mathsf{x})T(\mathsf{x})}\right]^2\frac{\mathsf{v}(\mathsf{x})^2}{\kappa(\mathsf{x})T(\mathsf{x})}. \tag{X.51}$$

Since $\mathsf{n}(\mathsf{x})$ is space-like and all three transport coefficients η, ζ, κ positive, the right hand side of this equation is positive, as it should.

The constitutive equations (X.50) only involve first order terms in the derivatives of velocity, temperature, or chemical potential. In keeping, the theory constructed with such Ansätze is referred to as *first order dissipative fluid dynamics*—which is the relativistic generalization of the set of laws valid for Newtonian fluids.

This simple analogy with the non-relativistic case, together with the fact that only 3 transport coefficients (for the case with a single conserved charge) are needed, makes the "traditional" for-

[69] The corresponding formulae for Π, $\pi^{\mu\nu}$ and q^μ valid in the Eckart frame, in which v vanishes, can be found e.g. in Sec. 2.4 of Ref. [45].

mulation of first-order dissipative relativistic fluid dynamics à la Landau presented here — or its variant in the Eckart frame — attractive. However the approaches suffer from an issue that does not affect the non-relativistic counterpart. Indeed, it has been shown that many solutions of the relativistic Navier–Stokes(–Fourier) equations in the Landau–Lifshitz or Eckart formulations are unstable against small perturbations [46]. Such disturbances will grow exponentially with time, on a microscopic typical time scale. As a result, the velocity of given modes can quickly exceed the speed of light, which is of course unacceptable in a relativistic theory. In addition, gradients also grow quickly, leading to the breakdown of the small-gradient assumption that implicitly underlies the construction of first-order dissipative fluid dynamics.

> Violations of causality actually occur for short-wavelength modes, which from a physical point of view should not be described by fluid dynamics since they involve length scales on which the system is not "continuous". As such, the issue is more mathematical than physical. These modes do however play a role in numerical computations, so that there is indeed a problem when one is not working with an analytical solution.

However, it was shown in 2019 [47] that there exist formulations of relativistic first-order dissipative fluid dynamics—still based on the gradients of temperature and chemical potentials,[(70)] as well as the 4-velocity, as in the non-relativistic case—, using more general classes of reference frames, that are causal and stable against small linear perturbations. This finding also holds in the general-relativistic context [48]. However, the corresponding equations involve more transport coefficients than the Landau or Eckart frame formulations, and the reference frames do not have as simple a physical interpretation.

X.4.5 Second order dissipative relativistic fluid dynamics

To remedy the instability of the usual Landau–Lifshitz or Eckart formulations of first-order dissipative relativistic fluid dynamics—which is especially a problem for numerical implementations, in which rounding errors will quickly propagate if the theory is unstable—, theories going beyond a first-order expansion in gradients were developed.

Coming back to an arbitrary 4-velocity $\mathsf{u}(\mathsf{x})$, the components of the entropy 4-current $\mathsf{S}(\mathsf{x})$ in a first-order dissipative theory read

$$S^\mu(\mathsf{x}) = \frac{\mathcal{P}(\mathsf{x})g^{\mu\nu}(\mathsf{x}) - T^{\mu\nu}(\mathsf{x})}{T(\mathsf{x})}u_\nu(\mathsf{x}) - \sum \frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}N_a^\mu(\mathsf{x}), \tag{X.52a}$$

or equivalently

$$S^\mu(\mathsf{x}) = s(\mathsf{x})u^\mu(\mathsf{x}) - \sum \frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}\nu_a^\mu(\mathsf{x}) + \frac{1}{T(\mathsf{x})}q^\mu(\mathsf{x}) \tag{X.52b}$$

which reduces to the expression between square brackets on the left hand side of Eq. (X.49b) with Landau's choice of 4-velocity.

This entropy 4-current is *linear* in the dissipative 4-currents $\nu(\mathsf{x})$ and $\mathsf{q}(\mathsf{x})$. In addition, it is independent of the velocity 3-gradients—encoded in the expansion rate $\boldsymbol{\nabla}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x})$ and the rate-of-shear tensor $\mathbf{S}(\mathsf{x})$—, which play a decisive role in dissipation. That is, the form (X.52) can be generalized. A more general form for the entropy 4-current is thus

$$\mathsf{S}(\mathsf{x}) = s(\mathsf{x})\mathsf{u}(\mathsf{x}) - \frac{\mu_N(\mathsf{x})}{T(\mathsf{x})}\boldsymbol{\nu}(\mathsf{x}) + \frac{1}{T(\mathsf{x})}\mathsf{q}(\mathsf{x}) + \frac{1}{T(\mathsf{x})}\mathsf{Q}(\mathsf{x}) \tag{X.53a}$$

or equivalently, component-wise,

$$S^\mu(\mathsf{x}) = s(\mathsf{x})u^\mu(\mathsf{x}) - \frac{\mu_N(\mathsf{x})}{T(\mathsf{x})}\nu^\mu(\mathsf{x}) + \frac{1}{T(\mathsf{x})}q^\mu(\mathsf{x}) + \frac{1}{T(\mathsf{x})}Q^\mu(\mathsf{x}), \tag{X.53b}$$

[(70)] A crucial point is precisely that of the definitions of these notions.

with $\mathsf{Q}(\mathsf{x})$ a 4-vector, with components $Q^\mu(\mathsf{x})$, that depends on the flow 4-velocity and its gradients—where $\boldsymbol{\nabla}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x})$ and $\mathsf{S}(\mathsf{x})$ are traditionally replaced by $\Pi(\mathsf{x})$ and $\boldsymbol{\pi}(\mathsf{x})$—and on the dissipative currents:

$$Q^\mu(\mathsf{x}) = Q^\mu\big(\mathsf{u}(\mathsf{x}), \nu(\mathsf{x}), \mathsf{q}(\mathsf{x}), \Pi(\mathsf{x}), \boldsymbol{\pi}(\mathsf{x})\big). \tag{X.53c}$$

In second order dissipative relativistic fluid dynamics with for simplicity a single conserved charge, the most general form for the additional 4-vector $\mathsf{Q}(\mathsf{x})$ contributing to the entropy density is [49, 50, 51]

$$\mathsf{Q}(\mathsf{x}) = \frac{\beta_0(\mathsf{x})\Pi(\mathsf{x})^2 + \beta_1(\mathsf{x})\mathsf{q}_{\scriptscriptstyle N}(\mathsf{x})^2 + \beta_2(\mathsf{x})\boldsymbol{\pi}(\mathsf{x}):\boldsymbol{\pi}(\mathsf{x})}{2T(\mathsf{x})}\mathsf{u}(\mathsf{x}) - \frac{\alpha_0(\mathsf{x})}{T(\mathsf{x})}\Pi(\mathsf{x})\mathsf{q}_{\scriptscriptstyle N}(\mathsf{x}) - \frac{\alpha_1(\mathsf{x})}{T(\mathsf{x})}\boldsymbol{\pi}(\mathsf{x})\cdot\mathsf{q}_{\scriptscriptstyle N}(\mathsf{x}), \tag{X.54a}$$

where

$$\mathsf{q}_{\scriptscriptstyle N}(\mathsf{x}) \equiv \mathsf{q}(\mathsf{x}) - \frac{\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})}{n(\mathsf{x})}\nu(\mathsf{x});$$

component-wise, this reads

$$Q^\mu(\mathsf{x}) = \frac{\beta_0(\mathsf{x})\Pi(\mathsf{x})^2 + \beta_1(\mathsf{x})\mathsf{q}_{\scriptscriptstyle N}(\mathsf{x})^2 + \beta_2(\mathsf{x})\pi_{\nu\rho}(\mathsf{x})\pi^{\nu\rho}(\mathsf{x})}{2T(\mathsf{x})}u^\mu(\mathsf{x}) - \frac{\alpha_0(\mathsf{x})}{T(\mathsf{x})}\Pi(\mathsf{x})q_{\scriptscriptstyle N}^\mu(\mathsf{x}) - \frac{\alpha_1(\mathsf{x})}{T(\mathsf{x})}\pi^\mu{}_\rho(\mathsf{x})q_{\scriptscriptstyle N}^\rho(\mathsf{x}). \tag{X.54b}$$

The 4-vector $\mathsf{Q}(\mathsf{x})$ is now quadratic ("of second order") in the dissipative currents—in the wider sense—$\mathsf{q}(\mathsf{x})$, $\nu(\mathsf{x})$, $\Pi(\mathsf{x})$ and $\boldsymbol{\pi}(\mathsf{x})$, and involves 5 additional coefficients depending on temperature and particle-number density, α_0, α_1, β_0, β_1, and β_2.

Substituting this form of $\mathsf{Q}(\mathsf{x})$ in the entropy 4-current (X.53), the simplest way to ensure that its 4-divergence should be positive is to postulate *linear* relationships between the dissipative currents and the gradients of velocity, chemical potential (or rather of $-\mu/T$), and temperature (or rather, $1/T$), as was done in Eqs. (X.50). This recipe yields differential equations for $\Pi(\mathsf{x})$, $\boldsymbol{\pi}(\mathsf{x})$, $\mathsf{q}_{\scriptscriptstyle N}(\mathsf{x})$, representing 9 coupled scalar equations of motion. These describe the relaxation—with appropriate characteristic time scales τ_Π, $\tau_{\boldsymbol{\pi}}$, $\tau_{\mathsf{q}_{\scriptscriptstyle N}}$ respectively proportional to β_0, β_2, β_1, while the involved "time derivative" is that in the local rest frame, $\mathsf{u}\cdot\mathsf{d}$—, of the dissipative currents towards their first-order expressions (X.50).

Adding up the new equations to the usual ones (X.2) and (X.7), the resulting set of equations, known as (Müller[bd]–)Israel[be]–Stewart[bf] theory, is no longer plagued by the issues that affects the relativistic Navier–Stokes–Fourier equations.

Bibliography for Chapter X

- Andersson & Comer [52];
- Landau–Lifshitz [4, 5], Chapter XV, § 133,134 (perfect fluid) and § 136 (dissipative fluid);
- Romatschke [53];
- Weinberg [54], Chapter 2, § 10 (perfect fluid) and § 11 (dissipative fluid).

[bd] I. Müller, born 1936 [be] W. Israel, 1931–2022 [bf] J. M. Stewart, 1943–2016

Appendices to Chapter X

X.A Microscopic formulation of the hydrodynamical fields

In Sec. X.1, we have taken common non-relativistic quantities—particle number density and flux density, energy density, momentum flux density, and so on—and claimed that they may be used to define a 4-vector resp. a Lorentz tensor, namely the particle number 4-current $\mathsf{N}(\mathsf{x})$ resp. the energy-momentum tensor $\mathsf{T}(\mathsf{x})$. However, we did not explicitly show that the latter are indeed a 4-vector resp. a tensor. For that purpose, the best is to turn the reasoning round and to introduce quantities which are manifestly, by construction, a Lorentz 4-vector or tensor. In turn, one investigates the physical interpretation of their components and shows that it coincides with known non-relativistic quantities.

Throughout this Appendix, we consider a system Σ of N "particles"—i.e. carriers of some conserved additive quantum number—labeled by $k \in \{1, \dots, N\}$ with world-lines $\mathsf{x}_k(\tau)$ and associated 4-velocities $\mathsf{u}_k(\tau) \equiv \mathrm{d}\mathsf{x}_k(\tau)/\mathrm{d}\tau$, where the scalar parameter τ along the world-line of a given particle is conveniently taken as its proper time.

X.A.1 Particle number 4-current

The particle-number 4-current associated with the collection of particles Σ is defined as

$$\mathsf{N}(\mathsf{x}) \equiv \sum_{k=1}^{N} \int \mathsf{u}_k(\tau)\, \delta^{(4)}\big(\mathsf{x}-\mathsf{x}_k(\tau)\big)\, \mathrm{d}(c\tau) \tag{X.A.1a}$$

or component-wise

$$N^{\mu}(\mathsf{x}) \equiv \sum_{k=1}^{N} \int u_k^{\mu}(\tau)\, \delta^{(4)}\big(x^{\nu}-x_k^{\nu}(\tau)\big)\, \mathrm{d}(c\tau) \quad \text{for } \mu = 0,1,2,3, \tag{X.A.1b}$$

where the k-th integral in either sum is along the world-line of particle k. The right hand sides of these equations clearly define a 4-vector resp. its components. For the latter, some simple algebra yields the identities

$$\frac{1}{c} N^0(t,\vec{r}) = \sum_{k=1}^{N} \delta^{(3)}\big(\vec{r} - \vec{x}_k(t)\big), \tag{X.A.2a}$$

$$N^i(t,\vec{r}) = \sum_{k=1}^{N} v_k^i(t)\, \delta^{(3)}\big(\vec{r} - \vec{x}_k(t)\big) \tag{X.A.2b}$$

with $\vec{x}_k(t)$ the spatial trajectory corresponding to the world-line $\mathsf{x}_k(\tau)$.

Using $u_k^0(\tau) = c\,\mathrm{d}t_k(\tau)/\mathrm{d}\tau$ and changing the parameter along the world-lines from τ to t, one finds

$$N^0(t,\vec{r}) = c\sum_{k=1}^{N} \int \delta\big(ct - ct_k(\tau)\big)\, \delta^{(3)}\big(\vec{x}-\vec{x}_k(\tau)\big)\, \frac{\mathrm{d}t_k(\tau)}{\mathrm{d}\tau}\, \mathrm{d}(c\tau) = c\sum_{k=1}^{N} \int \delta\big(t - t_k(t)\big)\, \delta^{(3)}\big(\vec{x}-\vec{x}_k(t)\big)\, \mathrm{d}t$$

i.e. $N^0(t,\vec{r}) = c\sum_{k=1}^{N} \delta^{(3)}\big(\vec{x}-\vec{x}_k(t)\big)$. The proof for Eq. (X.A.2b) is identical. □

Inspecting the right hand sides of relations (X.A.2), they obviously represent the particle number density and flux density for the system Σ, respectively.

X.A.2 Energy-momentum tensor

Denoting by p_k the 4-momentum carried by particle k, the energy-momentum tensor associated with the collection of particles Σ is defined as

$$\mathsf{T}(\mathsf{x}) \equiv \sum_{k=1}^{N} \int \mathsf{p}_k(\tau) \otimes \mathsf{u}_k(\tau)\, \delta^{(4)}\big(\mathsf{x}-\mathsf{x}_k(\tau)\big)\, \mathrm{d}(c\tau) \tag{X.A.3a}$$

where the k-th integral in the sum is along the world-line of particle k, as above; component-wise, this gives

$$T^{\mu\nu}(\mathsf{x}) \equiv \sum_{k=1}^{N} \int p_k^{\mu}(\tau)\, u_k^{\nu}(\tau)\, \delta^{(4)}\big(x^{\lambda}-x^{\lambda}(\tau)\big)\, \mathrm{d}(c\tau) \quad \text{for } \mu,\nu = 0,1,2,3. \tag{X.A.3b}$$

The members of these equations clearly define a Lorentz tensor of type $\binom{2}{0}$ resp. its components.

Repeating the same derivation as that leading to Eq. (X.A.2a), one shows that

$$T^{\mu 0}(t,\vec{r}) = \sum_{k=1}^{N} p_k^{\mu}(t) c\, \delta^{(3)}\big(\vec{r}-\vec{x}_k(t)\big). \tag{X.A.4a}$$

Recognizing in $p_k^0 c$ the energy of particle k, T^{00} represents the energy density of the system Σ—under the assumption that the potential energy associated with the interaction between particles is much smaller than their mass and kinetic energies—, while T^{i0} for $i = 1,2,3$ represents c times the density of the i-th component of momentum. In turn,

$$T^{0j}(t,\vec{r}) = \sum_{k=1}^{N} p_k^0(t)\, v_k^j(t)\, \delta^{(3)}\big(\vec{r}-\vec{x}_k(t)\big) \tag{X.A.4b}$$

with $j \in \{1,2,3\}$ is the $1/c$ times the j-th of the energy flux density of the collection of particles. Eventually, for $i,j = 1,2,3$

$$T^{ij}(t,\vec{r}) = \sum_{k=1}^{N} p_k^i(t)\, v_k^j(t)\, \delta^{(3)}\big(\vec{r}-\vec{x}_k(t)\big) \tag{X.A.4c}$$

is clearly the j-th component of the flux density of momentum along the i-th direction.

Remark: Invoking the relation $\mathsf{p} = m\mathsf{u}$ between the 4-momentum, mass and 4-velocity of a (massive!) particle shows at once that the energy-momentum tensor (X.A.3) is symmetric.

X.B Relativistic kinematics

Later

X.C Equations of state for relativistic fluids

CHAPTER XI

Flows of relativistic fluids

Throughout this Chapter $c = 1$.

XI.1 Relativistic fluids at rest

To be written!

... necessitate the presence of a gravitational field (no spontaneous "photon star" or similar).

"Dust" can be in hydrostatic equilibrium.

XI.2 One-dimensional relativistic flows

XI.2.1 Bjorken flow

To describe (part of the) system—often referred to as "fireball"—created in the collision of two heavy nuclei at extremely high energies, Bjorken[(bg)] proposed to treat it as a perfect fluid with a simple velocity field. In a reference frame $\mathcal{R}_0$ ("center-of-momentum frame") in which the total momentum of the colliding nuclei vanishes, and using Minkowski coordinates such that the momenta of the nuclei before their collision lie along the z-direction, the ansatz for the velocity reads [55]

$$\mathsf{v}^z(\mathsf{x}) = \frac{z}{t} \quad \text{for } |z| < t, \qquad \mathsf{v}^x(\mathsf{x}) = \mathsf{v}^y(\mathsf{x}) = 0, \tag{XI.1}$$

independent of the "transverse" coordinates x and y. Accordingly, the Lorentz factor of the local rest frame at point x is $\gamma(\mathsf{x}) = 1/\sqrt{1-\mathsf{v}^z(\mathsf{x})^2} = t/\sqrt{t^2-z^2}$, resulting in the 4-velocity field

$$\mathsf{u}^t(\mathsf{x}) = \frac{t}{\sqrt{t^2-z^2}}, \quad \mathsf{u}^x(\mathsf{x}) = \mathsf{u}^y(\mathsf{x}) = 0, \quad \mathsf{u}^z(\mathsf{x}) = \frac{z}{\sqrt{t^2-z^2}}. \tag{XI.2}$$

Note that Eq. (XI.1) coincides with the velocity distribution of *non-interacting* particles emitted at time $t = 0$ at $z = 0$ with a velocity along the z-direction.

XI.2.1 a Milne coordinates

A convenient coordinate system to investigate the properties of the flow defined by Eq. (XI.1) consists of the so-called *Milne coordinates*[(bh)]

$$\tau \equiv \sqrt{t^2-z^2} \quad , \quad \varsigma \equiv \frac{1}{2}\ln\frac{t+z}{t-z}, \tag{XI.3a}$$

[(bg)] J. D. Bjorken, born 1934 [(bh)] E. A. Milne, 1896–1950

called respectively "Bjorken proper time" and "space-time rapidity". Inverting these equations yield the simple relations

$$t = \tau \cosh \varsigma \quad , \quad z = \tau \sinh \varsigma . \tag{XI.3b}$$

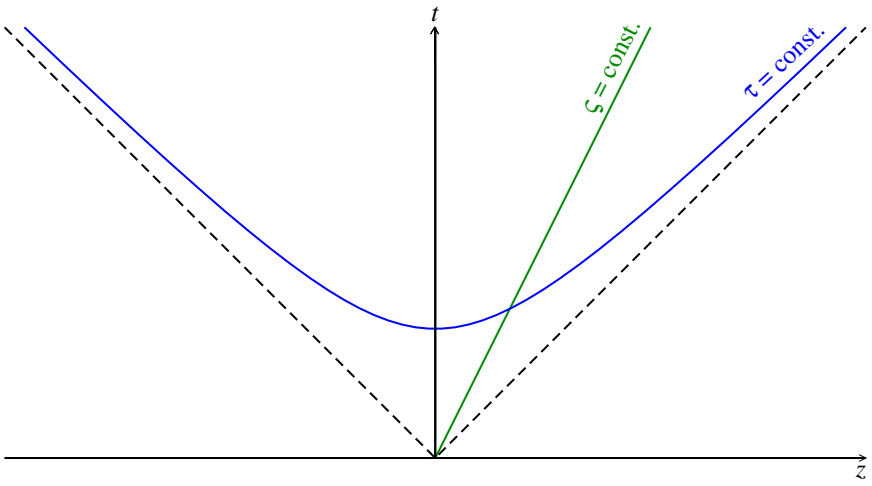

Figure XI.1 – Milne coordinates

Introducing the matrix with entries $\Lambda^{\mu'}{}_{\nu} \equiv \partial x^{\mu'}/\partial x^{\nu}$, where the primed resp. unprimed indices refer to Milne resp. Minkowski coordinates, one quickly finds that the covariant components of a 4-vector V in the two coordinate systems are related by

$$\begin{pmatrix} V^{\tau} \\ V^{\varsigma} \end{pmatrix} = \begin{pmatrix} \cosh \varsigma & -\sinh \varsigma \\ -\dfrac{1}{\tau}\sinh \varsigma & \dfrac{1}{\tau}\cosh \varsigma \end{pmatrix} \begin{pmatrix} V^{t} \\ V^{z} \end{pmatrix} . \tag{XI.4}$$

In particular, this transformation applied to the Bjorken flow 4-velocity (XI.2) yields

$$\mathsf{u}^{\tau}(\mathsf{x}) = 1, \quad \mathsf{u}^{\varsigma}(\mathsf{x}) = 0. \tag{XI.5}$$

In turn, the Minkowski components (XI.2) can be rewritten as

$$\mathsf{u}^{t}(\mathsf{x}) = \cosh \varsigma, \quad \mathsf{u}^{z}(\mathsf{x}) = \sinh \varsigma \tag{XI.6}$$

which is convenient for calculations.

Since the Milne coordinates (XI.3) are clearly curvilinear, the covariant derivatives d_{τ}, d_{ς} do not necessarily coincide with the respective partial derivatives ∂_{τ}, ∂_{ς} when acting on vector or more general tensor fields. Instead of working fully in Milne coordinates in the following,[(71)] we shall compute expressions involving covariant derivatives in Minkowksi coordinates, where $\mathrm{d}_t = \partial_t$ and $\mathrm{d}_z = \partial_z$. Using the chain rule $\partial_{\mu'} = \partial_{\mu'} t\, \partial_t + \partial_{\mu'} z\, \partial_z$ for $\mu' \in \{\tau, \varsigma\}$, one finds

$$\begin{pmatrix} \partial_{\tau} \\ \dfrac{1}{\tau}\partial_{\varsigma} \end{pmatrix} = \begin{pmatrix} \cosh \varsigma & \sinh \varsigma \\ \sinh \varsigma & \cosh \varsigma \end{pmatrix} \begin{pmatrix} \partial_t \\ \partial_z \end{pmatrix} \tag{XI.7}$$

and conversely

$$\begin{pmatrix} \partial_t \\ \partial_z \end{pmatrix} = \begin{pmatrix} \cosh \varsigma & -\sinh \varsigma \\ -\sinh \varsigma & \cosh \varsigma \end{pmatrix} \begin{pmatrix} \partial_{\tau} \\ \dfrac{1}{\tau}\partial_{\varsigma} \end{pmatrix} . \tag{XI.8}$$

From there and the 4-velocity components (XI.6), one arrives at once at the relations

(71) An appendix to this Chapter may be added at some point...

$$u^\mu(\mathsf{x})\partial_\mu = u^t(\mathsf{x})\partial_t + u^z(\mathsf{x})\partial_z = \partial_\tau \qquad \text{and} \qquad \partial_\mu u^\mu(\mathsf{x}) = \partial_t u^t(\mathsf{x}) + \partial_z u^z(\mathsf{x}) = \frac{1}{\tau}. \tag{XI.9}$$

Note that $u^\mu\partial_\mu$ coincides with $u^{\mu'}\partial_{\mu'} \equiv u^\tau\partial_\tau + u^\varsigma\partial_\varsigma$, while on the other hand $\partial_\mu u^\mu$ does not equal $\partial_{\mu'}u^{\mu'} \equiv \partial_\tau u^\tau + \partial_\varsigma u^\varsigma$—which trivially vanishes.

Eventually, the projector (X.19b) on the 3-space orthogonal to the flow velocity is readily computed, from which one then deduces the (contravariant) Minkowski components $\nabla^\mu(\mathsf{x}) \equiv \Delta^{\mu\nu}(\mathsf{x})\partial_\nu$ of the 3-gradient (X.45a)

$$\nabla^t = \sinh^2\varsigma\,\partial_t + \cosh\varsigma\sinh\varsigma\,\partial_z = \frac{\sinh\varsigma}{\tau}\partial_\varsigma, \quad \nabla^z = \cosh\varsigma\sinh\varsigma\,\partial_t + \cosh^2\varsigma\,\partial_z = \frac{\cosh\varsigma}{\tau}\partial_\varsigma$$

together with $\nabla^x = \partial_x$, $\nabla^y = \partial_y$, where Eq. (XI.7) was used. Invoking transformation (XI.4), the Milne components of the 3-gradient are

$$\nabla^\tau = 0 \quad , \quad \nabla^\varsigma = \frac{1}{\tau^2}\partial_\varsigma. \tag{XI.10}$$

Consistent with the fact that only u^τ is non-vanishing, ∇^τ vanishes and ∇^ς only involves ∂_ς.

> The reader worried by the appearance of the factor $1/\tau^2$ in ∇^ς will possibly be relieved when realizing that $\nabla_\varsigma \equiv g_{\mu'\varsigma}\nabla^{\mu'} = g_{\varsigma\varsigma}\nabla^\varsigma$—because the metric tensor is still diagonal in Milne coordinates—and that this equals ∂_ς thanks to $g_{\varsigma\varsigma} = \tau^2$.

XI.2.1 b Perfect fluid

For a perfect fluid, with energy-momentum tensor given by Eq. (X.17b), the conservation equation (X.7a) projected parallel resp. orthogonal to the flow 4-velocity leads to the general equations of motion

$$u^\mu(\mathsf{x})\mathrm{d}_\mu\epsilon(\mathsf{x}) + \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\mathrm{d}_\mu u^\mu(\mathsf{x}) = 0 \tag{XI.11a}$$

resp.

$$\big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]u^\mu(\mathsf{x})\mathrm{d}_\mu u^\rho(\mathsf{x}) + \nabla^\rho(\mathsf{x})\mathcal{P}(\mathsf{x}) = 0, \tag{XI.11b}$$

corresponding to Eqs. (X.46b)–(X.46c) with vanishing viscous tensor.

In the case of the Bjorken flow 4-velocity, for which we derived Eq. (XI.9), these equations become

$$\partial_\tau\epsilon(\mathsf{x}) + \frac{\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})}{\tau} = 0 \tag{XI.12a}$$

and

$$\big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\partial_\tau u^\rho(\mathsf{x}) + \nabla^\rho(\mathsf{x})\mathcal{P}(\mathsf{x}) = 0. \tag{XI.12b}$$

The second of these equations holds in any coordinate system, in particular with Milne coordinates. In the latter, we have found that the components u^τ, u^ς of the velocity are constant, see Eq. (XI.5), in particular independent of τ. That is, the first term on the left hand side of Eq. (XI.12b) vanishes, leaving only

$$\nabla^{\rho'}(\mathsf{x})\mathcal{P}(\mathsf{x}) = 0 \quad \text{for } \rho' \in \{\tau, x, y, \varsigma\}.$$

From Eq. (XI.10), the component $\rho' = \tau$ of this equation is trivial since $\nabla^\tau = 0$. In turn the spatial components read $\partial_x\mathcal{P} = \partial_y\mathcal{P} = 0$, which were obvious from the start since the problem was assumed to be independent of x and y, and

$$\partial_\varsigma\mathcal{P}(\mathsf{x}) = 0. \tag{XI.13}$$

That is, the pressure—and invoking an equation of state, the energy density— is also independent of rapidity.

Coming back to the first equation of motion (XI.12), it can also be rewritten in the form

$$\partial_\tau\big[\tau\epsilon(\mathsf{x})\big] = -\mathcal{P}(\mathsf{x}), \tag{XI.14}$$

which shows that it is the energy-balance equation: the change in the total energy (per unit transverse surface) of a comoving volume[(72)] is due to the work of pressure forces.

[(72)] $\mathrm{d}^4\mathsf{x}' = \tau\,\mathrm{d}\tau\,\mathrm{d}\varsigma\,\mathrm{d}x\,\mathrm{d}y$ grows proportionally to τ.

Remarks:

∗ In a perfect fluid the entropy is conserved: $\mathrm{d}_\mu[s(\mathsf{x})u^\mu(\mathsf{x})] = 0$, see Eq. (X.22), with s the entropy density. This equation can be recast in the form $u^\mu(\mathsf{x})\,\mathrm{d}_\mu s(\mathsf{x}) = -s(\mathsf{x})\,\mathrm{d}_\mu u^\mu(\mathsf{x})$, which using Eq. (XI.9) becomes

$$\partial_\tau s(\mathsf{x}) = -\frac{s(\mathsf{x})}{\tau}. \tag{XI.15}$$

This equation leads at once to $s(\mathsf{x}) \propto 1/\tau$, with the simple interpretation that the total entropy in a comoving fluid volume, proportional to $\tau s(\mathsf{x})$, remains constant in the evolution.

∗ Ditto for conserved charges: the conservation equation $\mathrm{d}_\mu N_a^\mu(\mathsf{x}) = 0$ [Eq. (X.2)] together with the constitutive relation $N_a^\mu(\mathsf{x}) = n_a(\mathsf{x})u^\mu(\mathsf{x})$ [Eq. (X.17a)] of perfect fluids result in

$$\partial_\tau n_a(\mathsf{x}) = -\frac{n_a(\mathsf{x})}{\tau}. \tag{XI.16}$$

∗ Bjorken's ansatz (XI.1) for the flow velocity means that an observer $\mathcal{O}_v$ comoving with the fluid at a given point—being say at time t_0 at position z_0 with velocity $v = \mathsf{v}^z(t_0, z_0) = z_0/t_0$—actually moves with constant velocity v with respect to the reference frame $\mathcal{R}_0$. If $\mathcal{R}_0$ is inertial, then $\mathcal{O}_v$ defines another inertial frame $\mathcal{R}_v$: systems of Minkowski coordinates (with parallel-oriented axes) in the two frames are related by a Lorentz boost along the z-direction with velocity v. Instead of v, such a boost is often characterized by its rapidity $\xi \equiv \operatorname{artanh} v = \frac{1}{2}\ln\frac{1+v}{1-v}$. One sees that the boost rapidity ξ is precisely the space-time rapidity ς of the point at which $\mathcal{O}_v$ is sitting. In turn, the statements that the fluid velocity is independent of ς [Eq.(XI.5)] and that this also holds for the locally-measured thermodynamic quantities [(XI.13)] means that all comoving observer $\mathcal{O}_v$, irrespective of their velocity v, view the flow in the same way. The Bjorken flow is thus said to be (longitudinally) boost invariant.

By assuming a simple equation of state, one can derive further results. Let us thus assume that the pressure and energy density are proportional to each other, with a constant—i.e. time- and position-independent—proportionality factor:

$$\mathcal{P}(\mathsf{x}) = c_s^2 \epsilon(\mathsf{x}). \tag{XI.17}$$

For instance, $\mathcal{P} = \epsilon/3$ for an ideal gas of ultrarelativistic particles without conserved charge (see Appendix X.C). The notation c_s^2 is not arbitrary but corresponds to the fact that c_s is indeed the (phase) velocity of sound waves in the fluid.

With this equation of state, Eq. (XI.12a) leads at once to

$$\epsilon(\mathsf{x}) \propto \frac{1}{\tau^{1+c_s^2}}, \tag{XI.18}$$

i.e. $\epsilon(\mathsf{x}) \propto 1/\tau^{4/3}$ for an ideal ultrarelativistic gas. That is, the energy density decreases faster than the entropy density—due to the work exerted by pressure.

If one now combines the equation of state (XI.17), the Gibbs–Duhem equation (in absence of conserved charge) $\mathrm{d}\mathcal{P} = s\,\mathrm{d}T$, and the fundamental relation $\epsilon = Ts - \mathcal{P}$, one finds

$$\mathrm{d}\mathcal{P} = c_s^2\,\mathrm{d}\epsilon = \frac{\epsilon + \mathcal{P}}{T}\,\mathrm{d}T.$$

Rewriting the numerator of the rightmost term as $(1 + c_s^2)\epsilon$, there comes

$$\frac{\mathrm{d}\epsilon}{\epsilon} = \frac{1 + c_s^2}{c_s^2}\frac{\mathrm{d}T}{T}.$$

This yields $\epsilon \propto T^{1+c_s^{-2}}$, which together with relation (XI.18) gives

$$T(\mathsf{x}) \propto \frac{1}{\tau^{c_s^2}}, \tag{XI.19}$$

i.e. $T(\mathsf{x}) \propto 1/\tau^{1/3}$ for an ideal ultrarelativistic gas. Since we found earlier that the energy density of such a system decreases as $\tau^{-4/3}$, the behavior of temperature is consistent with the thermal equation of state $\varepsilon \propto T^4$ (and with $s \propto T^3$).

XI.2.1 c First-order dissipative fluid

to be added soon

XI.2.2 Landau flow

[56, 57]

Appendices

APPENDIX A

Basic elements of thermodynamics

To be written!

$$U = TS - \mathcal{P}\mathcal{V} + \mu N \tag{A.1}$$

$$\mathrm{d}U = T\,\mathrm{d}S - \mathcal{P}\,\mathrm{d}\mathcal{V} + \mu\,\mathrm{d}N \tag{A.2}$$

$$e + \mathcal{P} = Ts + \mu n \tag{A.3}$$

$$\mathrm{d}e = T\,\mathrm{d}s + \mu\,\mathrm{d}n. \tag{A.4}$$

$$\mathrm{d}\mathcal{P} = s\,\mathrm{d}T + n\,\mathrm{d}\mu \tag{A.5}$$

The first of these relations follows from

$$\begin{aligned} \mathrm{d}e &= \mathrm{d}\left(\frac{U}{\mathcal{V}}\right) = \frac{1}{\mathcal{V}}\,\mathrm{d}U - \frac{U}{\mathcal{V}^2}\,\mathrm{d}\mathcal{V} \\ &= \frac{T}{\mathcal{V}}\,\mathrm{d}S - \frac{\mathcal{P}}{\mathcal{V}}\,\mathrm{d}\mathcal{V} + \frac{\mu}{\mathcal{V}}\,\mathrm{d}N - \frac{TS}{\mathcal{V}^2}\,\mathrm{d}\mathcal{V} + \frac{\mathcal{P}}{\mathcal{V}}\,\mathrm{d}\mathcal{V} - \frac{\mu N}{\mathcal{V}^2}\,\mathrm{d}\mathcal{V} = T\,\mathrm{d}\left(\frac{S}{\mathcal{V}}\right) + \mu\,\mathrm{d}\left(\frac{N}{\mathcal{V}}\right), \end{aligned}$$

in which use was made of Eq. (A.2). □

APPENDIX B

Elements on holomorphic functions of a complex variable

In this appendix, we give without proof nor any attempt at being exhaustive a few important results on holomorphic functions of a complex variables that are useful for the study of two-dimensional potential flows (§ IV.4.3).

B.1 Holomorphic functions

B.1.1 Definitions

A function $Z = f(z)$ is defined to be *complex-differentiable* at a point z_0 in its domain of definition if the limit

$$f'(z_0) \equiv \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{B.1}$$

exists independently of the direction along which z approaches z_0.
If f is complex-differentiable at every point of an open set $\mathcal{U}$ resp. of a neighborhood of a point z_0, it is said to be *holomorphic* on $\mathcal{U}$ resp. at z_0.

B.1.2 Some properties

B.1.2 a Cauchy–Riemann equations

Let $P(x, y)$ resp. $Q(x, y)$ denote the real resp. imaginary part of a function $f(z = x+\mathrm{i}y)$ of a complex variable:

$$f(x+\mathrm{i}y) = P(x, y) + \mathrm{i}Q(x, y). \tag{B.2}$$

Theorem: f is holomorphic if and only if the *Cauchy–Riemann equations*

$$\boxed{\frac{\partial P(x,y)}{\partial x} = \frac{\partial Q(x,y)}{\partial y} \quad \text{and} \quad \frac{\partial P(x,y)}{\partial y} = -\frac{\partial Q(x,y)}{\partial x}} \tag{B.3}$$

relating the first partial derivatives of its real and imaginary parts are satisfied.

Equivalently, the relations (B.3) can be recast as

$$\frac{\mathrm{d}f}{\mathrm{d}\bar{z}} = 0 \tag{B.4}$$

where $\bar{z} = x - \mathrm{i}y$.

Corollary: A function $f(z{=}x{+}\mathrm{i}y)$ is holomorphic on a domain if and only if its real and imaginary parts are *conjugate harmonic functions*, i.e. they obey the Cauchy–Riemann equations (B.3) and the Laplace equations

$$\triangle P(x,y) = 0 \quad , \quad \triangle Q(x,y) = 0 \tag{B.5}$$

on the domain.

B.1.2 b Integration of holomorphic functions

$$\int_{\mathscr{C}} f(z)\,\mathrm{d}z = \int_a^b f\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t \tag{B.6}$$

Cauchy's integral theorem

$$\oint_{\mathscr{C}} f(z)\,\mathrm{d}z = 0. \tag{B.7}$$

Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi\mathrm{i}} \oint_{\mathscr{C}} \frac{f(z)}{z - z_0}\,\mathrm{d}z. \tag{B.8}$$

B.2 Multivalued functions

B.3 Series expansions

B.3.1 Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{B.9}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi\mathrm{i}} \oint_{\mathscr{C}} \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z, \tag{B.10}$$

which generalizes the Cauchy integral formula (B.8) to the successive derivatives of f.

zeroes

B.3.2 Isolated singularities and Laurent series

B.3.2 a Definitions

isolated singularity

removable singularity

pole of order m

essential singularity

B.3.2 b Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{B.11}$$

with

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint_{\mathscr{C}} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z \tag{B.12}$$

where $\mathscr{C}$ denotes

a_{-1} residue

B.3.3 Singular points

B.4 Conformal maps

Function $Z = f(z)$ defines mapping from plane of complex variables $z = x + \mathrm{i}y$ to plane of complex Z. Such a function is said to be a *conformal map* if it preserves angles locally.

If a function $Z = f(z)$ is holomorphic at z_0 and such that $f'(z_0) \neq 0$, it is invertible in a neighborhood of z_0, and f and its inverse F define a conformal mapping between the planes z and Z.

Proof: $\mathrm{d}Z = |f'(z_0)| \, \mathrm{e}^{\mathrm{i} \arg f'(z_0)} \, \mathrm{d}z$.

Singular point: $f'(z_0) = 0$: if zero of n, angles are multiplied by $n + 1$ in transformation $z \to Z$.

Bibliography for Appendix B

- Cartan [58], Chapters II, III & VI;
- Whittaker & Watson [59], Chapters 4.6, 5.1–5.2 & 5.6–5.7.

APPENDIX C

Dimensional analysis

The purpose of this appendix is to introduce the so-called π-*theorem*, according to which a relation of the form

$$\mathcal{G} = f(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n) \tag{C.1}$$

between the mathematical representations $\mathcal{G}$, $\mathcal{G}_1$, $\mathcal{G}_2$, …, $\mathcal{G}_n$ of physical quantities can generically be simplified.

C.1 Buckingham π-theorem

C.1.1 Physical dimensions and dimensionally independent quantities

Before stating the theorem, we introduce a few definitions and notions.

As the reader most probably knows, physical quantities implicitly carry a (physical) *dimension*, which in particular dictates which units can be used to measure them. Given a quantity $\mathcal{G}$, the associated dimension is generally denoted $[\mathcal{G}]$.[(73)]

These physical dimensions can all be expressed as products of monomials of a handful of *base quantities*, namely length (symbol: L), mass (M), time (T), electric current (I), (thermodynamic) temperature (Θ), amount of substance (N), and luminous intensity (J). In fluid dynamics—and as long as one only considers uncharged fluids—, only the $q = 4$ quantities L, M, T and Θ are relevant, which is what we shall from now on assume.[(74)] For an arbitrary physical quantity $\mathcal{G}$, one may thus write

$$[\mathcal{G}] = \mathsf{L}^\alpha \, \mathsf{M}^\beta \, \mathsf{T}^\gamma \, \Theta^\delta \tag{C.2}$$

with rational coefficients α, β, γ, δ. In addition, quantities without physical dimension—such as pure numbers—are assigned the dimension $\mathsf{1}$.

A set of physical quantities $\mathcal{G}_1$, $\mathcal{G}_2$, …, $\mathcal{G}_n$ are said to be *dimensionally independent* when the product $[\mathcal{G}_1]^{\lambda_1}[\mathcal{G}_2]^{\lambda_2}\cdots[\mathcal{G}_n]^{\lambda_n}$ is dimensionless if and only if every exponent λ_i vanishes:

$$[\mathcal{G}_1]^{\lambda_1}[\mathcal{G}_2]^{\lambda_2}\cdots[\mathcal{G}_n]^{\lambda_n} = \mathsf{1} \quad \Leftrightarrow \quad \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0. \tag{C.3}$$

(73) Note that we refer to $\mathcal{G}$ as the "physical quantity", while it should rather be called "mathematical representation of the physical quantity".

(74) The reader will be able to generalize the argumentation to $q \neq 4$ by herself/himself.

By definition, the base quantities are dimensionally independent of each other.

Introducing the dimensional exponents α_i, β_i, γ_i, δ_i for each quantity $\mathcal{G}_i$:

$$[\mathcal{G}_i] = \mathsf{L}^{\alpha_i}\,\mathsf{M}^{\beta_i}\,\mathsf{T}^{\gamma_i}\,\Theta^{\delta_i}, \tag{C.4}$$

one defines the *dimension matrix* associated with the set $\{\mathcal{G}_i\}_{i=1,\dots,n}$ as the $q \times n$ matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & \beta_2 & \cdots & \beta_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \delta_1 & \delta_2 & \cdots & \delta_n \end{pmatrix}. \tag{C.5}$$

One easily finds that n physical quantities $\mathcal{G}_1$, $\mathcal{G}_2$, …, $\mathcal{G}_n$ are dimensionally independent if and only if the rank r of the corresponding dimension matrix equals n—which obviously implies $n \leq q$.

Proof: The logarithm of the relation $[\mathcal{G}_1]^{\lambda_1}[\mathcal{G}_2]^{\lambda_2}\cdots[\mathcal{G}_n]^{\lambda_n} = \mathsf{1}$ reads, at least symbolically[(75)]

$$\lambda_1 \ln[\mathcal{G}_1] + \lambda_2 \ln[\mathcal{G}_2] + \cdots + \lambda_n \ln[\mathcal{G}_n] = 0.$$

Replacing each physical dimension $[\mathcal{G}_i]$ by its expression (C.4) in terms of the basis quantities, one obtains the linear system

$$\begin{cases} \alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n = 0 \\ \beta_1\lambda_1 + \cdots + \beta_n\lambda_n = 0 \\ \gamma_1\lambda_1 + \cdots + \gamma_n\lambda_n = 0 \\ \delta_1\lambda_1 + \cdots + \delta_n\lambda_n = 0, \end{cases} \tag{C.6}$$

which can be recast in a matrix form involving the dimension matrix (C.5). Using basic results from linear algebra, the vector space spanned by the vectors $(\lambda_1, \dots, \lambda_n)^\mathsf{T}$ that satisfy the system is of dimension $n - r$, i.e. that space reduces to the zero vector iff $n = r$. □

If the rank r of the dimension matrix (C.5) is smaller than n, then the physical quantities $\mathcal{G}_1$, $\mathcal{G}_2$, …, $\mathcal{G}_n$ are not dimensionally independent, i.e. some of them can be expressed ("derived") in terms of the others.

Indeed, in that case, the linear system (C.6) is underdetermined. Given r independent coefficients among the $\{\lambda_i\}_{i=1,\dots,n}$, which up to relabeling can be chosen to be λ_1, …, λ_r, then the $n-r$ other coefficients λ_{r+1}, …, λ_n are linear combinations of the independent ones. Coming back to the physical quantities, $\{\mathcal{G}_1, \dots, \mathcal{G}_r\}$ form a "complete" set of dimensionally independent quantities, in terms of which the dimension of every quantity $\mathcal{G}_k$ with $k \in \{r+1, \dots, n\}$ can be expressed: there exist (rational) coefficients $a_{k,1}$, … $a_{k,r}$ such that

$$[\mathcal{G}_k] = [\mathcal{G}_1]^{a_{k,1}} \cdots [\mathcal{G}_r]^{a_{k,r}} \quad \text{for } k \in \{r+1, \dots, n\}. \tag{C.7}$$

Stated differently, the coefficients are such that the ratio

$$\pi_{k-r} \equiv \frac{\mathcal{G}_k}{\mathcal{G}_1^{a_{k,1}} \cdots \mathcal{G}_r^{a_{k,r}}} \quad \text{for } k \in \{r+1, \dots, n\} \tag{C.8}$$

is dimensionless: $[\pi_k] = \mathsf{1}$.

C.1.2 π-theorem

Let us come back to relation (C.1), with f some function. We assume that the n quantities $\mathcal{G}_1$, $\mathcal{G}_2$, …, $\mathcal{G}_n$ are *physically independent*, i.e. that the values they take can a priori be varied independently from each other. Denoting by r the rank of the dimension matrix associated with the set $\{\mathcal{G}_i\}_{i=1,\dots,n}$, we further assume that the first r quantities $\mathcal{G}_1$, …, $\mathcal{G}_r$ are *dimensionally* independent, while the dimensions of the remaining ones can be expressed by Eq. (C.7).

If relation (C.1) is physically meaningful, i.e. if it holds irrespective of the values (in a given system of units) of $\mathcal{G}$ and the quantities $\mathcal{G}_1$, …, $\mathcal{G}_n$, then necessarily $\mathcal{G}$ is not dimensionally independent

[(75)] … because taking the logarithm of a dimensionful quantity should upset you.

from the $\{\mathcal{G}_i\}_{i=1,\dots,n}$. One can thus find rational exponents $a_1, \dots, a_k$ such that

$$[\mathcal{G}] = [\mathcal{G}_1]^{a_1} \cdots [\mathcal{G}_r]^{a_r} \tag{C.9}$$

and accordingly define a dimensionless ratio

$$\pi \equiv \frac{\mathcal{G}}{\mathcal{G}_1^{a_1} \cdots \mathcal{G}_r^{a_r}}. \tag{C.10}$$

Relation (C.1) can then be rewritten as

$$\pi = \mathcal{G}_1^{a_1} \cdots \mathcal{G}_r^{a_r} f\big(\mathcal{G}_1, \dots, \mathcal{G}_r, \mathcal{G}_1^{a_{r+1,1}} \cdots \mathcal{G}_r^{a_{r+1,r}} \pi_1, \dots, \mathcal{G}_1^{a_{n,1}} \cdots \mathcal{G}_r^{a_{n,r}} \pi_{n-r}\big)$$

i.e., introducing an appropriate function F, as

$$\pi = F(\mathcal{G}_1, \dots, \mathcal{G}_r, \pi_1, \dots, \pi_{n-r}).$$

Now, since π is dimensionless, it is a number, whose value cannot depend on the system of units used to measure the dimensionful physical quantities $\mathcal{G}_1, \dots, \mathcal{G}_r$. Accordingly, the function F can actually not depend on its first r arguments, and one may replace it by a function f^* of the last $n-r$ arguments only and write

$$\pi = \mathsf{f}^*(\pi_1, \dots, \pi_{n-r}). \tag{C.11}$$

We have thus derived the *Buckingham π theorem*:

Any physically meaningful relation between $n+1$ dimensionful physical quantities $\mathcal{G}$, $\mathcal{G}_1, \dots, \mathcal{G}_n$ of the form $\mathcal{G} = f(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n)$ can be reduced to a relation $\pi = \mathsf{f}^*(\pi_1, \dots, \pi_{n-r})$ between $n+1-r$ dimensionless quantities, where r is the rank of the dimension matrix associated with the physical quantities $\mathcal{G}_1, \dots, \mathcal{G}_n$.

Remarks:

∗ The dimensionless quantities π, π_k are sometimes referred to as *Pi groups* or (less obscure, but more seldom) *similitude parameters*.

∗ The π theorem gives no information on the functional form of f^*.

C.2 Examples of application

Let us give a few examples of application of the π theorem in fluid dynamics.

C.2.1 Velocity of sea surface waves

We begin with examples corresponding to the physical situation discussed in Sec. VI.3. We consider a perfect fluid — characterized by its mass density ρ, dimension $\mathsf{M}\,\mathsf{L}^{-3}$ — in a constant gravitational field with acceleration g (dimension $\mathsf{L}\,\mathsf{T}^{-2}$). In the following, we want to find of the velocity c_w of waves — either phase or group velocity, it does not matter — depends on ρ, g, the wave number k ($[k] = \mathsf{L}^{-1}$) and other possibly relevant dimensionful parameters of the problem.

C.2.1 a Gravity-induced surface waves on an infinitely deep ocean

Since the fluid is assumed to be infinitely deep (and wide), there is no associated characteristic length scale. Accordingly, the velocity of the gravity-induced waves may be expected to depend on k, ρ, g only, i.e. there exists a function f of the $n = 3$ quantities such that

$$c_w = f(k, g, \rho)$$

which plays the role of relation (C.1). Note that the three quantities k, ρ, g are clearly dimensionally independent, so that the corresponding dimension matrix has rank $r = 3$.

Using k, ρ, g, the only combination with the dimension $\mathsf{L}\,\mathsf{T}^{-1}$ of a velocity is $\sqrt{g/k}$. The dimensionless "similitude parameter" associated with c_w [Eq. (C.10)] is thus $\pi = c_w/\sqrt{g/k}$.

Turning now to Eq. (C.11), we find that since here $n = r$ the relation formally reads

$$\frac{c_w}{\sqrt{g/k}} = \mathsf{f}^*() = \text{constant}, \tag{C.12a}$$

where $\mathsf{f}^*()$ designates a "function without argument". Reorganizing Eq. (C.12a) yields

$$c_w \propto \sqrt{\frac{g}{k}} \tag{C.12b}$$

which is indeed correct, see Eq. (VI.42) (the purely numerical proportionality factor is 1 for the phase velocity, $\frac{1}{2}$ for the group velocity).

Remark: The mass density ρ plays no role here, nor in the following example. This could have been anticipated: in both cases there is no other physical quantity involved in the problem with a non-zero dimensional exponent for M, making it impossible to construct a dimensionless quantity involving ρ. This ultimately reflects the fact that the waves under consideration are induced by gravity only, so that the resulting acceleration of a fluid element is independent of its mass.

C.2.1 b Gravity-induced surface waves on an ocean with finite depth

Let us now consider the case of gravity induced waves on the surface of an ocean with depth h_0, which thus adds up to the list of physical quantities on which a wave velocity can depend, and we may write

$$c_w = f(k, g, \rho, h_0).$$

Since only 3 of the $n = 4$ quantities are dimensionally independent — say k, g and ρ again —, the similitude parameter $c_w/\sqrt{g/k}$ depends on $n - r = 1$ dimensionless parameter. The latter should involve the remaining dimensionful quantity h_0, and is quickly found to be kh_0, leading to

$$\frac{c_w}{\sqrt{g/k}} = \mathsf{f}^*(kh_0). \tag{C.13a}$$

This leads to

$$c_w \propto \sqrt{\frac{g}{k}}\, \mathsf{f}^*(kh_0). \tag{C.13b}$$

One can check starting from Eq. (VI.41) that this is again correct — with $\mathsf{f}^* = \tanh$ for the phase velocity and a slightly more complicated form for the group velocity.

C.2.1 c Capillary waves on an infinitely deep ocean

Let us now discuss waves that are driven not by gravity, but by the surface tension σ (dimension $\mathsf{M}\,\mathsf{T}^{-2}$) at the water/air interface: σ represents the energy per unit surface[(76)] which is necessary to increase the size of the interface between the two fluids. When a flat ocean surface is deformed (by the action of wind), surface tension will tend to drive it back to flatness, to minimize the size of the interface, leading to so-called capillary waves.

In the case of an infinitely deep and wide ocean, the velocity of the capillary waves can depend on k, g, ρ — which will form our "basis" of dimensionally independent quantities —, and σ:

$$c_w = f(k, g, \rho, \sigma).$$

Using the basic quantities k, g, ρ, the unique combination with dimension $\mathsf{M}\,\mathsf{T}^{-2}$ is $\rho g/k^2$, so that relation (C.11) here reads

$$\frac{c_w}{\sqrt{g/k}} = \mathsf{f}^*\left(\frac{\sigma k^2}{\rho g}\right), \tag{C.14a}$$

[(76)] Energy has dimension $[E] = \mathsf{M}\,\mathsf{L}^2\,\mathsf{T}^{-2}$, surface L^2, leading at once to the dimension of σ.

or equivalently

$$c_w \propto \sqrt{\frac{g}{k}}\, \mathsf{f}^*\left(\frac{\sigma k^2}{\rho g}\right). \tag{C.14b}$$

If gravity is to play no role in the waves,[(77)] i.e. in the case of pure capillary waves, then g should drop out of the relation. This is only possible if f^* is (proportional to) the square root of its argument, leading to

$$c_w \propto \sqrt{\frac{\sigma k}{\rho}}.$$

Indeed, the dispersion relation for pure capillary waves under the conditions considered here is $\omega^2 = \sigma k^3/\rho$, leading to the above form for the phase and group velocities.

Remarks:

∗ The relative influence of gravity and surface tension on surface waves with wave vector k is quantified by a dimensionless number, the so-called Bond[(bi)] or Eötvös number[(bj)] Eo

$$\mathrm{Bo} = \mathrm{Eo} \equiv \frac{\rho g}{\sigma k^2}. \tag{C.15}$$

As could be expected, the argument of of the dimensionless function f^* is precisely (the inverse of) this number.

∗ Since σ is related to the interface between the water and the air, we should have considered not only the mass density ρ of the water, but also that of the air in the reasoning. Here we implicitly assumed that ρ_{air} is negligible, which reflects $\rho_{\mathrm{air}} \ll \rho_{\mathrm{water}}$. For the capillary waves at the boundary between two immiscible fluids — like water and oil — with similar mass densities, both of them should play a role in the velocity.

C.2.2 Expansion velocity of a shock front

To be completed...

[(77)] ... apart from ensuring the flatness of the ocean in the absence of waves.

[(bi)] W. N. Bond, 1897–1937 [(bj)] L. Eötvös, 1848–1919

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