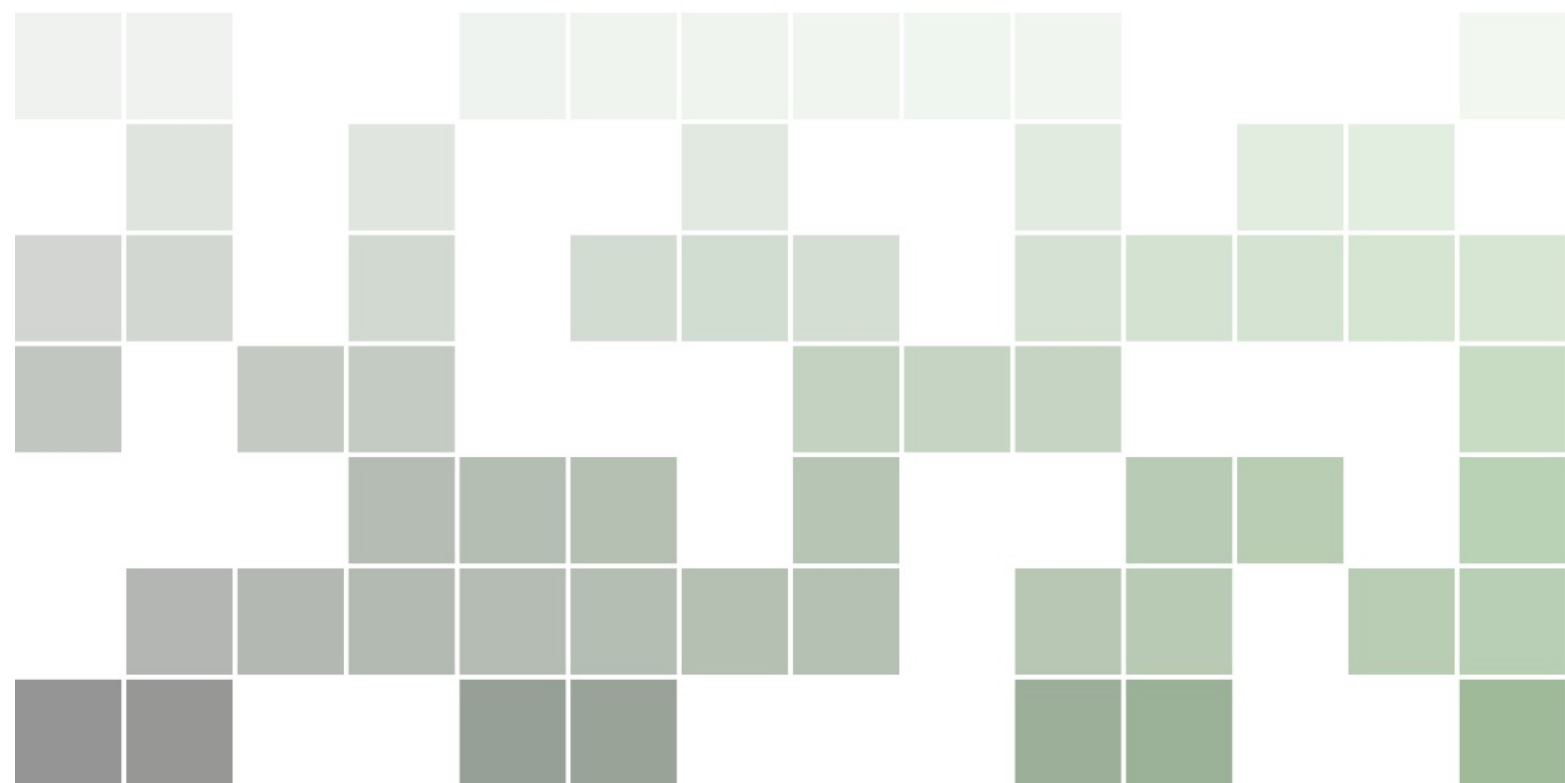# Elements of Hydrodynamics

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Foreword

The following pages were originally not designed to fall under your eyes. They grew up from handwritten notes for myself, listing the important points which I should not forget in the lecture room. As time went by, more and more remarks or developments were added, which is why I started to replace the growingly dirty sheets of paper by an electronic version—that could then also be easily uploaded on the web page of my lecture, for the benefit(?) of the students.

Again, additional results, calculations, comments, paragraphs or even whole chapters accumulated, leading to the temporary outcome which you are reading now: a not necessarily optimal overall outline; at times, unfinished sentences; not fully detailed proofs or calculations—because the missing steps are obvious to me—; insufficient discussions of the physics of some results—which I hopefully provide in the classroom—; not-so-good-looking figures; incomplete bibliography; etc. You may also expect a few solecisms, inconsistent notations, and the usual, unavoidable typos.[*] Eventually, you will have to cope with the many idiosyncrasies in my writing, as for instance my immoderate use of footnotes, dashes or parentheses, quotation marks, which are not considered as "good practice".

In short: the following chapters may barely be called "lecture notes"; they cannot replace a textbook[†] and the active participation in a course and in the corresponding tutorial/exercise sessions.

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[*]Comments and corrections are welcome!
[†]... which is one of several good reasons why you should think at least twice before printing a hard copy!

Contents

Introduction

General introduction and outline.
Notations, conventions, etc.

General references

(in alphabetical order)

- Faber, *Fluid dynamics for physicists* [1]

- Guyon, Hulin, Petit & Mitescu, *Physical hydrodynamics* [2]

- Greiner & Stock, *Hydrodynamik* [3]

- Landau & Lifshitz, *Course of theoretical physics. Vol. 6: Fluid mechanics* [4]
  = Landau & Lifschitz, *Lehrbuch der theoretischen Physik. Band VI: Hydrodynamik* [5]

- Rieutord, *Fluid dynamics — An introduction* [6]

- Sommerfeld, *Lectures on theoretical physics. Vol. II: Mechanics of deformable bodies* [7]
  = *Vorlesungen über theoretische Physik. Band II: Mechanik der deformierbaren Medien* [8]

CHAPTER I

Basic notions on continuous media

A system of many microscopic degrees of freedom is often more conveniently described as a material body that fills some region of space continuously, rather than as a collection of discrete point particles (Sec. I.1). This theoretical approach, which is especially suited to represent systems whose internal deformations are relevant, is an instance of physical *modeling*, originally motivated by the agreement of its predictions with experimental observations. Like every model, that of a continuous medium is valid only in some range of physical conditions, in particular on macroscopic scales.

Mathematically, a classical continuous medium at a given instant is described as a volume—or more generally a manifold—in usual Euclidean space. The infinitesimal elements of this volume constitute the elementary "material points", which are entirely characterized by their position.

To describe the time evolution of the physical system modeled as a continuous medium, two equivalent approaches are available. The first one consists in following the trajectories of the material points as time passes by (Sec. I.2). The physical picture of continuousness is then enforced by requesting that the mapping between the position of a given point at some reference initial time and its position at any later instant is continuous.

The second point of view, which will mostly be adopted in the remainder of these notes, focuses on the change in the various physical quantities at a fixed position as time elapses (Sec. I.3). The reference for the medium evolution between successive instants t and $t + \mathrm{d}t$ is the "current" configuration of the material points, i.e. at time t, instead of their positions in the (far) past. In that description, the spatial variables are no longer dynamical, but only labels for the position at which some observable is considered. Accordingly, the dynamical quantities in the system are now time-dependent fields; the desired continuousness of the medium translates into continuity conditions on those fields.

Eventually, the mathematical object that models internal forces in a continuous medium, i.e. the influence from neighboring material points on each other, is shortly introduced (Sec. I.4). This allows the classification of deformable continuous media into two traditional large classes, and in particular the definition of fluids.

I.1 Continuous medium: a model for many-body systems

In this Section, we first spell out a few arguments which lead to the introduction of the model of a continuous medium (§ I.1.1). The basic ingredients of the mathematical implementation of the model are then presented and a few notions are defined (§ I.1.2). Eventually, the physical assumptions underlying the modeling are reexamined in greater detail, and some more or less obvious limitations of the continuous description are indicated (§ I.1.3).

I.1.1 Basic ideas and concepts

The actual structure of matter at the microscopic scale is discrete, and involves finite "elementary" entities: electrons, atoms, ions, molecules, ..., which in the remainder of these notes will be collectively referred to as "atoms". Any macroscopic sample of matter contains a large amount of these atoms. For instance, the number density in an ideal gas under normal conditions is about $2.7 \times 10^{25} \text{ m}^{-3}$, so that one cubic millimeter still contains 2.7×10^{16} atoms. Similarly, even though the number density in the interstellar medium might be as low as 10^2 m^{-3}, any volume relevant for astrophysics, i.e. with at least a kilometer-long linear size, involves a large number of atoms.

Additionally, these atoms are in constant chaotic motion, with individual velocities of order 10^2–$10^3 \text{ m} \cdot \text{s}^{-1}$ for a system at thermal equilibrium at temperature $T \simeq 300$ K. Given a mean free path[(i)] of order 10^{-7} m in a gas under normal conditions, each atom undergoes 10^9–10^{10} times per second, i.e. its trajectory changes direction constantly.

As in Statistical Mechanics, it is in general unnecessary to know the details of the motion of each atom in a macroscopic system: as a matter of fact, there emerge global characteristics, which can be predicted to a high degree of accuracy thanks to the large number of degrees of freedom involved in their definition (mostly as averages), despite the chaoticity of the individual atomic behaviors. The macroscopic properties of systems at (global) thermodynamic equilibrium are thus entirely determined by a handful of collective variables, either extensive—like entropy, internal energy, volume, particle number, total momentum...—, or intensive—as e.g. the respective densities of the various extensive variables, or temperature, pressure, chemical potential, average velocity...—, where the latter take the same value throughout the system.

When thermodynamic equilibrium does not hold globally in a system, there is still the possibility that one may consider that it is valid locally, "at each point" in space. In that situation—whose underlying assumptions will be specified in greater detail in § I.1.3—the intensive thermodynamic variables characterizing the system macroscopically become *fields*, which can vary from point to point. More generally, experience shows that it is fruitful to describe a large amount of characteristics—not only thermodynamic, but also of mechanical nature, like forces and the displacements or deformations they induce—of macroscopic bodies as fields. A "continuous medium" is then intuitively a system described by such fields, which should satisfy some (mathematical) continuity property with respect to the spatial variables that parameterize the representation of the physical system as a geometrical quantity. This picture will be better specified in Secs. I.2 and I.3.

Assuming the relevance of the model of a medium whose properties are described by continuous fields is often referred to as *continuum hypothesis*[(ii)].

The reader should keep in mind that the modeling of a given macroscopic system as a continuous medium does not invalidate the existence of its underlying discrete atomic structure. Specific

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[(i)]*mittlere freie Weglänge*   [(ii)]*Kontinuumshypothese*

phenomena will still directly probe the latter, as e.g. X-ray scattering experiments for the gases or liquids of our everyday life. That is, the model has limitations to its validity, especially at small wavelengths or high frequencies, where "small" or "high" implies a comparison to some microscopic physical scale characteristic of the system under consideration. Turning the reasoning the other way around, the continuous-medium picture is often referred to as a long-wavelength, low-frequency approximation to a more microscopic description—from which it can actually be shown to emerge in the corresponding limits.

> It is important to realize that the model itself is blind to its own limitations, i.e. there is no a priori criterion *within* the mathematical continuous-medium description that signals the breakdown of the relevance of the picture to actual physics. In practice, there might be hints that the equations of the continuous model are being applied in a regime where they should not, as for instance if they yield negative values for a quantity which should be positive, but such occurrences are not the general rule.

Remarks:

∗ The model of a continuous model is not only applicable—and applied—to obvious cases like gases, liquids or (deformable) solids, it may also be used to describe the behaviors of large crowds, fish schools, car traffic. . . provided the number of "elementary" constituents is large and the system is studied on a large enough scale.

∗ Even if the continuous description is valid on "long wavelengths", it remains obvious that any physical system, viewed on a scale much larger than its spatial extent, is to first approximation best described as pointlike.
Consider for instance a molecular cloud of interstellar medium with a 10 parsec radius and about 10^{10} H_2 molecules per cubic meter. For a star forming at its core, it behaves a continuous medium; 1 kpc away, however, the inner degrees of the cloud are most likely already irrelevant and it is best described as a mere point.

I.1.2 General mathematical framework

Consider a non-relativistic classical macroscopic physical system Σ, described by Newtonian physics. The positions of its individual atoms, viewed as pointlike, at a given instant t—which is the same for all observers—are points in a three-dimensional Euclidean space $\mathscr{E}_3$.

In the description as a continuous medium, the system Σ is represented by a geometrical manifold in $\mathscr{E}_3$, which for the sake of simplicity will be referred to as a "volume" and denoted by $\mathcal{V}$. The basic constituents of $\mathcal{V}$ are its infinitesimal elements $d^3\mathcal{V}$, called *material points*[(iii)] or *continuous medium particles*[(iv)]—which explains a posteriori our designating the discrete constituents of matter as "atoms"—, or, in the specific case of the elementary subdivisions of a fluid, *fluid particles*[(v)]. As we shall state more explicitly in § I.1.3, these infinitesimal elements are assumed to have the same physical properties as a finite macroscopic piece.

Associated with the physical picture attached to the notion of continuousness is the requirement that neighboring material points in the medium remain close to each other throughout the system evolution. We shall see below how this picture is implemented in the mathematical description.

Remark: The volume $\mathcal{V}$—with the topology inherited from $\mathscr{E}_3$—need not be simply connected. For instance, one may want to describe the flow of a river around a bridge pier: the latter represents a physical region which water cannot penetrate, which is modeled as a hole throughout the volume $\mathcal{V}$ occupied by fluid particles.

To characterize the position of a given material point, as well as some of the observables relative to the physical system Σ, one still needs to specify the *reference frame* in which the system is

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[(iii)] *Materielle Punkte*   [(iv)] *Mediumteilchen*   [(v)] *Fluidteilchen*

studied, corresponding to the point of view of a given observer, and to choose a *coordinate system* in that reference frame. This choice allows one to define vectors—like position vectors, velocities, or forces—and tensors.

The basis vectors of the coordinate system will generically be designated as $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, while the components of a given vector will be denoted with upper ("contravariant") indices, as e.g. $\vec{c} = c^i\,\vec{e}_i$, where the summation convention over repeated upper and lower indices was used.

Once the reference frame and coordinate system are determined, the macroscopic state of the physical system at time t is mapped onto a corresponding *configuration κ_t* of the medium, consisting of the continuous set of the position vectors $\vec{r} = x^i\,\vec{e}_i$ of its constituting material points. Since the volume occupied by the latter may also depends on time, it will also be labeled by t: $\mathcal{V}_t$.

To be able to formalize the necessary continuity conditions in the following Sections, one also introduces a reference time t_0—conveniently taken as the origin of the time axis, $t_0 = 0$—and the corresponding *reference configuration κ_0* of the medium, which occupies a volume $\mathcal{V}_0$. The generic position vector of a material point in this reference configuration will be denoted as $\vec{R} = X^i\,\vec{e}_i$.

Remark: In so-called "classical" continuous media, as have been introduced here, the material points are entirely characterized by their position vector. In particular, they have no intrinsic angular momentum.

### I.1.3 Local thermodynamic equilibrium

In a more bottom-up approach to the modeling of a system Σ of discrete constituents as a continuous medium, one should first divide Σ (in thought) into small cells of fixed—yet not necessarily universal—size fulfilling two conditions:

(i) each individual cell can meaningfully be treated as a thermodynamic system, i.e. it must be large enough that the relative fluctuations of the usual extensive thermodynamic quantities computed for the content of the cell are negligible;

(ii) the thermodynamic properties vary little over the cell scale, i.e. cells cannot be too large, so that (approximate) homogeneity is ensured.

The rationale behind these two requirements is illustrated by Fig. I.1, which represents schematically how the value of a local macroscopic quantity, e.g. a density, depends on the resolution of the apparatus with which it is measured, i.e. equivalently on the length scale on which it is defined. If the apparatus probes too small a length scale, so that the discrete degrees of freedom become relevant, the measured value strongly fluctuates from one observation to the next one, as hinted at by the displayed envelope of possible results of measurements: this is the issue addressed by condition (i). Simultaneously, a small change in the measurement resolution, even with the apparatus still centered on the same point in the system, can lead to a large variation in the measured value of the observable, corresponding to the erratic behavior of the curve at small scales shown in Fig. I.1. This fluctuating pattern decreases with increasing size of the observation scale, since this increase leads to a growth in the number of atoms inside the probed volume, and thus a drop in the size of relative fluctuations. At the other end of the curve, one reaches a regime where the low resolution of the observation leads to encompassing domains with enough atoms to be rid of fluctuations, yet with inhomogeneous macroscopic properties, in a single probed region—in violation of condition (ii). As a result, the measured value of the density under consideration slowly evolves with the observation scale.

In between these two domains of strong statistical fluctuations and slow macroscopic variations lies a regime where the value measured for an observable barely depends on the scale over which it is determined. This represents the appropriate regime for meaningfully defining—and measuring—a local density, and more general local quantities.

It is important to note that this intermediate "mesoscopic" interval may not always exist. There are physical systems in which strong macroscopic variations are already present in a range of scales
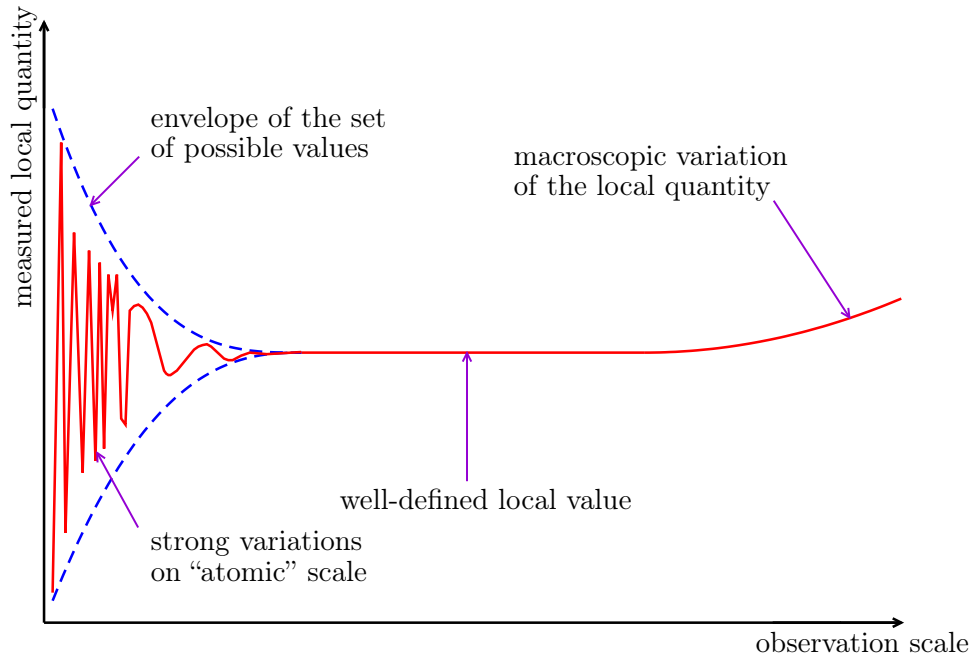
Figure I.1 – Typical variation of the measured value for a "local" macroscopic observable as a function of the size scale over which it is determined.

where microscopic fluctuations are still sizable. For such systems, one cannot find scale-independent local variables. That is, the proper definition of local quantities implicitly relies on the existence of a clear separation of scales in the physical system under consideration, which is what will be assumed in the remainder of these notes.

Remark: The smallest volume over which meaningful local quantities can be defined is sometimes called *representative volume element* (RVE), or *representative elementary volume*.

When conditions (i) and (ii) hold, one may in particular define *local thermodynamic variables*, corresponding to the values taken in each intermediate-size cell—labeled by its position $\vec{r}$—by the usual extensive parameters: internal energy, number of atoms... Since the separation between cells is immaterial, nothing prevents energy or matter from being transported from a cell to its neighbors, even if the global system is isolated. Accordingly, the local extensive variables in any given cell are actually time-dependent in the general case. In addition, it becomes important to add linear momentum—with respect to some reference frame—to the set of local extensive variables characterizing the content of a cell.

The size of each cell is physically irrelevant, as long as it satisfies the two key requirements; there is thus no meaningful local variable corresponding to volume. Similarly, the values of the extensive variables in a given cell, which are by definition proportional to the cell size, are as arbitrary as the latter. They are thus conveniently replaced by the respective local densities: internal energy density $e(t,\vec{r})$, number density $n(t,\vec{r})$, linear momentum density $\rho(t,\vec{r})\vec{v}(t,\vec{r})$, where ρ denotes the mass density, entropy density $s(t,\vec{r})$...

Remark: Rather than considering the densities of extensive quantities, some authors—in particular Landau & Lifshitz [4, 5]—prefer to work with *specific quantities*, i.e. their respective amounts per unit mass, instead of per unit volume. The relation between densities and specific quantities is trivial: denoting by χ_j resp. $\chi_{j,\mathrm{m}}$ a generic local density resp. specific amount for the same physical quantity, one has the identity

$$\chi_j(t,\vec{r}) = \rho(t,\vec{r})\,\chi_{j,\mathrm{m}}(t,\vec{r}) \tag{I.1}$$

in every cell—labeled by $\vec{r}$—and at every time t.

Once the local extensive variables have been meaningfully defined, one can develop the usual formalism of thermodynamics in each cell. In particular, one introduces the conjugate intensive variables, as e.g. local temperature $T(t,\vec{r})$ and pressure $\mathcal{P}(t,\vec{r})$. The underlying, important hypothesis is the assumption of a *local thermodynamic equilibrium*. According to the latter, the equation(s) of state of the system inside the small cell, expressed with local thermodynamic quantities, is the same as for a macroscopic system in the actual thermodynamic limit of infinitely large volume and particle number.

Consider for instance a non-relativistic classical ideal gas: its (mechanical) equation of state reads $\mathcal{P}\mathcal{V} = Nk_{\mathrm{B}}T$, with N the number of atoms, which occupy a volume $\mathcal{V}$ at uniform pressure $\mathcal{P}$ and temperature T, while k_{B} is the Boltzmann[a] constant. This is trivially recast as $\mathcal{P} = nk_{\mathrm{B}}T$, with n the number density of atoms. The local thermodynamic equilibrium assumption then states that under non-uniform conditions of temperature and pressure, the equation of state in a local cell at position $\vec{r}$ is given by

$$\mathcal{P}(t,\vec{r}) = n(t,\vec{r})k_{\mathrm{B}}T(t,\vec{r}) \tag{I.2}$$

at every time t.

The last step towards the continuous-medium model is to promote $\vec{r}$, which till now was simply the discrete label attached to a given cell, to be a continuous variable taking its values in $\mathbb{R}^3$— or rather, in the volume $\mathcal{V}_t$ attached to the system at the corresponding instant t. Accordingly, taking into account the time-dependence of physical quantities, the local variables, in particular the thermodynamic parameters, become *fields* on $\mathbb{R} \times \mathbb{R}^3$.

The replacement of the fine-resolution description, in which atoms are the relevant degrees of freedom, by the lower-resolution model which assimilates small finite volumes of the former to structureless points is called *coarse graining*[vi].

> This is a quite generic procedure in theoretical physics, whereby the finer degrees of freedom of a more fundamental description are smoothed away—technically, this is often done by performing averages or integrals, so that these degrees of freedom are "integrated out"—and replaced by novel, effective variables in a theory with a more limited range of applicability, but which is more tractable for "long-range" phenomena.

Coming back to condition (ii), we already stated that it implicitly involves the existence of at least one large length scale L, over which the macroscopic physical properties of the system may vary. This scale can be a characteristic dimension of the system under consideration, as e.g. the diameter of the tube in which a liquid is flowing. In the case of periodic waves propagating in the continuous medium, L also corresponds to their wavelength. More generally, if $\mathcal{G}$ denotes a macroscopic physical quantity, one may consider

$$L \cong \left[\frac{|\vec{\nabla}\mathcal{G}(t,\vec{r})|}{|\mathcal{G}(t,\vec{r})|} \right]^{-1}, \tag{I.3}$$

where $\vec{\nabla}$ denotes the (spatial) gradient.

Condition (i) in particular implies that the typical size of the cells which are later coarse grained should be significantly larger than the *mean free path* ℓ_{mfp} of atoms, so that thermodynamic equilibrium holds in the local cells. Since on the other hand this same typical size should be significantly smaller than the scale L of macroscopic variations, one deduces the condition

$$\boxed{\mathrm{Kn} \equiv \frac{\ell_{\mathrm{mfp}}}{L} \ll 1} \tag{I.4}$$

on the dimensionless *Knudsen number* Kn.[b]

[vi] *Vergröberung*

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[a]L. BOLTZMANN, 1844–1906   [b]M. KNUDSEN, 1871–1949

In air under normal conditions $\mathscr{P} = 10^5$ Pa and $T = 300$ K, the mean free path is $\ell_{\mathrm{mfp}} \approx 0.1$ μm. In the study of phenomena with variations on a characteristic scale $L \approx 10$ cm, one finds Kn $\approx 10^{-6}$, so that air can be meaningfully treated as a continuous medium.

> The opposite regime Kn > 1 is that of a *rarefied medium*, as for instance of the so-called Knudsen gas, in which the collisions between atoms are negligible—and in particular insufficient to ensure thermal equilibrium as an ideal gas. The flow of such systems is not well described by hydrodynamics, but necessitates alternative descriptions like molecular dynamics, in which the degrees of freedom are explicitly atoms.

I.2 Lagrangian description

The *Lagrangian*[(c)] perspective, which generalizes the approach usually adopted in the description of the motion of a (few) point particle(s), focuses on the trajectories of the material points, where the latter are labeled by their position in the reference configuration. Accordingly, physical quantities are expressed as functions of time t and initial position vectors $\vec{R}$, and any continuity condition has to be formulated with respect to these variables.

I.2.1 Lagrangian coordinates

Consider a material point M in a continuous medium. Given a reference frame $\mathcal{R}$, which allows the definition of its position vector at any time t, one can follow its *trajectory* $\vec{r}(t)$. With a choice of coordinate system, that trajectory is equivalently characterized by the functions $\{x^i(t)\}$ for $i = 1, 2, 3$.

Let $\vec{R}$ resp. $\{X^i\}$ denote the position resp. coordinates of the material point M at t_0. The trajectory obviously depends on this "initial" position, and $\vec{r}$ can thus be viewed as a function of t and $\vec{R}$, where the latter refers to the reference configuration κ_0:

$$\vec{r} = \vec{r}(t, \vec{R}) \tag{I.5a}$$

with the consistency condition

$$\vec{r}(t = t_0, \vec{R}) = \vec{R}. \tag{I.5b}$$

In the *Lagrangian description*, also referred to as *material description* or *particle description*, this point of view is generalized, and the various physical quantities $\mathcal{G}$ characterizing a continuous medium are viewed at *any* time as mathematical functions of the variables t and $\vec{R}$:

$$\mathcal{G} = \mathcal{G}(t, \vec{R}), \tag{I.6}$$

where the mapping $\mathcal{G}$—which as often in physics will be denoted with the same notation as the physical quantity represented by its value—is defined for every t on the initial volume $\mathcal{V}_0$ occupied by the reference configuration κ_0.

Together with the time t, the position vector $\vec{R}$—or equivalently its coordinates X^1, X^2, X^3 in a given system—are called *Lagrangian coordinates*.

I.2.2 Continuity assumptions

An important example of physical quantity, function of t and $\vec{R}$, is simply the (vector) position in the reference frame $\mathcal{R}$ of material points at time t, i.e. $\vec{r}$ or equivalently its coordinates $\{x^i\}$, as given by relation (I.5a), which thus relates the configurations κ_0 and κ_t.

[(c)] J.-L. LAGRANGE, 1736–1813

More precisely, $\vec{r}(t,\vec{R})$ maps for every t the initial volume $\mathcal{V}_0$ *onto* $\mathcal{V}_t$. To implement mathematically the physical picture of continuity, it will be assumed that the mapping $\vec{r}(t,\cdot):\ \mathcal{V}_0 \to \mathcal{V}_t$ is also one-to-one for every t—i.e. all in all bijective—, and that the function $\vec{r}$ and its inverse

$$\vec{R} = \vec{R}(t,\vec{r}) \tag{I.7}$$

are *continuous* with respect to both time and space variables. This requirement in particular ensures that neighboring points remain close to each other as time elapses. It also preserves the connectedness of volumes, (closed) surfaces or curves along the evolution: one may then define *material domains*, i.e. connected sets of material points which are transported together in the evolution of the continuous medium.

For the sake of simplicity, it will be assumed that the mapping $\vec{r}$ and its inverse, and more generally every mathematical function $\mathcal{G}$ representing a physical quantity, is at least twice continuously differentiable (i.e. of class $\mathscr{C}^2$). To be able to accommodate for important phenomena that are better modeled with discontinuities, like shock waves in fluids (Sec. **??**) or ruptures in solids—for instance, in the Earth's crust—, the $\mathscr{C}^2$-character of functions under consideration may hold only piecewise.

I.2.3 Velocity and acceleration of a material point

As mentioned above, for a fixed reference position $\vec{R}$ the function $t \mapsto \vec{r}(t,\vec{R})$ is the trajectory of the material point which passes through $\vec{R}$ at the reference time t_0. As a consequence, the velocity at time t of this material point, measured in the reference frame $\mathcal{R}$, is simply

$$\vec{v}(t,\vec{R}) = \frac{\partial \vec{r}(t,\vec{R})}{\partial t}. \tag{I.8}$$

Since the variable $\vec{R}$ is independent of t, one could actually also write $\vec{v}(t,\vec{R}) = \mathrm{d}\vec{r}(t,\vec{R})/\mathrm{d}t$. In turn, the acceleration of the material point in $\mathcal{R}$ is given at time t by

$$\vec{a}(t,\vec{R}) = \frac{\partial \vec{v}(t,\vec{R})}{\partial t}. \tag{I.9}$$

Remark: The trajectory (or *pathline*[(vii)]) of a material point can be visualized, by tagging the point at its position $\vec{R}$ at time t_0, for instance with a fluorescent or radioactive marker, and then imaging the positions at later times $t > t_0$.

On the other hand, if one regularly—say for every instant $t_0 \le t' \le t$—injects some marker at a fixed geometrical point P, the resulting tagged curve at time t is the locus of the geometrical points occupied by medium particles which passed through P in the past. This locus is referred to as *streakline*.[(viii)] Denoting by $\vec{r}_P$ the position vector of point P, the streakline is the set of geometrical points with position vectors

$$\vec{r} = \vec{r}\big(t,\vec{R}(t',\vec{r}_P)\big) \quad \text{for } t_0 \le t' \le t. \tag{I.10}$$

I.3 Eulerian description

The Lagrangian approach introduced in the previous Section is actually not commonly used in fluid dynamics, at least not in its original form, except for specific problems.

> One reason is that physical quantities at a given time are expressed in terms of a reference configuration in the (far) past: a small uncertainty on this initial condition may actually yield

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[(vii)] *Bahnlinie*   [(viii)] *Streichlinie*

after a finite duration a large uncertainty on the present state of the system, which is problematic. On the other hand, this line of argument explains why the Lagrangian point of view is adopted to investigate *chaos* in many-body systems!

The more usual description is the so-called *Eulerian*[(d)] perspective, in which the evolution between instants t and $t + dt$ takes the system configuration at time t as a reference.

I.3.1 Eulerian coordinates. Velocity field

In contrast to the "material" Lagrangian point of view, which identifies the medium particles in a reference configuration and follows them in their motion, in the Eulerian description the emphasis is placed on the *geometrical* points. Thus, the *Eulerian coordinates* are time t and a spatial vector $\vec{r}$, where the latter does not label the position of a material point, but rather that of a geometrical point. Accordingly, the physical quantities in the Eulerian specification are described by *fields* on space-time.

Thus, the fundamental field that entirely characterizes the motion of a continuous medium in a given reference frame $\mathcal{R}$ is the *velocity field* $\vec{\mathsf{v}}_t(t, \vec{r})$. The latter is defined such that it gives the value of the Lagrangian velocity $\vec{v}$ [Eq. (I.8)] of a material point passing through $\vec{r}$ at time t:

$$\vec{v} = \vec{\mathsf{v}}_t(t, \vec{r}) \quad \forall t, \ \forall \vec{r} \in \mathcal{V}_t. \tag{I.11}$$

More generally, the value taken at given time and position by a physical quantity $\mathcal{G}$, whether attached to a material point or not, is expressed as a mathematical function $\mathcal{G}_t$ of the same Eulerian variables:

$$\mathcal{G} = \mathcal{G}_t(t, \vec{r}) \quad \forall t, \ \forall \vec{r} \in \mathcal{V}_t. \tag{I.12}$$

Note that the mappings $(t, \vec{R}) \mapsto \mathcal{G}(t, \vec{R})$ in the Lagrangian approach and $(t, \vec{r}) \mapsto \mathcal{G}_t(t, \vec{r})$ in the Eulerian description are in general different. For instance, the domains in $\mathbb{R}^3$ over which their spatial variables take their values differ: constant ($\mathcal{V}_0$) in the Lagrangian specification, time-dependent ($\mathcal{V}_t$) in the case of the Eulerian quantities. Accordingly the latter will be denoted with a subscript t in the next subsection.

I.3.2 Equivalence between the Eulerian and Lagrangian viewpoints

Despite the different choices of variables, the Lagrangian and Eulerian descriptions are fully equivalent. Accordingly, the prevalence in practice of the one over the other is more a technical issue than a conceptual one.

Thus, it is rather clear that the knowledge of the Lagrangian specification can be used to obtain the Eulerian formulation at once, using the mapping $\vec{r} \mapsto \vec{R}(t, \vec{r})$ between present and reference positions of a material point. For instance, the Eulerian velocity field can be expressed as

$$\vec{\mathsf{v}}_t(t, \vec{r}) = \vec{v}\big(t, \vec{R}(t, \vec{r})\big). \tag{I.13a}$$

This identity in particular shows that $\vec{\mathsf{v}}_t$ automatically inherits the smoothness properties of $\vec{v}$: if the mapping $(t, \vec{R}) \mapsto \vec{r}(t, \vec{R})$ and its inverse are piecewise $\mathscr{C}^2$ (cf. § I.2.2), then $\vec{\mathsf{v}}_t$ is (at least) piecewise $\mathscr{C}^1$ in both its variables.

For a generic physical quantity, the transition from the Lagrangian to the Eulerian point of view similarly reads

$$\mathcal{G}_t(t, \vec{r}) = \mathcal{G}\big(t, \vec{R}(t, \vec{r})\big). \tag{I.13b}$$

Reciprocally, given a (well-enough behaved) Eulerian velocity field $\vec{\mathsf{v}}_t$ on a continuous medium, one can uniquely obtain the Lagrangian description of the medium motion by solving the *initial*

[(d)]L. Euler, 1707–1783

value problem

$$\begin{cases} \dfrac{\partial \vec{r}(t,\vec{R})}{\partial t} = \vec{\mathrm{v}}_t\big(t,\vec{r}(t,\vec{R})\big) \\[2mm] \vec{r}(t_0,\vec{R}) \ = \vec{R}, \end{cases} \tag{I.14a}$$

where the second line represents the initial condition. That is, one actually reconstructs the pathline of every material point of the continuous medium. Introducing differential notations, the above system can also be rewritten as

$$\mathrm{d}\vec{r} = \vec{\mathrm{v}}_t(t,\vec{r})\,\mathrm{d}t \quad \text{with} \quad \vec{r}(t_0,\vec{R}) = \vec{R}. \tag{I.14b}$$

Once the pathlines $\vec{r}(t,\vec{R})$ are known, one obtains the Lagrangian function $\mathcal{G}(t,\vec{R})$ for a given physical quantity $\mathcal{G}$ by writing

$$\mathcal{G}(t,\vec{R}) = \mathcal{G}_t\big(t,\vec{r}(t,\vec{R})\big). \tag{I.14c}$$

Since both Lagrangian and Eulerian descriptions are equivalent, we shall from now on drop the subscript t on the mathematical functions representing physical quantities in the Eulerian point of view.

I.3.3 Streamlines

At a given time t, the *streamlines*[(ix)] of the motion are defined as the field lines of $\vec{\mathrm{v}}$. That is, these are curves whose tangent is everywhere parallel to the instantaneous velocity field at the same geometrical point.

Let $\vec{x}(\lambda)$ denote a streamline, parameterized by λ. The definition can be formulated as

$$\frac{\mathrm{d}\vec{x}(\lambda)}{\mathrm{d}\lambda} = \alpha(\lambda)\vec{\mathrm{v}}\big(t,\vec{x}(\lambda)\big) \tag{I.15a}$$

with $\alpha(\lambda)$ a scalar function. Equivalently, denoting by $\mathrm{d}\vec{x}(\lambda)$ a differential line element tangent to the streamline, one has the condition

$$\mathrm{d}\vec{x} \times \vec{\mathrm{v}}\big(t,\vec{x}(\lambda)\big) = \vec{0}. \tag{I.15b}$$

Introducing a Cartesian system of coordinates, the equation for a streamline is conveniently rewritten as

$$\frac{\mathrm{d}x^1(\lambda)}{\mathrm{v}^1\big(t,\vec{x}(\lambda)\big)} = \frac{\mathrm{d}x^2(\lambda)}{\mathrm{v}^2\big(t,\vec{x}(\lambda)\big)} = \frac{\mathrm{d}x^3(\lambda)}{\mathrm{v}^3\big(t,\vec{x}(\lambda)\big)} \tag{I.15c}$$

at a point where none of the component v^i of the velocity field vanishes—if one of the v^i is zero, then so is the corresponding $\mathrm{d}x^i$, thanks to Eq. (I.15b).

Remark: Since the velocity field $\vec{\mathrm{v}}$ depends on the choice of reference frame, this is also the case of its streamlines at a given instant!

Consider now a closed geometrical curve in the volume $\mathcal{V}_t$ occupied by the continuous medium at time t. The streamlines tangent to this curve form in the generic case a tube-like surface, called *stream tube*.[(x)]

Let us introduce two further definitions related to properties of the velocity field:

- If $\vec{\mathrm{v}}(t,\vec{r})$ has at some t the same value at every geometrical point $\vec{r}$ of a (connected) domain $\mathcal{D} \subset \mathcal{V}_t$, then the velocity field is said to be *uniform* across $\mathcal{D}$.
  In that case, the streamlines are parallel to each other over $\mathcal{D}$.

[(ix)]*Stromlinien*   [(x)]*Stromröhre*

- If $\vec{v}(t,\vec{r})$ only depends on the position, not on time, then the velocity field and the corresponding motion of the continuous medium are said to be *steady* or equivalently *stationary*. In that case, the streamlines coincide with the pathlines and the streaklines.

  Indeed, one checks that Eq. (I.14b) for the pathlines, in which the velocity becomes time-independent, can then be recast (at a point where all v^i are non-zero) as

  $$\frac{\mathrm{d}x^1}{v^1(t,\vec{r})} = \frac{\mathrm{d}x^2}{v^2(t,\vec{r})} = \frac{\mathrm{d}x^3}{v^3(t,\vec{r})},$$

  where the variable t plays no role: this is exactly the system (I.15c) defining the streamlines at time t. The equivalence between pathlines and streaklines is also trivial. □

I.3.4 Material derivative

Consider a material point M in a continuous medium, described in a reference frame $\mathcal{R}$. Let $\vec{r}$ resp. $\vec{r} + \mathrm{d}\vec{r}$ denote its position vectors at successive instants t resp. $t + \mathrm{d}t$. The velocity of M at time t resp. $t + \mathrm{d}t$ is by definition equal to the value of the velocity field at that time and at the respective position, namely $\vec{v}(t,\vec{r})$ resp. $\vec{v}(t+\mathrm{d}t,\vec{r}+\mathrm{d}\vec{r})$. For small enough $\mathrm{d}t$, the displacement $\mathrm{d}\vec{r}$ of the material point between t and $t + \mathrm{d}t$ is related to its velocity at time t by $\mathrm{d}\vec{r} = \vec{v}(t,\vec{r})\,\mathrm{d}t$.

Let $\mathrm{d}\vec{v} \equiv \vec{v}(t+\mathrm{d}t,\vec{r}+\mathrm{d}\vec{r}) - \vec{v}(t,\vec{r})$ denote the change in the material point velocity between t and $t + \mathrm{d}t$. Assuming that $\vec{v}(t,\vec{r})$ is differentiable (cf. § I.3.2) and introducing for simplicity a system of Cartesian coordinates, a Taylor expansion to lowest order yields

$$\mathrm{d}\vec{v} \simeq \frac{\partial\vec{v}(t,\vec{r})}{\partial t}\,\mathrm{d}t + \frac{\partial\vec{v}(t,\vec{r})}{\partial x^1}\,\mathrm{d}x^1 + \frac{\partial\vec{v}(t,\vec{r})}{\partial x^2}\,\mathrm{d}x^2 + \frac{\partial\vec{v}(t,\vec{r})}{\partial x^3}\,\mathrm{d}x^3,$$

up to terms of higher order in $\mathrm{d}t$ or $\mathrm{d}\vec{r}$. Introducing the differential operator

$$\mathrm{d}\vec{r} \cdot \vec{\nabla} \equiv \mathrm{d}x^1\,\frac{\partial}{\partial x^1} + \mathrm{d}x^2\,\frac{\partial}{\partial x^2} + \mathrm{d}x^3\,\frac{\partial}{\partial x^3},$$

this can be recast in the more compact form

$$\mathrm{d}\vec{v} \simeq \frac{\partial\vec{v}(t,\vec{r})}{\partial t}\,\mathrm{d}t + \left(\mathrm{d}\vec{r}\cdot\vec{\nabla}\right)\vec{v}(t,\vec{r}). \tag{I.16}$$

In the second term on the right-hand side, $\mathrm{d}\vec{r}$ can be replaced by $\vec{v}(t,\vec{r})\,\mathrm{d}t$. On the other hand, the change in velocity of the material point between t and $t + \mathrm{d}t$ is simply the product of its acceleration $\vec{a}(t)$ at time t by the length $\mathrm{d}t$ of the time interval, at least to lowest order in $\mathrm{d}t$. Dividing both sides of Eq. (I.16) by $\mathrm{d}t$ and taking the limit $\mathrm{d}t \to 0$, in particular in the ratio $\mathrm{d}\vec{v}/\mathrm{d}t$, yield

$$\vec{a}(t) = \frac{\partial\vec{v}(t,\vec{r})}{\partial t} + \left[\vec{v}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{v}(t,\vec{r}). \tag{I.17}$$

That is, the acceleration of the material point consists of two terms:

- the *local acceleration* $\dfrac{\partial\vec{v}}{\partial t}$, which follows from the non-stationarity of the velocity field;

- the *convective acceleration* $\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}$, due to the non-uniformity of the motion.

More generally, one finds by repeating the same derivation as above that the time derivative of a physical quantity $\mathcal{G}$ attached to a material point or domain, expressed in terms of Eulerian fields, is the sum of a local ($\partial\mathcal{G}/\partial t$) and a convective $[(\vec{v}\cdot\vec{\nabla})\mathcal{G}]$ part, irrespective of the tensorial nature of $\mathcal{G}$. Accordingly, one introduces the operator

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \vec{v}(t,\vec{r})\cdot\vec{\nabla}} \tag{I.18}$$

called *material derivative*[xi] or (between others) *substantial derivative*,[xii] *derivative following the motion*, *hydrodynamic derivative*. Relation (I.17) can thus be recast as

$$\vec{a}(t) = \frac{\mathrm{D}\vec{\mathrm{v}}(t,\vec{r})}{\mathrm{D}t}. \tag{I.19}$$

Remarks:

∗ Equation (I.17) shows that even in the case of a steady motion, the acceleration of a material point may be non-vanishing, thanks to the convective part.

∗ The material derivative (I.18) is also often denoted (and referred to) as total derivative $\mathrm{d}/\mathrm{d}t$.

∗ One also finds in the literature the denomination *convective derivative*.[xiii] To the eyes and ears of the author of these lines, that name has the drawback that it does not naturally evoke the local part, but only...the convective one, which comes from the fact that matter is being transported, "conveyed", with a non-vanishing velocity field $\vec{\mathrm{v}}(t,\vec{r})$.

∗ The two terms in Eq. (I.18) actually "merge" together when considering the motion of a material point in Galilean space-time $\mathbb{R} \times \mathbb{R}^3$. As a matter of fact, one easily shows that $\mathrm{D}/\mathrm{D}t$ is the (Lie[e]) derivative along the world-line of the material point

> The world-line element corresponding to the motion between t and $t+\mathrm{d}t$ goes from (t, x^1, x^2, x^3) to $(t+\mathrm{d}t, x^1+\mathrm{v}^1\,\mathrm{d}t, x^2+\mathrm{v}^2\,\mathrm{d}t, x^3+\mathrm{v}^3\,\mathrm{d}t)$. The tangent vector to this world-line thus has components $(1, \mathrm{v}^1, \mathrm{v}^2, \mathrm{v}^3)$, i.e. the derivative along the direction of this vector is $\partial_t + \mathrm{v}^1\partial_1 + \mathrm{v}^2\partial_2 + \mathrm{v}^3\partial_3$, with the usual shorthand notations $\partial_t \equiv \partial/\partial t$ and $\partial_i \equiv \partial/\partial x^i$. □

I.4 Mechanical stress

I.4.1 Forces in a continuous medium

Consider a closed material domain $\mathcal{V}$ inside the volume $\mathcal{V}_t$ occupied by a continuous medium, and let $\mathcal{S}$ denote the (geometric) surface enclosing $\mathcal{V}$. One distinguishes between two classes of forces acting on this domain:

- *Volume* or *body forces*,[xiv] which act at each point of the bulk volume of $\mathcal{V}$.
  Examples are weight, long-range electromagnetic forces or, in non-inertial reference frames, fictitious forces (Coriolis, centrifugal).
  For such forces, which tend to be proportional to the volume they act on, it will later be more convenient to introduce the corresponding volumic force density.

- *Surface* or *contact forces*,[xv] which act on the surface $\mathcal{S}$, like friction. These will be now discussed in further detail.

Consider an infinitesimally small geometrical surface element $\mathrm{d}^2\mathcal{S}$ at point P. Let $\mathrm{d}^2\vec{F_s}$ denote the surface force through $\mathrm{d}^2\mathcal{S}$. That is, $\mathrm{d}^2\vec{F_s}$ is the contact force, due to the medium exterior to $\mathcal{V}$, that a "test" material surface coinciding with $\mathrm{d}^2\mathcal{S}$ would experience. The vector

$$\vec{T}_\mathrm{s} \equiv \frac{\mathrm{d}^2\vec{F_s}}{\mathrm{d}^2\mathcal{S}}, \tag{I.20}$$

representing the surface density of contact forces, is called *(mechanical) stress vector*[xvi] on $\mathrm{d}^2\mathcal{S}$.

[xi] *Materielle Ableitung*   [xii] *Substantielle Ableitung*   [xiii] *Konvektive Ableitung*   [xiv] *Volumenkräfte*
[xv] *Oberflächenkräfte*
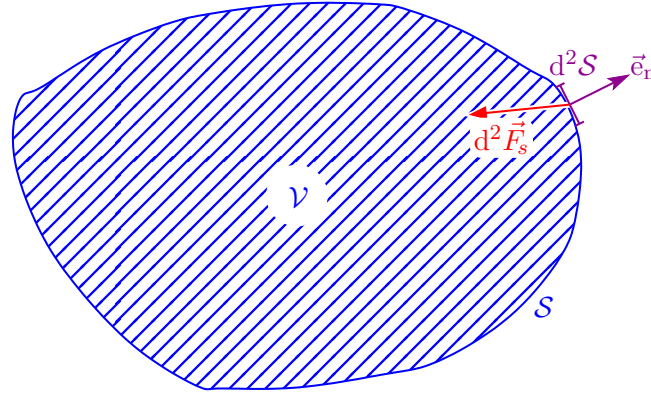
---

[e] S. LIE, 1842–1899

Figure I.2

The corresponding unit in the SI system is the Pascal, with $1\,\mathrm{Pa} = 1\,\mathrm{N}\cdot\mathrm{m}^{-2}$.

Purely geometrically, the stress vector $\vec{T}_s$ on a given surface element $\mathrm{d}^2\mathcal{S}$ at a given point can be decomposed into two components, namely

- a vector orthogonal to the plane tangent at P to $\mathrm{d}^2\mathcal{S}$, the so-called *normal stress*[(xvii)]; when it is directed towards the interior resp. exterior of the medium domain being acted on, it is also referred to as *compression*[(xviii)] resp. *tension*[(xix)];

- a vector in the tangent plane at P, called *shear stress*[(xx)] and often denoted as $\vec{\tau}$.

Despite the short notation adopted in Eq. (I.20), the stress vector depends not only on the position of the geometrical point P where the infinitesimal surface element $\mathrm{d}^2\mathcal{S}$ lies, but also on the orientation of the surface. Let $\vec{e}_n$ denote the normal unit vector to the surface element, directed towards the exterior of the volume $\mathcal{V}$ (cf. Fig. I.2), and let $\vec{r}$ denote the position vector of P in a given reference frame. The relation between $\vec{e}_n$ and the stress vector $\vec{T}_s$ on $\mathrm{d}^2\mathcal{S}$ is then linear:

$$\vec{T}_s = \boldsymbol{\sigma}(\vec{r}) \cdot \vec{e}_n, \tag{I.21a}$$

with $\boldsymbol{\sigma}(\vec{r})$ a symmetric tensor of rank 2, the so-called (*Cauchy*[(f)]) *stress tensor.*[(xxi)]

In a given coordinate system, relation (I.21a) yields

$$T_s^i = \sum_{j=1}^{3} \boldsymbol{\sigma}^i{}_j\, e_n^j \tag{I.21b}$$

with T_s^i resp. e_n^j the coordinates of the vectors $\vec{T}_s$ resp. $\vec{e}_n$, and $\boldsymbol{\sigma}^i{}_j$ the $\binom{1}{1}$-components of the stress tensor.

> While valid in the case of a three-dimensional position space, Eq. (I.21a) should actually be better formulated to become valid in arbitrary dimension. Thus, the unit-length "normal vector" to a surface element at point P is rather a *1-form* acting on the vectors of the tangent space to the surface at P. As such, it should be represented as the transposed of a vector $[(\vec{e}_n)^{\mathsf{T}}]$, which multiplies the stress tensor from the left:

$$\boxed{\vec{T}_s = (\vec{e}_n)^{\mathsf{T}} \cdot \boldsymbol{\sigma}(\vec{r}).} \tag{I.21c}$$

[(xvi)] *Mechanischer Spannungsvektor*   [(xvii)] *Normalspannung*   [(xviii)] *Druckspannung*   [(xix)] *Zugsspannung*   [(xx)] *Scher-, Tangential-* oder *Schubspannung*   [(xxi)] (*Cauchy'scher*) *Spannungstensor*

[(f)] A.L. Cauchy, 1789–1857

This shows that the Cauchy stress tensor is a $\binom{2}{0}$-tensor (a "bivector"), which maps 1-forms onto vectors. In terms of coordinates, this gives, using Einstein's summation convention

$$T_s^j = e_{n,i}\,\boldsymbol{\sigma}^{ij}\,, \tag{I.21d}$$

which thanks to the symmetry of $\boldsymbol{\sigma}$ is equivalent to the relation given above.

Remark: The symmetry property of the Cauchy stress tensor is intimately linked to the assumption that the material points constituting the continuous medium have no intrinsic angular momentum.

I.4.2 Fluids

With the help of the notion of mechanical stress, we may now introduce the definition of a *fluid*, which is the class of continuous media whose motion is described by hydrodynamics:

> *A fluid is a continuous medium that deforms itself as long as it is submitted to shear stresses.*

$$\tag{I.22}$$

Turning this definition around, one sees that in a fluid *at rest*—or, to be more accurate, studied in a reference frame with respect to which it is at rest—the mechanical stresses are necessarily normal. That is, the stress tensor is in each point diagonal.

More precisely, for a locally isotropic fluid—which means that the material points are isotropic, which is the case throughout these notes—the stress $\binom{2}{0}$-tensor is everywhere proportional to the inverse metric tensor:

$$\boxed{\boldsymbol{\sigma}(t,\vec{r}) = -\mathcal{P}(t,\vec{r})\,\mathbf{g}^{-1}(t,\vec{r})} \tag{I.23}$$

with $\mathcal{P}(t,\vec{r})$ the *hydrostatic pressure* at position $\vec{r}$ at time t.

> Going back to relation (I.21b), the stress vector will be parallel to the "unit normal vector" in any coordinate system if the square matrix of the $\binom{1}{1}$-components $\boldsymbol{\sigma}^i{}_j$ is proportional to the identity matrix, i.e. $\boldsymbol{\sigma}^i{}_j \propto \delta^i{}_j$, where we have introduced the Kronecker symbol. To obtain the $\binom{2}{0}$-components $\boldsymbol{\sigma}^{ik}$, one has to multiply $\boldsymbol{\sigma}^i{}_j$ by the component g^{jk} of the inverse metric tensor, summing over k, which precisely gives Eq. (I.23).

Remarks:

∗ Definition (I.22), as well as the two remarks hereafter, rely on an intuitive picture of "deformations" in a continuous medium. To support this picture with some mathematical background, we shall introduce in Sec. II.A an appropriate strain tensor, which quantifies these deformations, at least as long as they remain small.

∗ A deformable solid will also deform itself when submitted to shear stress! However, for a given fixed amount of tangential stress, the solid will after some time reach a new, deformed equilibrium position—otherwise, it is not a solid, but a fluid.

∗ The previous remark is actually a simplification, valid on the typical time scale of human beings. Thus, materials which in our everyday experience are solids—as for instance those forming the mantle of the Earth—will behave on a longer time scale as fluids—in the previous example, on geological time scales. Whether a given substance behaves as a fluid or a deformable solid is sometimes characterized by the dimensionless *Deborah number* [9], which compares the typical time scale for the response of the substance to a mechanical stress and the observation time.

∗ Even nicer, the fluid vs. deformable solid behavior may actually depend on the intensity of the applied shear stress: ketchup!

Bibliography for Chapter I

- National Committee for Fluid Mechanics films & film notes on *Eulerian Lagrangian description* and on *Flow visualization*;[1]

- Faber [1] Chapter 1.1–1.3;

- Feynman [10, 11] Chapter 31–6;

- Guyon *et al.* [2] Chapter 1.1;

- Sedov [12] Chapters 1 & 2.1–2.2;

- Sommerfeld [7, 8] beginning of Chapter II.5.

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[1]The visualization techniques have evolved since the 1960s, yet pathlines, streaklines or streamlines are still defined in the same way.

CHAPTER II

Kinematics of a continuous medium

The goal of fluid dynamics is to investigate the motion of fluids under consideration of the forces at play, as well as to study the mechanical stresses exerted by moving fluids on bodies with which they are in contact. The description of the motion itself, irrespective of the forces, is the object of *kinematics*.

The possibilities for the motion of a deformable continuous medium, in particular of a fluid, are richer than for a mere point particle or a rigid body: besides translations and global rotations, a deformable medium may also rotate locally and undergo... deformations! The latter term actually encompasses two different yet non-exclusive possibilities, namely either a change of shape or a variation of the volume. All these various types of motion are encoded in the local properties of the velocity field at each instant (Sec. II.1). Generic fluid motions are then classified according to several criteria, especially taking into account kinematics (Sec. II.2).

For the sake of reference, the characterization of deformations themselves, complementing that of their rate of change, is briefly presented in Sec. II.A. That formalism is not needed within fluid dynamics, but rather for the study of deformable solids, like elastic ones.

II.1 Generic motion of a continuous medium

Let $\vec{\mathrm{v}}(t,\vec{r})$ denote the velocity field in a continuous medium, measured with respect to some reference frame $\mathcal{R}$. To illustrate (some of) the possible motions that occur in a deformable body, Fig. II.1 shows the positions at successive instants t and $t + \delta t$ of a small "material vector" $\delta\vec{\ell}(t)$, that is, a continuous set of material points distributed along the straight line element stretching between two neighboring geometrical points. Let $\vec{r}$ and $\vec{r} + \delta\vec{\ell}(t)$ denote the geometrical endpoints of this material vector at time t.

Thanks to the continuity of the mappings $\vec{R} \mapsto \vec{r}(t,\vec{r})$ and its inverse $\vec{r} \mapsto \vec{R}(t,\vec{r})$, the material vector defined at instant t remains a connected set of material points as time evolves, in particular at $t + \delta t$. Assuming that both the initial length $|\delta\vec{\ell}(t)|$ as well as δt are small enough, the evolved set at $t + \delta t$ remains approximately along a straight line, and constitutes a new material vector, denoted $\delta\vec{\ell}(t + \mathrm{d}t)$. The position vectors of its endpoints simply follow from the initial positions of the corresponding material points: $\vec{r}$ resp. $\vec{r} + \delta\vec{\ell}(t)$, to which should be added the respective
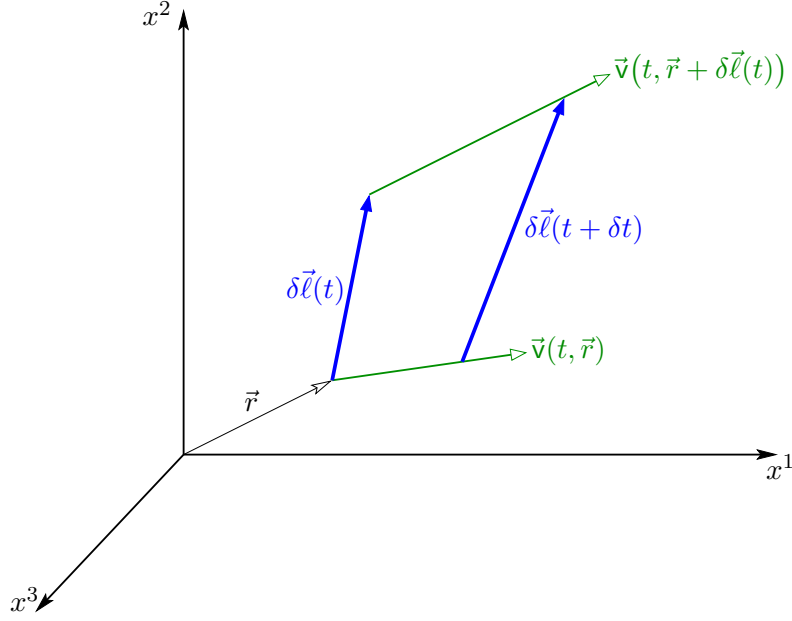
Figure II.1 – Positions of a material line element $\delta\vec{\ell}$ at successive times t and $t + \delta t$.

displacement vectors between t and $t+\delta t$, namely the product by δt of the initial velocity $\vec{v}(t,\vec{r})$ resp. $\vec{v}(t,\vec{r} + \delta\vec{\ell}(t))$. That is, one finds

$$\delta\,\vec{\ell}(t + \delta t) = \delta\vec{\ell}(t) + \left[\vec{v}(t,\vec{r} + \delta\vec{\ell}(t)) - \vec{v}(t,\vec{r})\right]\delta t + \mathcal{O}(\delta t^2). \tag{II.1}$$

Figure II.1 already suggests that the motion of the material vector consists not only of a translation, but also of a rotation, as well as an "expansion"—the change in length of the vector.

II.1.1 **Local distribution of velocities in a continuous medium**

Considering first a fixed time t, let $\vec{v}(t,\vec{r})$ resp. $\vec{v}(t,\vec{r}) + \delta\vec{v}$ be the velocity at the geometric point situated at position $\vec{r}$ resp. at $\vec{r} + \delta\vec{r}$ in $\mathcal{R}$.

Introducing for simplicity a system of Cartesian coordinates (x^1, x^2, x^3) in $\mathcal{R}$, the Taylor expansion of the i-th component of the velocity field—which is at least piecewise $\mathscr{C}^1$ in its variables, see § I.3.2—gives to first order

$$\delta v^i \simeq \sum_{j=1}^{3} \frac{\partial v^i(t,\vec{r})}{\partial x^j}\,\delta x^j, \tag{II.2a}$$

where $\{\delta x^j\}$ denote the components of $\delta\vec{r}$. Introducing the $\binom{1}{1}$-tensor $\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})$ whose components in the coordinate system used here are the partial derivatives $\partial v^i(t,\vec{r})/\partial x^j$, the above relation can be recast in the coordinate-independent form

$$\delta\vec{v} \simeq \vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r}) \cdot \delta\vec{r}. \tag{II.2b}$$

Like every rank 2 tensor, the *velocity gradient tensor* $\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})$ at time t and position $\vec{r}$ can be decomposed into the sum of the symmetric and an antisymmetric part:

$$\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r}) = \mathbf{D}(t,\vec{r}) + \mathbf{R}(t,\vec{r}), \tag{II.3a}$$

where one conventionally writes

$$\mathbf{D}(t,\vec{r}) \equiv \frac{1}{2}\left(\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r}) + \left[\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\right]^{\mathsf{T}}\right), \qquad \mathbf{R}(t,\vec{r}) \equiv \frac{1}{2}\left(\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r}) - \left[\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\right]^{\mathsf{T}}\right) \tag{II.3b}$$

with $\left[\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\right]^{\mathsf{T}}$ the transposed tensor to $\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})$. These definitions are to be understood as follows:

Using the same Cartesian coordinate system as above, the components of the two tensors **D**, **R**, viewed for simplicity as $\binom{0}{2}$-tensors, respectively read

$$\mathbf{D}_{ij}(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial \mathsf{v}_i(t,\vec{r})}{\partial x^j} + \frac{\partial \mathsf{v}_j(t,\vec{r})}{\partial x^i}\right], \qquad \mathbf{R}_{ij}(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial \mathsf{v}_i(t,\vec{r})}{\partial x^j} - \frac{\partial \mathsf{v}_j(t,\vec{r})}{\partial x^i}\right]. \qquad \text{(II.3c)}$$

Note that here we have silently used the fact that for Cartesian coordinates, the position—subscript or superscript—of the index does not change the value of the component, i.e. numerically $\mathsf{v}_i = \mathsf{v}^i$ for every $i \in \{1,2,3\}$.

> Relations (II.3c) clearly represent the desired symmetric and antisymmetric parts. However, one sees that the definitions would not appear to fulfill their task if the indices were not both either up or down, as e.g.
>
> $$\mathbf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial \mathsf{v}^i(t,\vec{r})}{\partial x^j} + \frac{\partial \mathsf{v}_j(t,\vec{r})}{\partial x_i}\right],$$
>
> in which the symmetry is no longer obvious. The trick is to rewrite the previous identity as
>
> $$\mathbf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}\delta^{ik}\delta^l{}_j\left[\frac{\partial \mathsf{v}_k(t,\vec{r})}{\partial x^l} + \frac{\partial \mathsf{v}_l(t,\vec{r})}{\partial x^k}\right] = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\partial \mathsf{v}_k(t,\vec{r})}{\partial x^l} + \frac{\partial \mathsf{v}_l(t,\vec{r})}{\partial x^k}\right],$$
>
> where we have used the fact that the metric tensor of Cartesian coordinates coincides with the Kronecker symbol. To fully generalize to curvilinear coordinates, the partial derivatives in the rightmost term should be replaced by the covariant derivatives discussed in Appendix **??**, leading eventually to
>
> $$\mathbf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}_k(t,\vec{r})}{\mathrm{d}x^l} + \frac{\mathrm{d}\mathsf{v}_l(t,\vec{r})}{\mathrm{d}x^k}\right] \qquad \text{(II.4a)}$$
>
> $$\mathbf{R}^i{}_j(t,\vec{r}) = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}_k(t,\vec{r})}{\mathrm{d}x^l} - \frac{\mathrm{d}\mathsf{v}_l(t,\vec{r})}{\mathrm{d}x^k}\right] \qquad \text{(II.4b)}$$
>
> With these new forms, which are valid in any coordinate system, the raising or lowering of indices does not affect the visual symmetric or antisymmetric aspect of the tensor.

Using the tensors **D** and **R** we just introduced, whose physical meaning will be discussed at length in § II.1.2–II.1.3, relation (II.2b) can be recast as

$$\vec{\mathsf{v}}(t,\vec{r}+\delta\vec{r}) = \vec{\mathsf{v}}(t,\vec{r}) + \mathbf{D}(t,\vec{r})\cdot\delta\vec{r} + \mathbf{R}(t,\vec{r})\cdot\delta\vec{r} + \mathcal{O}(|\delta\vec{r}|^2) \qquad \text{(II.5)}$$

where as stated at the beginning every field is considered at the same time.

Under consideration of relation (II.5) with $\delta\vec{r} = \delta\vec{\ell}(t)$, Eq. (II.1) for the time evolution of the material line element becomes

$$\delta\vec{\ell}(t+\delta t) = \delta\vec{\ell}(t) + \left[\mathbf{D}(t,\vec{r})\cdot\delta\vec{\ell}(t) + \mathbf{R}(t,\vec{r})\cdot\delta\vec{\ell}(t)\right]\delta t + \mathcal{O}(\delta t^2).$$

Subtracting $\delta\vec{\ell}(t)$ from both sides, dividing by δt and taking the limit $\delta t \to 0$, one finds for the rate of change of the material vector:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{\ell}(t) = \mathbf{D}(t,\vec{r})\cdot\delta\vec{\ell}(t) + \mathbf{R}(t,\vec{r})\cdot\delta\vec{\ell}(t) \qquad \text{(II.6)}$$

In the following two subsections, we shall investigate the physical content of each of the tensors $\mathbf{R}(t,\vec{r})$ and $\mathbf{D}(t,\vec{r})$.

II.1.2 Rotation rate tensor and vorticity vector

The tensor $\mathbf{R}(t,\vec{r})$ defined by Eq. (II.3b) is called, for reasons that will become clearer below, *rotation rate tensor*.[(xxii)]

By construction, this tensor is antisymmetric. Accordingly, one can naturally associate with it a dual (pseudo)-vector $\vec{\Omega}(t,\vec{r})$, such that for any vector $\vec{V}$

$$\mathbf{R}(t,\vec{r}) \cdot \vec{V} = \vec{\Omega}(t,\vec{r}) \times \vec{V} \quad \forall \vec{V} \in \mathbb{R}^3. \tag{II.7}$$

In Cartesian coordinates, the components of $\vec{\Omega}(t,\vec{r})$ are related to those of the rotation rate tensor by

$$\Omega^i(t,\vec{r}) \equiv -\frac{1}{2}\sum_{j,k=1}^{3} \epsilon^{ijk}\,\mathbf{R}_{jk}(t,\vec{r}) \tag{II.8a}$$

with ϵ^{ijk} the totally antisymmetric Levi-Civita symbol. Using the antisymmetry of $\mathbf{R}(t,\vec{r})$, this equivalently reads

$$\Omega^1(t,\vec{r}) \equiv -\mathbf{R}_{23}(t,\vec{r}), \quad \Omega^2(t,\vec{r}) \equiv -\mathbf{R}_{31}(t,\vec{r}), \quad \Omega^3(t,\vec{r}) \equiv -\mathbf{R}_{12}(t,\vec{r}). \tag{II.8b}$$

Comparing with Eq. (II.3c), one finds

$$\vec{\Omega}(t,\vec{r}) = \frac{1}{2}\vec{\nabla} \times \vec{\mathsf{v}}(t,\vec{r}). \tag{II.9}$$

> Proof of Eqs. (II.7), (II.9): introducing the Cartesian components $\{V^j\}$ of $\vec{V}$ and dropping for brevity the $(t,\vec{r})$-dependence of fields, the i-th component of $\mathbf{R} \cdot \vec{V}$ reads
>
> $$\mathbf{R}_{ij}V^j = \frac{1}{2}\big(\partial_j\mathsf{v}_i - \partial_i\mathsf{v}_j\big)V^j,$$
>
> where we used the summation convention over the repeated index j and the shorthand notation ∂_i for the partial derivative with respect to x^i. This may further be rewritten as
>
> $$\mathbf{R}_{ij}V^j = -\frac{1}{2}\big(\delta_i^k\delta_j^l - \delta_j^k\delta_i^l\big)(\partial_k\mathsf{v}_l)V^j,$$
>
> which now involves three sums. The term with the four Kronecker symbols is in fact the sum (over a fifth index m) of the product $\epsilon_{ijm}\epsilon^{mkl}$ of Levi-Civita symbols:
>
> $$\mathbf{R}_{ij}V^j = -\frac{1}{2}\epsilon_{ijm}\epsilon^{mkl}(\partial_k\mathsf{v}_l)V^j.$$
>
> On the right hand side of this identity, $\epsilon^{mkl}\partial_k\mathsf{v}_l$ is the m-th component of the curl $\vec{\nabla} \times \vec{\mathsf{v}}$, i.e. using definition (II.9):
> $$\mathbf{R}_{ij}V^j = -\epsilon_{ijm}\Omega^m V^j = \epsilon_{imj}\Omega_m V^j,$$
> which is precisely the i-th component of $\Omega \times \vec{\mathsf{v}}$.   □

Let us now rewrite relation (II.6) with the help of the vector $\vec{\Omega}(t,\vec{r})$, assuming that $\mathbf{D}(t,\vec{r})$ vanishes so as to isolate the effect of the remaining term. Under this assumption, the rate of change of the material vector between two neighboring points reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{\ell}(t) = \mathbf{R}(t,\vec{r}) \cdot \delta\vec{\ell}(t) = \vec{\Omega}(t,\vec{r}) \times \delta\vec{\ell}(t). \tag{II.10}$$

The term on the right hand side is then exactly the rate of rotation of a vector $\delta\vec{\ell}(t)$ in the motion of a rigid body with instantaneous angular velocity $\vec{\Omega}(t,\vec{r})$. Accordingly, the pseudovector $\vec{\Omega}(t,\vec{r})$ is referred to as *local angular velocity*.[(xxiii)] This a posteriori justifies the denomination *rotation rate tensor* for the antisymmetric tensor $\mathbf{R}(t,\vec{r})$.

[(xxii)] *Wirbeltensor*   [(xxiii)] *Wirbelvektor*

Remarks:

∗ Besides the local angular velocity $\vec{\Omega}(t,\vec{r})$, one also defines the *vorticity vector*[(xxiv)] as the curl of the velocity field

$$\boxed{\vec{\omega}(t,\vec{r}) \equiv \vec{\nabla} \times \vec{\mathsf{v}}(t,\vec{r})} = 2\vec{\Omega}(t,\vec{r}). \tag{II.11}$$

In fluid mechanics, the vorticity is actually more often used than the local angular velocity.

∗ The local angular velocity $\vec{\Omega}(t,\vec{r})$ or equivalently the vorticity vector $\vec{\omega}(t,\vec{r})$ define, at fixed t, divergence-free (pseudo)vector fields, since obviously $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathsf{v}}) = 0$. The corresponding field lines are called *vorticity lines*[(xxv)] and are given by [cf. Eq. (I.15)]

$$\mathrm{d}\vec{x} \times \vec{\omega}(t,\vec{r}) = \vec{0} \tag{II.12a}$$

or equivalently, at a point where none of the components of the vorticity vector vanishes,

$$\frac{\mathrm{d}x_1}{\omega^1(t,\vec{r})} = \frac{\mathrm{d}x_2}{\omega^2(t,\vec{r})} = \frac{\mathrm{d}x_3}{\omega^3(t,\vec{r})}. \tag{II.12b}$$

II.1.3 Strain rate tensor

According to the previous subsection, the local rotational motion of a material vector is governed by the (local and instantaneous) rotation rate tensor $\mathbf{R}(t,\vec{r})$. In turn, the translational motion is simply the displacement—which must be described in an affine space, not a vector one—of one of the endpoints of $\delta\vec{\ell}$ by an amount given by the product of velocity and length of time interval. That is, both components of the motion of a rigid body are already accounted for without invoking the symmetric tensor $\mathbf{D}(t,\vec{r})$.

In other words, the tensor $\mathbf{D}(t,\vec{r})$ characterizes the local deviation between the velocity fields in a deformable body, in particular a fluid, and in a rigid body rotating with angular velocity $\vec{\Omega}(t,\vec{r})$. Accordingly, it is called *strain rate tensor* or *deformation rate tensor*.[(xxvi)]

As we shall now see, the diagonal and off-diagonal components of $\mathbf{D}(t,\vec{r})$ actually describe the rates of change of different kinds of deformation. For simplicity, we assume throughout this subsection that $\vec{\Omega}(t,\vec{r}) = \vec{0}$.

II.1.3 a Diagonal components

We first assume that all off-diagonal terms in the strain rate tensor vanish: $\mathbf{D}_{ij}(t,\vec{r}) = 0$ for $i \neq j$, so as to isolate the meaning of the diagonal components.

Going back to Eq. (II.1), let us simply project it along one of the axes of the coordinate system, say along direction i. Denoting $\delta\ell^i$ the i-th component of $\delta\vec{\ell}$, one thus finds

$$\delta\ell^i(t+\delta t) = \delta\ell^i(t) + \left[\mathsf{v}^i\big(t,\vec{r}+\delta\vec{\ell}(t)\big) - \mathsf{v}^i\big(t,\vec{r}\big)\right]\delta t + \mathcal{O}\big(\delta t^2\big).$$

Taylor-expanding the term between square brackets to first order then yields

$$\delta\ell^i(t+\delta t) - \delta\ell^i(t) \simeq \sum_{j=1}^{3} \frac{\partial \mathsf{v}^i(t,\vec{r})}{\partial x^j}\,\delta\ell^j(t)\,\delta t,$$

up to terms of higher order in $|\delta\vec{\ell}(t)|$ or δt. Since we have assumed that both $\vec{\Omega}(t,\vec{r})$—or equivalently the components $\mathbf{R}_{ij}(t,\vec{r})$ of the rotation rate tensor—and the off-diagonal $\mathbf{D}_{ij}(t,\vec{r})$ with $i \neq j$ vanish, one checks that the partial derivative $\partial \mathsf{v}^i(t,\vec{r})/\partial x^j$ vanishes for $i \neq j$. That is, the only non-zero term in the sum is that with $j = i$, so that the equation simplifies to

$$\delta\ell^i(t+\delta t) - \delta\ell^i(t) \simeq \frac{\partial \mathsf{v}^i(t,\vec{r})}{\partial x^i}\,\delta\ell^i(t)\,\delta t = \mathbf{D}^i{}_i(t,\vec{r})\,\delta\ell^i(t)\,\delta t.$$

[(xxiv)] *Wirbligkeit*   [(xxv)] *Wirbellinien*   [(xxvi)] *Verzerrungsgeschwindigkeitstensor, Deformationsgeschwindigkeitstensor*

Thus, the relative elongation in δt of the i-th component—remember that there is no local rotation, so that the change in $\delta\ell^i$ is entirely due to a variation of the length of the material vector—is given by

$$\frac{\delta\ell^i(t+\delta t) - \delta\ell^i(t)}{\delta\ell^i(t)} = \mathbf{D}^i_{\ i}(t,\vec{r})\,\delta t \qquad (\text{II.13})$$

or alternatively, taking the limit $\delta t \to 0$

$$\frac{1}{\delta\ell^i(t)}\frac{\mathrm{d}}{\mathrm{d}t}\delta\ell^i(t) = \mathbf{D}^i_{\ i}(t,\vec{r}). \qquad (\text{II.14})$$

This equation means that the diagonal component $\mathbf{D}^i_{\ i}(t,\vec{r})$ represents the local rate of linear elongation in direction i.

Volume expansion rate

Instead of considering a one-dimensional material vector, one can study the evolution of a small "material rectangular parallelepiped" of continuous medium, situated at time t at position $\vec{r}$ with instantaneous edge lengths $\delta L^1(t)$, $\delta L^2(t)$, $\delta L^3(t)$—where for simplicity the coordinate axes are taken along the parallelepiped edges—, so that its volume at time t is simply $\delta\mathcal{V}(t) = \delta L^1(t)\,\delta L^2(t)\,\delta L^3(t)$.

Taking into account Eq. (II.13) for the relative elongation of each edge length, one finds that the relative change in volume between t and $t+\delta t$ is

$$\frac{\delta\mathcal{V}(t+\delta t) - \delta\mathcal{V}(t)}{\delta\mathcal{V}(t)} = \frac{\delta L^1(t+\delta t) - \delta L^1(t)}{\delta L^1(t)} + \frac{\delta L^2(t+\delta t) - \delta L^2(t)}{\delta L^2(t)} + \frac{\delta L^3(t+\delta t) - \delta L^3(t)}{\delta L^3(t)}$$
$$= \left[\mathbf{D}^1_{\ 1}(t,\vec{r}) + \mathbf{D}^2_{\ 2}(t,\vec{r}) + \mathbf{D}^3_{\ 3}(t,\vec{r})\right]\delta t.$$

In the second line, one recognizes the *trace* of the tensor $\mathbf{D}(t,\vec{r})$, which going back to the definition of the latter is equal to the divergence of the velocity fluid:

$$\mathbf{D}^1_{\ 1}(t,\vec{r}) + \mathbf{D}^2_{\ 2}(t,\vec{r}) + \mathbf{D}^3_{\ 3}(t,\vec{r}) = \frac{\partial \mathsf{v}^1(t,\vec{r})}{\partial x^1} + \frac{\partial \mathsf{v}^2(t,\vec{r})}{\partial x^2} + \frac{\partial \mathsf{v}^3(t,\vec{r})}{\partial x^3} = \vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}).$$

That is, this divergence represents the local and instantaneous *volume expansion rate* of the continuous medium. Accordingly, the flow of a fluid is referred to as *incompressible* in some region when the velocity field in that region is divergence-free:

$$\boxed{\text{incompressible flow} \quad \Leftrightarrow \quad \vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}) = 0.} \qquad (\text{II.15})$$

We shall comment on this definition in § II.2.3.

II.1.3 b Off-diagonal components

Let us now assume that $\mathbf{D}_{12}(t,\vec{r})$, and thereby automatically $\mathbf{D}_{21}(t,\vec{r})$, is the only non-vanishing component of the strain rate tensor. To see the influence of that component, we need to consider the time evolution of a different object than a material vector, since anything that can affect the latter—translation, rotation, dilatation—has already been described above.

Accordingly, we now look at the change between successive instants t and $t+\delta t$ of an elementary "material rectangle", as pictured in Fig. II.2. We denote by $\vec{\mathsf{v}}$ resp. $\vec{\mathsf{v}}+\delta\vec{\mathsf{v}}$ the velocity at time t at the lower left resp. upper right corner of the rectangle. Taylor expansions give for the Cartesian components of the shift $\delta\vec{\mathsf{v}}$

$$\delta\mathsf{v}_1 = \frac{\partial \mathsf{v}_1(t,\vec{r})}{\partial x^2}\,\delta\ell^2, \qquad \delta\mathsf{v}_2 = \frac{\partial \mathsf{v}_2(t,\vec{r})}{\partial x^1}\,\mathrm{d}\ell^1.$$

Figure II.2 shows that what is a right angle at time t becomes an angle $\pi/2 - \delta\alpha$ at $t+\mathrm{d}t$, where $\delta\alpha = \delta\alpha_1 - \delta\alpha_2$. In the limit of small δt, both $\delta\alpha_1$ and $\delta\alpha_2$ will be small and thus approximately
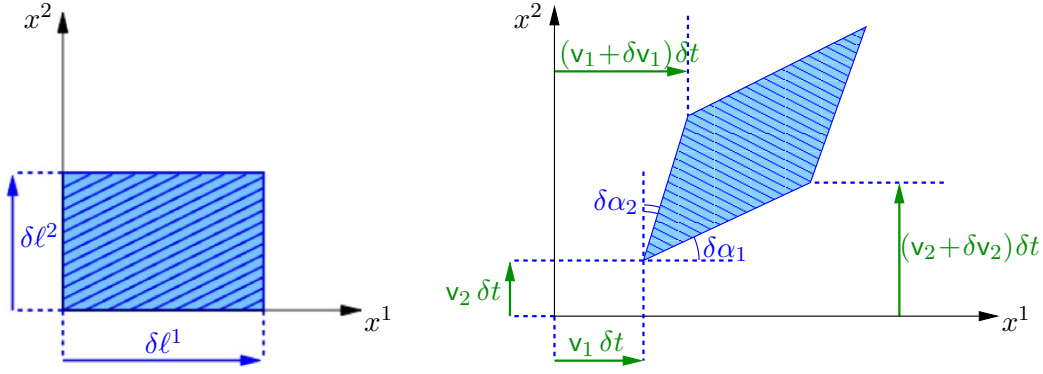
Figure II.2 – Evolution of a material rectangle caught in the motion of a continuous medium between times t (left) and $t + \delta t$ (right).

equal to their respective tangents. Using the fact that the parallelogram still has the same area—since the diagonal components of **D** vanish—the projection of any side of the deformed rectangle at time $t + \delta t$ on its original direction at time t keeps approximately the same length, up to corrections of order δt. One thus finds for the oriented angles

$$\delta\alpha_1 \simeq \frac{\delta v_2\, \delta t}{\delta\ell^1} \quad \text{and} \quad \delta\alpha_2 \simeq -\frac{\delta v_1\, \delta t}{\delta\ell^2}.$$

With the Taylor expansions given above, this leads to

$$\delta\alpha_1 \simeq \frac{\partial v_2(t,\vec{r})}{\partial x^1}\,\delta t, \qquad \delta\alpha_2 \simeq -\frac{\partial v_1(t,\vec{r})}{\partial x^2}\,\delta t.$$

Gathering all pieces, one finds

$$\frac{\delta\alpha}{\delta t} \simeq \frac{\partial v_2(t,\vec{r})}{\partial x^1} + \frac{\partial v_1(t,\vec{r})}{\partial x^2} = 2\,\mathbf{D}_{12}(t,\vec{r}). \tag{II.16}$$

In the limit $\delta t \to 0$, one sees that the off-diagonal component $\mathbf{D}_{12}(t,\vec{r})$ represents half the local velocity of the "angular deformation"—the *shear*—around direction x^3.

Remark: To separate the two physical effects present in the strain rate tensor, the latter is often written as the sum of a diagonal *rate-of-expansion tensor* proportional to the identity **1**—which is in fact the $\binom{1}{1}$-form of the metric tensor **g** of Cartesian coordinates—and a traceless *rate-of-shear tensor* **S**:

$$\mathbf{D}(t,\vec{r}) = \frac{1}{3}\big[\vec{\nabla}\cdot\vec{v}(t,\vec{r})\big]\mathbf{1} + \mathbf{S}(t,\vec{r}) \tag{II.17a}$$

with

$$\mathbf{S}(t,\vec{r}) \equiv \frac{1}{2}\left(\vec{\nabla}\vec{v}(t,\vec{r}) + \big[\vec{\nabla}\vec{v}(t,\vec{r})\big]^{\mathsf{T}} - \frac{2}{3}\big[\vec{\nabla}\cdot\vec{v}(t,\vec{r})\big]\mathbf{1}\right). \tag{II.17b}$$

Component-wise, and generalizing to curvilinear coordinates, this reads

$$\mathbf{D}_{ij}(t,\vec{r}) = \frac{1}{3}\big[\vec{\nabla}\cdot\vec{v}(t,\vec{r})\big]g_{ij}(t,\vec{r}) + \mathbf{S}_{ij}(t,\vec{r}) \tag{II.17c}$$

with [cf. Eq. (II.4a)]

$$\mathbf{S}_{ij}(t,\vec{r}) \equiv \frac{1}{2}\left[g^k_{\ i}(t,\vec{r})g^l_{\ j}(t,\vec{r})\left(\frac{\mathrm{d}v_k(t,\vec{r})}{\mathrm{d}x^l} + \frac{\mathrm{d}v_l(t,\vec{r})}{\mathrm{d}x^k}\right) - \frac{2}{3}\big[\vec{\nabla}\cdot\vec{v}(t,\vec{r})\big]g_{ij}(t,\vec{r})\right]. \tag{II.17d}$$

Summary

Gathering the findings of this Section, the most general motion of a material volume element inside a continuous medium, in particular in a fluid, can be decomposed in four elements:

- a translation;

- a rotation, with a local angular velocity $\vec{\Omega}(t,\vec{r})$ given by Eq. (II.9)—i.e. related to the anti-symmetric part $\mathbf{R}(t,\vec{r})$ of the velocity gradient—and equal to twice the (local) vorticity vector $\vec{\omega}(t,\vec{r})$;

- a local dilatation or contraction, in which the geometric form of the material volume element remains unchanged, whose rate is given by the divergence of the velocity field $\vec{\nabla}\cdot\vec{v}(t,\vec{r})$, i.e. encoded in the diagonal elements of the strain rate tensor $\mathbf{D}(t,\vec{r})$;

- a change of shape ("deformation") of the material volume element at constant volume, controlled by the rate-of-shear tensor $\mathbf{S}(t,\vec{r})$ [Eqs. (II.17b), (II.17d)], obtained by taking the traceless symmetric part of the velocity gradient.

Remark: In the case of a uniform motion, all spatial derivatives are by definition zero, so that the vorticity $\vec{\omega}(t,\vec{r})$, the expansion rate $\vec{\nabla}\cdot\vec{v}(t,\vec{r})$ and the rate-of-shear tensor $\mathbf{S}(t,\vec{r})$ actually vanish everywhere in the flow. Accordingly, the motion of a material element in that case is simply a pure translation, without deformation or rotation.

II.2 Classification of fluid flows

The motion, or *flow*[(xxvii)], of a fluid can be characterized according to several criteria, either purely geometrical (§ II.2.1), kinematic (§ II.2.2), or of a more physical nature (§ II.2.3), that takes into account the physical behavior of the flowing fluid in its evolution.

II.2.1 Geometrical criteria

In the general case, the quantities characterizing the properties of a fluid flow will depend on time as well as on three spatial coordinates.

For some more or less idealized models of actual flows, it may turn out that only two spatial coordinates play a role, in which case one talks of a *two-dimensional flow*. An example is the flow of air around the wing of an airplane, which in first approximation is "infinitely" long compared to its transverse profile: the (important!) effects at the ends of the wing, which introduce the dependence on the spatial dimension along the wing, may be left aside in a first approach, then considered in a second, more detailed step.

In some cases, e.g. for fluid flows in pipes, one may even assume that the properties only depend on a single spatial coordinate, so that the flow is *one-dimensional*. In that approximation, the physical local quantities are actually often replaced by their average value over the cross section of the pipe.

On a different level, one also distinguishes between *internal* und *external* fluid flows, according to whether the fluid is enclosed inside solid walls—e.g. in a pipe—or flowing around a body—e.g. around an airplane wing.

II.2.2 Kinematic criteria

The notions of *uniform*—that is independent of position—and *steady*—independent of time—motions were already introduced at the end of § I.3.3. Accordingly, there are *non-uniform* and *unsteady* fluids flows.

If the vorticity vector $\vec{\omega}(t,\vec{r})$ vanishes at every point $\vec{r}$ of a flowing fluid, then the corresponding motion is referred to as an *irrotational flow*[(xxviii)] or, for reasons that will be clarified in Sec. **??**, *potential flow*. The opposite case is that of a *vortical* or *rotational flow*.[(xxix)]

[(xxvii)]*Strömung*   [(xxviii)]*wirbelfreie Strömung*   [(xxix)]*Wirbelströmung*

According to whether the flow velocity v is smaller or larger than the (local) speed of sound c_s in the fluid, one talks of *subsonic* or *supersonic* motion[(xxx)], corresponding respectively to a dimensionless *Mach number*[(g)]

$$\mathrm{Ma} \equiv \frac{\mathsf{v}}{c_s} \tag{II.18}$$

smaller or larger than 1. Note that the Mach number can a priori be defined, and take different values $\mathrm{Ma}(t, \vec{r})$, at every point in a flow.

When the fluid flows in layers that do not mix with each other, so that the streamlines remain parallel, the flow is referred to as *laminar*. In the opposite case the flow is *turbulent*.

II.2.3 Physical criteria

All fluids are compressible, more or less according to the substance and its thermodynamic state. Nevertheless, this compressibility is sometimes irrelevant for a given motion, in which case it may be fruitful to consider that the fluid flow is *incompressible*, which, as seen in § II.1.3 a, technically means that its volume expansion rate vanishes, $\vec{\nabla} \cdot \vec{\mathsf{v}} = 0$. In the opposite case ($\vec{\nabla} \cdot \vec{\mathsf{v}} \neq 0$), the flow is said to be *compressible*. It is however important to realize that the statement is more a kinematic one, than really reflecting the thermodynamic compressibility of the fluid.

> In practice, flows are compressible in regions where the fluid velocity is "large", namely where the Mach number (II.18) is not much smaller than 1, i.e. roughly speaking $\mathrm{Ma} \gtrsim 0.2$.

In an analogous manner, one speaks of viscous resp. non-viscous flows to express the fact that the fluid under consideration is modeled as viscous resp. inviscid—which leads to different equations of motion—, irrespective of the fact that every real fluid has a non-zero viscosity.

Other thermodynamic criteria are also used to characterize possible fluid motions: isothermal flows—i.e. in which the temperature is uniform and remains constant—, isentropic flows—i.e. without production of entropy—, and so on.

Bibliography for Chapter II

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- Feynman [10, 11] Chapter 39–1;
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- Sommerfeld [7, 8] Chapter I.

[(xxx)] *Unterschall-* bzw. *Überschallströmung*

[(g)]E. Mach, 1838–1916

Appendix to Chapter II

II.A Deformations in a continuous medium

Strain tensor...

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