VIII.3.3 Non-relativistic limit

We shall now consider the low-velocity limit $|\vec{\mathrm{v}}| \ll c$ of the relativistic equations of motion (VIII.2) and (VIII.7), in the case when the conserved currents are those of perfect fluids, namely given by constitutive relations (VIII.17a) and (VIII.17b). Anticipating on the result, we shall recover the equations governing the dynamics of non-relativistic perfect fluids presented in Chapter III, as could be expected for the sake of consistency.

In the small-velocity limit, the typical velocity of the "atoms" forming the fluid is also much smaller than the speed of light, which has two consequences. On the one hand, the available energies are too low to allow the creation of particle–antiparticle pairs—while their annihilation remains possible—, so that the fluid consists of either particles or antiparticles. Assuming that there is a single type of particles in the fluid, the various charges labeled by the index a are all redundant, and the charge density $\mathcal{n}_a(\mathsf{x})$, which is proportional to the difference of the amounts of particles and antiparticles in a unit volume, actually coincides with the "true" particle number density, which will be more shortly denoted $\mathcal{n}(\mathsf{x})$.

On the other hand, the relativistic energy density ϵ can now be expressed as the sum of the contribution from the masses of the particles and of a kinetic energy term. By definition, the latter is the local internal energy density $e(\mathsf{x})$ of the fluid, while the former is simply the number density of particles multiplied by their mass energy:

$$\epsilon(\mathsf{x}) = \mathcal{n}(\mathsf{x})mc^2 + e(\mathsf{x}) = \rho(\mathsf{x})c^2 + e(\mathsf{x}), \tag{VIII.26}$$

with $\rho(\mathsf{x})$ the mass density of the fluid constituents. It is important to note that the internal energy density e is of order $\vec{\mathrm{v}}^2/c^2$ with respect to the mass-energy term. The same holds for the pressure $\mathcal{P}$, which is of the same order of magnitude as e.[(59)]

Eventually, a Taylor expansion of the Lorentz factor associated with the flow velocity yields

$$\gamma(\mathsf{x}) \underset{|\vec{\mathrm{v}}|\ll c}{\sim} 1 + \frac{1}{2}\frac{\vec{\mathrm{v}}(\mathsf{x})^2}{c^2} + \mathcal{O}\bigg(\frac{\vec{\mathrm{v}}(\mathsf{x})^4}{c^4}\bigg). \tag{VIII.27}$$

Accordingly, to leading order in $\vec{\mathrm{v}}^2/c^2$, the components (VIII.11) of the flow 4-velocity read

$$u^\mu(\mathsf{x}) \underset{|\vec{\mathrm{v}}|\ll c}{\sim} \begin{pmatrix} c \\ \vec{\mathrm{v}}(\mathsf{x}) \end{pmatrix}. \tag{VIII.28}$$

Throughout this subsection, we shall omit for the sake of brevity the variables x resp. $(t, \vec{r})$ of the various fields. In addition, we adopt for simplicity a system of Minkowski coordinates.

(59) This is exemplified for instance by the non-relativistic classical ideal gas, in which the internal energy density is $e = \mathcal{n}c_{\mathcal{V}}k_{\mathrm{B}}T$ with $c_{\mathcal{V}}$ a number of order 1—this results e.g. from the *equipartition theorem*—while its pressure is $\mathcal{P} = \mathcal{n}k_{\mathrm{B}}T$.

VIII.3.3 a Particle number conservation

The 4-velocity components (VIII.28) give for those of the particle number 4-current (VIII.17a)

$$N^\mu \underset{|\vec{\mathrm{v}}|\ll c}{\sim} \begin{pmatrix} nc \\ n\vec{\mathrm{v}} \end{pmatrix}.$$

Accordingly, the particle number conservation equation (VIII.2) becomes

$$0 = \partial_\mu N^\mu \approx \frac{1}{c}\frac{\partial(nc)}{\partial t} + \sum_{i=1}^{3} \frac{\partial(n\mathrm{v}^i)}{\partial x^i} = \frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\,\vec{\mathrm{v}}). \tag{VIII.29}$$

That is, one recovers the non-relativistic continuity equation (III.13).

VIII.3.3 b Momentum and energy conservation

The components of the energy-momentum tensor of a perfect fluid are given by Eq. (VIII.17b). Performing a Taylor expansion including the leading and next-to-leading terms in $|\vec{\mathrm{v}}|/c$ yields, under consideration of relation (VIII.26)

$$T^{00} = -\mathcal{P} + \gamma^2(\rho c^2 + e + \mathcal{P}) \underset{|\vec{\mathrm{v}}|\ll c}{\sim} \rho c^2 + e + \rho\,\vec{\mathrm{v}}^2 + \mathcal{O}\bigg(\frac{\vec{\mathrm{v}}^2}{c^2}\bigg); \tag{VIII.30a}$$

$$T^{0j} = T^{j0} = \gamma^2(\rho c^2 + e + \mathcal{P})\frac{\mathrm{v}^j}{c} \underset{|\vec{\mathrm{v}}|\ll c}{\sim} \rho c\mathrm{v}^j + \big(e + \mathcal{P} + \rho\,\vec{\mathrm{v}}^2\big)\frac{\mathrm{v}^j}{c} + \mathcal{O}\bigg(\frac{|\vec{\mathrm{v}}|^3}{c^3}\bigg); \tag{VIII.30b}$$

$$T^{ij} = \mathcal{P}g^{ij} + \gamma^2(\rho c^2 + e + \mathcal{P})\frac{\mathrm{v}^i\mathrm{v}^j}{c^2} \underset{|\vec{\mathrm{v}}|\ll c}{\sim} \mathcal{P}g^{ij} + \rho\,\mathrm{v}^i\mathrm{v}^j + \mathcal{O}\bigg(\frac{\vec{\mathrm{v}}^2}{c^2}\bigg) = \mathbf{T}^{ij} + \mathcal{O}\bigg(\frac{\vec{\mathrm{v}}^2}{c^2}\bigg). \tag{VIII.30c}$$

In the last line, we have introduced the components $\mathbf{T}^{ij}$, defined in Eq. (III.24b), of the three-dimensional momentum flux-density tensor for a perfect non-relativistic fluid. As emphasized below Eq. (VIII.26), the internal energy density and pressure in the rightmost terms of the first or second equations are of the same order of magnitude as the term $\rho\,\vec{\mathrm{v}}^2$ with which they appear, i.e. they are always part of the next-to-leading order term.

Momentum conservation

Considering first the components (VIII.30b), (VIII.30c), the low-velocity limit of the relativistic momentum-conservation equation $\partial_\mu T^{\mu j} = 0$ for $j = 1, 2, 3$ reads

$$0 = \frac{1}{c}\frac{\partial(\rho c\mathrm{v}^j)}{\partial t} + \sum_{i=1}^{3}\frac{\partial\mathbf{T}^{ij}}{\partial x^i} + \mathcal{O}\bigg(\frac{\vec{\mathrm{v}}^2}{c^2}\bigg) = \frac{\partial(\rho\mathrm{v}^j)}{\partial t} + \sum_{i=1}^{3}\frac{\partial\mathbf{T}^{ij}}{\partial x^i} + \mathcal{O}\bigg(\frac{\vec{\mathrm{v}}^2}{c^2}\bigg). \tag{VIII.31}$$

This is precisely the conservation-equation formulation (III.28a) of the Euler equation in absence of external volume forces.

Energy conservation

Given the physical interpretation of the components T^{00}, T^{i0} with $i = 1, 2, 3$, the component $\nu = 0$ of the energy-momentum conservation equation (VIII.7), $\partial_\mu T^{\mu 0} = 0$, should represent the conservation of energy.

As was mentioned several times, the relativistic energy density and flux density actually also contain a term from the rest mass of the fluid constituents. Thus, the leading order contribution to $\partial_\mu T^{\mu 0} = 0$, coming from the first terms in the right members of Eqs. (VIII.30a) and (VIII.30b), is

$$0 = \frac{\partial(\rho c)}{\partial t} + \sum_{i=1}^{3}\frac{\partial(\rho c\mathrm{v}^i)}{\partial x^i} + \mathcal{O}\bigg(\frac{|\vec{\mathrm{v}}|}{c}\bigg),$$

that is, up to a factor c, exactly the continuity equation (III.12), which was already shown to be the low-velocity limit of the conservation of the particle-number 4-current.

To isolate the internal energy contribution, it is thus necessary to subtract that of mass energy. In the fluid local rest frame, relation (VIII.26) shows that one must subtract ρc^2 from ϵ. The former simply equals $\rho c u^0|_{\rm LR}$, while the latter is the component $\mu = 0$ of $T^{\mu 0}|_{\rm LR}$, whose space-like components vanish in the local rest frame. To fully subtract the mass energy contribution in any frame from both the energy density and flux density, one should thus consider the 4-vector $T^{\mu 0} - \rho c u^\mu$.

Accordingly, instead of simply using $\partial_\mu T^{\mu 0} = 0$, one should start from the equivalent—thanks to Eq. (VIII.2) and the relation $\rho = m n$—equation $\partial_\mu(T^{\mu 0} - \rho c u^\mu) = 0$. With the approximations

$$\rho c u^0 = \gamma \rho c^2 = \rho c^2 + \frac{1}{2}\rho \vec{\mathsf{v}}^2 + \mathcal{O}\bigg(\frac{\vec{\mathsf{v}}^2}{c^2}\bigg)$$

and

$$\rho c u^j = \gamma \rho c \mathsf{v}^j = \rho c \mathsf{v}^j + \bigg(\frac{1}{2}\rho \vec{\mathsf{v}}^2\bigg)\frac{v^j}{c} + \mathcal{O}\bigg(\frac{|\vec{\mathsf{v}}|^5}{c^3}\bigg)$$

one finds

$$0 = \partial_\mu\big(T^{\mu 0} - \rho c u^\mu\big) = \frac{1}{c}\frac{\partial}{\partial t}\bigg(\frac{1}{2}\rho\vec{\mathsf{v}}^2 + e\bigg) + \sum_{j=1}^{3}\frac{\partial}{\partial x^j}\bigg[\bigg(\frac{1}{2}\rho\vec{\mathsf{v}}^2 + e + \mathcal{P}\bigg)\frac{\mathsf{v}^j}{c}\bigg] + \mathcal{O}\bigg(\frac{\vec{\mathsf{v}}^2}{c^2}\bigg),$$

that is

$$\frac{\partial}{\partial t}\bigg(\frac{1}{2}\rho\vec{\mathsf{v}}^2 + e\bigg) + \vec{\nabla}\cdot\bigg[\bigg(\frac{1}{2}\rho\vec{\mathsf{v}}^2 + e + \mathcal{P}\bigg)\vec{\mathsf{v}}\bigg] \approx 0. \tag{VIII.32}$$

This is the non-relativistic local formulation of energy conservation (III.38) for a perfect fluid in absence of external volume forces. Since that equation had been postulated in Section III.4.1, the above derivation may be seen as its belated proof.

VIII.3.3 c Entropy conservation

Using the approximate 4-velocity components (VIII.28), the entropy conservation equation (VIII.22) becomes in the low-velocity limit

$$0 = \partial_\mu(s u^\mu) \approx \frac{1}{c}\frac{\partial(sc)}{\partial t} + \sum_{i=1}^{3}\frac{\partial(s\mathsf{v}^i)}{\partial x^i} = \frac{\partial s}{\partial t} + \vec{\nabla}\cdot(s\,\vec{\mathsf{v}}), \tag{VIII.33}$$

i.e. gives the non-relativistic equation (III.39), expressing locally the conservation of entropy in a fluid without dissipative effects.