## VIII.2 Four-velocity of a fluid flow. Local rest frame

The four-velocity of a flow is a field, defined at each point $\mathsf{x}$ of a space-time domain $\mathcal{D}$, of time-like 4-vectors $\mathsf{u}(\mathsf{x})$ with constant magnitude $c$, i.e. such that

$$[\mathsf{u}(\mathsf{x})]^2 = u_\mu(\mathsf{x})u^\mu(\mathsf{x}) = -c^2 \quad \forall\mathsf{x}, \tag{VIII.9}$$

with $u^\mu(\mathsf{x})$ the (contravariant) components of $\mathsf{u}(\mathsf{x})$.

At each point $\mathsf{x}$ of the fluid, one can define a proper reference frame, the so-called *local rest frame*,[lxxviii] hereafter abbreviated as LR($\mathsf{x}$), in which the space-like Minkowski components of the local flow 4-velocity vanish:

$$u^\mu(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = (c,0,0,0). \tag{VIII.10}$$

Let $\vec{\mathsf{v}}(\mathsf{x})$ denote the instantaneous velocity of (an observer at rest in) the local rest frame LR($\mathsf{x}$) with respect to a fixed reference frame $\mathcal{R}$, which hereafter will be referred to as the "laboratory frame". In the latter, the components of the flow four-velocity are

$$u^\mu(\mathsf{x})\big|_{\mathcal{R}} = \begin{pmatrix} \gamma(\mathsf{x})c \\ \gamma(\mathsf{x})\vec{\mathsf{v}}(\mathsf{x}) \end{pmatrix}, \tag{VIII.11}$$

with $\gamma(\mathsf{x}) = 1/\sqrt{1-\vec{\mathsf{v}}(\mathsf{x})^2/c^2}$ the corresponding Lorentz factor.

In a situation where the system is locally close to thermodynamic equilibrium, the local rest frame represents the reference frame in which the local thermodynamic variables—charge densities $n_a(\mathsf{x})$ and energy density $\epsilon(\mathsf{x})$—are defined in their usual sense:

$$n_a(\mathsf{x}) \equiv n_a(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})}, \qquad \epsilon(\mathsf{x}) \equiv T^{00}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})}. \tag{VIII.12}$$

For the remaining local thermodynamic variables, it is assumed that they are related to $n_a(\mathsf{x})$ and $\epsilon(\mathsf{x})$ in the same way as when the fluid is at thermodynamic equilibrium. Thus, the pressure $\mathcal{P}(\mathsf{x})$ is given by the mechanical equation of state

$$\mathcal{P}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = \mathcal{P}(\epsilon(\mathsf{x}),\{n_a(\mathsf{x})\}); \tag{VIII.13}$$

the temperature $T(\mathsf{x})$ is given by the thermal equation of state; the entropy density $s(\mathsf{x})$ is defined by the Gibbs fundamental relation, and so on.

**Remarks:**

∗ A slightly more formal approach to define the 4-velocity and the local rest frame is to turn the reasoning round. Namely, one first introduces the latter as a reference frame LR($\mathsf{x}$) in which "physics at point $\mathsf{x}$ is easy", which in particular means that the fluid should be locally "motionless".[58] Introducing then an instantaneous inertial reference frame that momentarily coincides with LR($\mathsf{x}$), one considers an observer $\mathscr{O}$ at rest in that inertial frame. The fluid four-velocity $\mathsf{u}(\mathsf{x})$ with respect to the laboratory frame $\mathcal{R}$ is then the four-velocity of $\mathscr{O}$ (assumed to be pointlike) in $\mathcal{R}$—defined as the derivative of $\mathscr{O}$'s space-time trajectory with respect to $\mathscr{O}$'s proper time.

∗ The relativistic energy density $\epsilon$ differs from its at first sight obvious non-relativistic counterpart, the internal energy density $e$. The reason is that $\epsilon$ also contains the contribution from the mass energy of the particles and antiparticles—$mc^2$ per (anti)particle—, which is conventionally not taken into account in the non-relativistic internal energy density.

[58] As we shall discuss in Ref. VIII.4.2, this requirement may not define a unique reference frame.

[lxxviii] *lokales Ruhesystem*

∗ To distinguish between the frame-dependent quantities, like charge densities $n_a(\mathsf{x})$ or energy density $T^{00}(\mathsf{x})$, and the corresponding quantities measured in the local rest frame, namely $\mathcal{n}_a(\mathsf{x})$ or $\epsilon(\mathsf{x})$, the latter are referred to as *comoving*.

The comoving quantities can actually be computed easily within any reference frame and coordinate system. Let us thus write

$$\mathcal{n}_a(\mathsf{x}) \equiv n_a(\mathsf{x})\big|_{\mathrm{LR(x)}} = \frac{1}{c}N_a^0(\mathsf{x})\big|_{\mathrm{LR(x)}} = \frac{N_a^0(\mathsf{x})u^0(\mathsf{x})}{[u^0(\mathsf{x})]^2}\bigg|_{\mathrm{LR(x)}} = \frac{N_a^0(\mathsf{x})u_0(\mathsf{x})}{g_{00}(\mathsf{x})[u^0(\mathsf{x})]^2}\bigg|_{\mathrm{LR(x)}} = \frac{N_a^\mu(\mathsf{x})u_\mu(\mathsf{x})}{u^\nu(\mathsf{x})u_\nu(\mathsf{x})}\bigg|_{\mathrm{LR(x)}},$$

where we used the fact that $u_0(\mathsf{x}) = g_{00}(\mathsf{x})u^0(\mathsf{x})$ in the local rest frame. The rightmost term of the above equation is the ratio of two Lorentz-invariant scalars, and thus is itself a Lorentz scalar field, independent of the reference frame in which it is computed:

$$\mathcal{n}_a(\mathsf{x}) = \frac{N_a^\mu(\mathsf{x})u_\mu(\mathsf{x})}{u^\nu(\mathsf{x})u_\nu(\mathsf{x})} = \frac{\mathsf{N}_a(\mathsf{x})\cdot\mathsf{u}(\mathsf{x})}{[\mathsf{u}(\mathsf{x})]^2}. \tag{VIII.14}$$

Similarly one finds

$$\epsilon(\mathsf{x}) \equiv T^{00}(\mathsf{x})\big|_{\mathrm{LR(x)}} = c^2\,\frac{u_\mu(\mathsf{x})T^{\mu\nu}(\mathsf{x})u_\nu(\mathsf{x})}{[u^\rho(\mathsf{x})u_\rho(\mathsf{x})]^2}\bigg|_{\mathrm{LR(x)}} = \frac{1}{c^2}u_\mu(\mathsf{x})T^{\mu\nu}(\mathsf{x})u_\nu(\mathsf{x}) = \frac{1}{c^2}\mathsf{u}(\mathsf{x})\cdot\mathbf{T}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x}), \tag{VIII.15}$$

where the normalization of the 4-velocity was used.

In the following Sections, we introduce fluid models, defined by the relations between the conserved currents—charge 4-currents $\mathsf{N}_a(\mathsf{x})$ and energy-momentum tensor $\mathbf{T}(\mathsf{x})$—and the fluid 4-velocity $\mathsf{u}(\mathsf{x})$ and comoving thermodynamic quantities.

# VIII.3 Perfect relativistic fluid

By definition, a fluid is perfect when there is no dissipative current in it, see definition (III.19a). As a consequence, one can at each point $\mathsf{x}$ of the fluid find a reference frame in which the local properties in the neighborhood of $\mathsf{x}$ are spatially *isotropic* [cf. definition (III.27)]. This reference frame represents the natural choice for the local rest frame at point $\mathsf{x}$, LR(x).

The expressions of the particle-number 4-current and the energy-momentum tensor of a perfect fluid are first introduced in § VIII.3.1. It is then shown that the postulated absence of dissipative current automatically leads to the conservation of entropy in the motion (§ VIII.3.2). Eventually, the low-velocity limit of the dynamical equations is investigated in § VIII.3.3.

## VIII.3.1 Charge four-current and energy-momentum tensor of a perfect fluid

To express the defining feature of the local rest frame LR(x), namely the spatial isotropy of the local fluid properties, it is convenient to adopt a Cartesian coordinate system for the space-like directions in LR(x): since the fluid characteristics are the same in all spatial directions, this in particular holds along the three mutually perpendicular axes defining Cartesian coordinates.

Adopting momentarily such a system—and accordingly Minkowski coordinates on space-time—the local-rest-frame values of the charge flux density $\vec{j}_a(\mathsf{x})$, the $j$-th component $cT^{0j}(\mathsf{x})$ of the energy flux density, and the density $c^{-1}T^{i0}(\mathsf{x})$ of the $i$-th component of momentum should all vanish. In addition, the momentum flux-density 3-tensor $\mathbf{T}(\mathsf{x})$ in LR(x) should be diagonal, and even proportional to the three-dimensional identity. All in all, one thus necessarily has

$$N_a^0(\mathsf{x})\big|_{\mathrm{LR(x)}} = c\,\mathcal{n}_a(\mathsf{x}), \qquad \vec{j}_a(\mathsf{x})\big|_{\mathrm{LR(x)}} = \vec{0}, \tag{VIII.16a}$$

and

$$\begin{aligned} T^{00}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} &= \epsilon(\mathsf{x}), \\ T^{ij}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} &= \mathcal{P}(\mathsf{x})\delta^{ij}, \qquad \forall i,j = 1,2,3 \\ T^{i0}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} &= T^{0j}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} = 0, \qquad \forall i,j = 1,2,3 \end{aligned} \tag{VIII.16b}$$

where the definitions (VIII.12) were taken into account, while $\mathcal{P}(\mathsf{x})$ denotes the pressure. Representing the energy-momentum tensor as a matrix in which $T^{\mu\nu}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})}$ is the entry in the $\mu+1$-th line and $\nu+1$-th column, Eq. (VIII.16b) yields

$$T^{\mu\nu}(\mathsf{x})\big|_{\mathrm{LR}(\mathsf{x})} \cong \begin{pmatrix} \epsilon(\mathsf{x}) & 0 & 0 & 0 \\ 0 & \mathcal{P}(\mathsf{x}) & 0 & 0 \\ 0 & 0 & \mathcal{P}(\mathsf{x}) & 0 \\ 0 & 0 & 0 & \mathcal{P}(\mathsf{x}) \end{pmatrix}. \tag{VIII.16c}$$

**Remark:** The identification of the diagonal spatial components with a "pressure" term is warranted by the physical interpretation of $T^{ii}(\mathsf{x})$. Referring to it as "the" pressure anticipates the fact that it behaves as the thermodynamic quantity that is related to energy density and particle number by the mechanical equation of state of the fluid.

In an arbitrary reference frame $\mathcal{R}$ and allowing for the possible use of curvilinear coordinates, the components of the charge 4-currents and the energy-momentum tensor of a perfect fluid are

$$N_a^\mu(\mathsf{x}) = n_a(\mathsf{x})u^\mu(\mathsf{x}) \tag{VIII.17a}$$

and

$$T^{\mu\nu}(\mathsf{x}) = \mathcal{P}(\mathsf{x})g^{\mu\nu}(\mathsf{x}) + \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\frac{u^\mu(\mathsf{x})\,u^\nu(\mathsf{x})}{c^2} \tag{VIII.17b}$$

respectively, with $u^\mu(\mathsf{x})$ the components of the fluid 4-velocity with respect to $\mathcal{R}$.

Relation (VIII.17a) resp. (VIII.17b) is an identity between the components of two 4-vectors resp. two 4-tensors, which transform identically under Lorentz transformations—i.e. changes of reference frame—and coordinate basis changes. Since the components of these 4-vectors resp. 4-tensors are equal in a given reference frame—the local rest frame—and a given basis—that of Minkowski coordinates—, they remain equal in any coordinate system in any reference frame.□

In geometric formulation, the particle number 4-current and energy-momentum tensor respectively read

$$\mathsf{N}_a(\mathsf{x}) = n_a(\mathsf{x})\mathsf{u}(\mathsf{x}) \tag{VIII.18a}$$

and

$$\mathbf{T}(\mathsf{x}) = \mathcal{P}(\mathsf{x})\,\mathbf{g}^{-1}(\mathsf{x}) + \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\frac{\mathsf{u}(\mathsf{x})\otimes\mathsf{u}(\mathsf{x})}{c^2}. \tag{VIII.18b}$$

The latter is very reminiscent of the 3-dimensional non-relativistic momentum flux density (III.25); similarly, the reader may also compare their component-wise expressions (III.24b) and (VIII.17a).

**Remarks:**

∗ The energy-momentum tensor is obviously symmetric—which is a non-trivial physical statement. For instance, the identity $T^{i0} = T^{0i}$ means that ($1/c$ times) the energy flux density in direction $i$ equals ($c$ times) the density of the $i$-th component of momentum—where one may rightly argue that the factors of $c$ are historical accidents in the choice of units. This is possible in a relativistic theory only because the energy density also contains the mass energy.

∗ In Eq. (VIII.17b) or (VIII.18b), the sum $\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})$ is equivalently the *enthalpy density* $w(\mathsf{x})$.

∗ Equation (VIII.17b), (VIII.18b) or (VIII.19a) below represents the most general symmetric $\binom{2}{0}$-tensor that can be constructed using only the metric tensor and the 4-velocity.

The component form (VIII.17b) of the energy-momentum tensor can trivially be recast as

$$T^{\mu\nu}(\mathsf{x}) = \epsilon(\mathsf{x}) \frac{u^\mu(\mathsf{x})\, u^\nu(\mathsf{x})}{c^2} + \mathcal{P}(\mathsf{x}) \Delta^{\mu\nu}(\mathsf{x}) \tag{VIII.19a}$$

with

$$\Delta^{\mu\nu}(\mathsf{x}) \equiv g^{\mu\nu}(\mathsf{x}) + \frac{u^\mu(\mathsf{x})\, u^\nu(\mathsf{x})}{c^2} \tag{VIII.19b}$$

the components of a tensor $\boldsymbol{\Delta}$ which—in its $\binom{1}{1}$-form—is actually a projector on the 3-dimensional space orthogonal to the 4-velocity $\mathsf{u}(\mathsf{x})$, while $u^\mu(\mathsf{x})\, u^\nu(\mathsf{x})/c^2$ projects on the time-like direction of the 4-velocity.

One easily checks the identities $\Delta^\mu{}_\nu(\mathsf{x})\Delta^\nu{}_\rho(\mathsf{x}) = \Delta^\mu{}_\rho(\mathsf{x})$ and $\Delta^\mu{}_\nu(\mathsf{x}) u^\nu(\mathsf{x}) = 0$.

From Eq. (VIII.19a) follows at once that the comoving pressure $\mathcal{P}(\mathsf{x})$ can be found in any reference frame as

$$\mathcal{P}(\mathsf{x}) = \frac{1}{3}\Delta_{\mu\nu}(\mathsf{x}) T^{\mu\nu}(\mathsf{x}). \tag{VIII.20}$$

which complements relations (VIII.14) and (VIII.15) for the charge density and energy density.

**Remark:** Contracting the energy-momentum tensor $\mathbf{T}$ with the metric tensor twice yields a scalar, the so-called *trace* of $\mathbf{T}$

$$\mathbf{T}(\mathsf{x}) : \mathbf{g}(\mathsf{x}) = T^{\mu\nu}(\mathsf{x}) g_{\mu\nu}(\mathsf{x}) = T^\mu{}_\mu(\mathsf{x}) = 3\mathcal{P}(\mathsf{x}) - \epsilon(\mathsf{x}). \tag{VIII.21}$$

## VIII.3.2 Entropy in a perfect fluid

Let $s(\mathsf{x})$ denote the (comoving) entropy density of the fluid, as defined in the local rest frame $\mathrm{LR}(\mathsf{x})$ at point $\mathsf{x}$.

### VIII.3.2 a Entropy conservation

For a perfect fluid, the fundamental equations of motion (VIII.2) and (VIII.7) lead automatically to the *local conservation of entropy*

$$\boxed{\mathrm{d}_\mu\big[s(\mathsf{x}) u^\mu(\mathsf{x})\big] = 0} \tag{VIII.22}$$

with $s(\mathsf{x})u^\mu(\mathsf{x})$ the *entropy four-current*.

Proof: The relation $U = TS - \mathcal{P}\mathcal{V} + \sum \mu_a N_a$ with $U$ resp. $\mu_a$ the internal energy resp. the chemical potential associated to the conserved charge of type $a$, gives for the local thermodynamic densities $\epsilon = Ts - \mathcal{P} + \sum \mu_a n_a$. Inserting this expression of the energy density in Eq. (VIII.17b) yields (dropping the $\mathsf{x}$ variable for the sake of brevity):

$$T^{\mu\nu} = \mathcal{P} g^{\mu\nu} + \big(Ts + \sum \mu_a n_a\big) \frac{u^\mu u^\nu}{c^2} = \mathcal{P} g^{\mu\nu} + \big[T(su^\mu) + \sum \mu_a (n_a u^\mu)\big] \frac{u^\nu}{c^2}.$$

Letting the 4-gradient $\mathrm{d}_\mu$ act on both sides of this identity gives

$$\mathrm{d}_\mu T^{\mu\nu} = \mathrm{d}^\nu \mathcal{P} + \big[T(su^\mu) + \sum \mu_a (n_a u^\mu)\big] \frac{\mathrm{d}_\mu u^\nu}{c^2} + \big[s\, \mathrm{d}_\mu T + \sum n_a\, \mathrm{d}_\mu \mu_a\big] \frac{u^\mu u^\nu}{c^2}$$
$$+ \big[T\, \mathrm{d}_\mu(su^\mu) + \sum \mu_a \mathrm{d}_\mu(n_a u^\mu)\big] \frac{u^\nu}{c^2}.$$

Invoking the energy-momentum conservation equation (VIII.7), the left member of this identity vanishes. The second term between square brackets on the right hand side can be rewritten with the help of the Gibbs–Duhem relation as $s\,\mathrm{d}_\mu T + \sum n_a\,\mathrm{d}_\mu \mu_a = \mathrm{d}_\mu \mathcal{P}$. Eventually, the charge conservation equation (VIII.7) can be used in the rightmost term. Multiplying everything by $u_\nu$ and summing over $\nu$ yields

$$0 = u_\nu\,\mathrm{d}^\nu\mathcal{P} + \left[T(su^\mu) + \sum \mu_a(n_a u^\mu)\right]\frac{u_\nu\,\mathrm{d}_\mu u^\nu}{c^2} + (\mathrm{d}_\mu\mathcal{P})\frac{u^\mu u^\nu u_\nu}{c^2} + \left[T\,\mathrm{d}_\mu(su^\mu)\right]\frac{u_\nu u^\nu}{c^2}.$$

The constant normalization $u_\nu u^\nu = -c^2$ of the 4-velocity implies $u_\nu\,\mathrm{d}_\mu u^\nu = 0$ for $\mu = 0,\dots,3$, so that the equation becomes

$$0 = u_\nu\,\mathrm{d}^\nu\mathcal{P} - (\mathrm{d}_\mu\mathcal{P})u^\mu - T\,\mathrm{d}_\mu(su^\mu),$$

leading to $\mathrm{d}_\mu(su^\mu) = 0$ since the first two terms cancel each other. □

### VIII.3.2 b Isentropic distribution

Let $n(\mathsf{x})$ denote the (local) comoving density of one of the conserved charges. The local conservation of entropy (VIII.22) implies the conservation of the ratio $s(\mathsf{x})/n(\mathsf{x})$, conveniently referred to as the entropy per particle, along the motion.

Proof: The total time derivative of the entropy per particle reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{s}{n}\right) = \frac{\partial}{\partial t}\left(\frac{s}{n}\right) + \vec{\mathrm{v}}\cdot\vec{\nabla}\left(\frac{s}{n}\right) = \frac{1}{\gamma}\mathsf{u}\cdot\mathsf{d}\left(\frac{s}{n}\right),$$

where the second identity makes use of Eq. (VIII.11), with $\gamma$ the Lorentz factor. The rightmost term is then

$$\mathsf{u}\cdot\mathsf{d}\left(\frac{s}{n}\right) = \frac{1}{n}\mathsf{u}\cdot\mathsf{d}s - \frac{s}{n^2}\mathsf{u}\cdot\mathsf{d}n = \frac{1}{n}\left(\mathsf{u}\cdot\mathsf{d}s - \frac{s}{n}\mathsf{u}\cdot\mathsf{d}n\right).$$

The continuity equation $\mathsf{d}\cdot(n\mathsf{u}) = 0$ gives $\mathsf{u}\cdot\mathsf{d}n = -n\,\mathsf{d}\cdot\mathsf{u}$, implying

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{s}{n}\right) = \frac{1}{\gamma}\mathsf{u}\cdot\mathsf{d}\left(\frac{s}{n}\right) = \frac{1}{\gamma n}\left(\mathsf{u}\cdot\mathsf{d}s + s\,\mathsf{d}\cdot\mathsf{u}\right) = \frac{1}{\gamma n}\mathsf{d}\cdot(s\mathsf{u}) = 0,$$

where the last identity expresses the conservation of entropy (VIII.22). □