VI.1.4 Absorption of sound waves

In chapter V, we only considered incompressible motions of Newtonian fluids, so that bulk viscosity could from the start play no role. The simplest example of compressible flow is that of sound waves. As in § VI.1.1, we consider small adiabatic perturbations of a fluid initially at rest and with uniform properties—which implies that external volume forces like gravity are neglected. Accordingly, the dynamical fields characterizing the fluid are

$$\rho(t,\vec{r}) = \rho_0 + \delta\rho(t,\vec{r}), \quad \mathcal{P}(t,\vec{r}) = \mathcal{P}_0 + \delta\mathcal{P}(t,\vec{r}), \quad \vec{\mathrm{v}}(t,\vec{r}) = \vec{0} + \delta\vec{\mathrm{v}}(t,\vec{r}), \tag{VI.11a}$$

with

$$|\delta\rho(t,\vec{r})| \ll \rho_0, \quad |\delta\mathcal{P}(t,\vec{r})| \ll \mathcal{P}_0, \quad \left|\delta\vec{\mathrm{v}}(t,\vec{r})\right| \ll c_s, \tag{VI.11b}$$

where c_s denotes the quantity which in the perfect-fluid case was found to coincide with the phase velocity of similar small perturbations, i.e. the "speed of sound", defined by Eq. (VI.5)

$$c_s^2 \equiv \left(\frac{\partial\mathcal{P}}{\partial\rho}\right)_{S,N}. \tag{VI.11c}$$

As in § VI.1.1, this partial derivative will allow us to relate the pressure perturbation $\delta\mathcal{P}$ to the variation of mass density $\delta\rho$.

Remark: Anticipating on later findings, the perturbations must actually fulfill a further condition, related to the size of their spatial variations [cf. Eq. (VI.21)]. This is nothing but the assumption of "small gradients" that underlies the description of their propagation with the Navier–Stokes equation, i.e. with first-order dissipative fluid dynamics.

For the sake of simplicity, we consider a one-dimensional problem, i.e. perturbations propagating along the x-direction and independent of y and z—as are the properties of the underlying background fluid. Under this assumption, the continuity equation (III.12) reads

$$\frac{\partial\rho(t,x)}{\partial t} + \rho(t,x)\frac{\partial\delta\mathsf{v}(t,x)}{\partial x} + \delta\mathsf{v}(t,x)\frac{\partial\rho(t,x)}{\partial x} = 0, \tag{VI.12a}$$

while the Navier–Stokes equation (III.35) becomes

$$\rho(t,x)\left[\frac{\partial\delta\mathsf{v}(t,x)}{\partial t} + \delta\mathsf{v}(t,x)\frac{\partial\delta\mathsf{v}(t,x)}{\partial x}\right] = -\frac{\partial\delta\mathcal{P}(t,x)}{\partial x} + \left(\frac{4}{3}\eta+\zeta\right)\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial x^2}. \tag{VI.12b}$$

Substituting the fields (VI.11a) in these equations and linearizing the resulting equations so as to keep only the leading order in the small perturbations, one finds

$$\frac{\partial\delta\rho(t,x)}{\partial t} + \rho_0\frac{\partial\delta\mathsf{v}(t,x)}{\partial x} = 0, \tag{VI.13a}$$

$$\rho_0\frac{\partial\delta\mathsf{v}(t,x)}{\partial t} = -\frac{\partial\delta\mathcal{P}(t,x)}{\partial x} + \left(\frac{4}{3}\eta+\zeta\right)\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial x^2}. \tag{VI.13b}$$

In the second equation, the derivative $\partial(\delta\mathcal{P})/\partial x$ can be replaced by $c_s^2\,\partial(\delta\rho)/\partial x$. Let us in addition introduce the (traditional) notation[(36)]

$$\bar{\nu} \equiv \frac{1}{\rho_0}\left(\frac{4}{3}\eta+\zeta\right), \tag{VI.14}$$

so that Eq. (VI.13b) can be rewritten as

$$\rho_0\frac{\partial\delta\mathsf{v}(t,x)}{\partial t} + c_s^2\frac{\partial\delta\rho(t,x)}{\partial x} = \rho_0\bar{\nu}\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial x^2}. \tag{VI.15}$$

Subtracting c_s^2 times the derivative of Eq. (VI.13a) with respect to x from the time derivative of Eq. (VI.15) and dividing the result by ρ_0 then yields

$$\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial t^2} - c_s^2\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial x^2} = \bar{\nu}\frac{\partial^3\delta\mathsf{v}(t,x)}{\partial t\,\partial x^2}. \tag{VI.16a}$$

One easily checks that the mass density variation obeys a similar equation

$$\boxed{\frac{\partial^2\delta\rho(t,x)}{\partial t^2} - c_s^2\frac{\partial^2\delta\rho(t,x)}{\partial x^2} = \bar{\nu}\frac{\partial^3\delta\rho(t,x)}{\partial t\,\partial x^2}.} \tag{VI.16b}$$

In the perfect-fluid case $\bar{\nu}=0$, one recovers the traditional wave equation (VI.10a).

Equations (VI.16) are homogeneous linear partial differential equations, whose solutions can be written as superposition of plane waves. Accordingly, let us substitute the Fourier ansatz

$$\delta\rho(t,x) = \widetilde{\delta\rho}(\omega,k)\,\mathrm{e}^{-\mathrm{i}(\omega t-kx)} \tag{VI.17}$$

in Eq. (VI.16b). This yields after some straightforward algebra the dispersion relation

$$\boxed{\omega^2 = c_s^2k^2 - \mathrm{i}\omega k^2\bar{\nu}.} \tag{VI.18}$$

Obviously, ω and k cannot be simultaneously (non-zero) real numbers.

(36) Introducing the kinetic shear resp. bulk viscosity coefficients ν resp. ν', one has $\bar{\nu} = \frac{4}{3}\nu + \nu'$, hence the notation.

Let us assume $k \in \mathbb{R}$ and $\omega = \omega_r + \mathrm{i}\omega_i$, where ω_r, ω_i are real. The dispersion relation becomes

$$\omega_r^2 - \omega_i^2 + 2\mathrm{i}\omega_r\omega_i = c_s^2 k^2 - \mathrm{i}\omega_r k^2 \bar{\nu} + \omega_i k^2 \bar{\nu},$$

which can only hold if both the real and imaginary parts are separately equal. The identity between the imaginary parts reads (for $\omega_r \neq 0$)

$$\omega_i = -\frac{1}{2}\bar{\nu}k^2, \tag{VI.19}$$

which is always negative, since $\bar{\nu}$ is non-negative. This term yields in the Fourier ansatz (VI.17) an exponentially decreasing factor $\mathrm{e}^{-\mathrm{i}(\mathrm{i}\omega_i)t} = \mathrm{e}^{-\bar{\nu}k^2t/2}$ which represents the *damping* or *absorption* of the sound wave. The perturbations with larger wave number k, i.e. corresponding to smaller length scales, are more dampened that those with smaller k. This is quite natural, since a larger k also means a larger gradient, thus an increased influence of the viscous term in the Navier–Stokes equation.

In turn, the identity between the real parts of the dispersion relation yields

$$\omega_r^2 = c_s^2 k^2 - \frac{1}{4}\bar{\nu}^2 k^4. \tag{VI.20}$$

This gives for the phase velocity $c_\varphi \equiv \omega/k$ of the traveling waves

$$c_\varphi^2 = c_s^2 - \frac{1}{4}\bar{\nu}^2 k^2. \tag{VI.21}$$

That is, the "speed of sound" actually depends on its wave number k, and is smaller for small wavelength, i.e. high-k, perturbations—which are also those which are more damped out.

Relation (VI.21) also shows that the whole linear description adopted below Eqs. (VI.12) requires that the perturbations should have a relatively large wavelength, namely such that $k \ll 2c_s/\bar{\nu}$, so that c_φ remain real-valued. This is equivalent to requesting that the dissipative term $\bar{\nu}\triangle\delta\mathsf{v} \sim k^2\bar{\nu}\delta\mathsf{v}$ in the Navier–Stokes equation (VI.13b) should be much smaller than the term describing the local acceleration, $\partial_t\delta\mathsf{v} \sim \omega\delta\mathsf{v} \sim c_s k\delta\mathsf{v}$.

Remarks:

∗ Instead of considering "temporal damping" as was done above by assuming $k \in \mathbb{R}$ but $\omega \in \mathbb{C}$, one may investigate "spatial damping", i.e. assume $\omega \in \mathbb{R}$ and put the whole complex dependence in the wave number $k = k_r + \mathrm{i}k_i$. For (angular) frequencies ω much smaller than the inverse of the typical time scale $\tau_\nu \equiv \bar{\nu}/c_s^2$, one finds

$$\omega^2 \simeq c_s^2 k_r^2 \left(1 + \frac{3}{4}\omega^2\tau_\nu^2\right) \quad \Leftrightarrow \quad c_\varphi \equiv \frac{\omega}{k_r} \simeq c_s\left(1 + \frac{3}{8}\omega^2\tau_\nu^2\right)$$

i.e. the phase velocity increases with the frequency, and on the other hand

$$k_i \simeq \frac{\bar{\nu}\omega^2}{2c_s^3}. \tag{VI.22}$$

The latter relation is known as *Stokes' law of sound attenuation*, k_i representing the inverse of the typical distance over which the sound wave amplitude decreases, due to the factor $\mathrm{e}^{\mathrm{i}(\mathrm{i}k_i)x} = \mathrm{e}^{-k_i x}$ in the Fourier ansatz (VI.17). Larger frequencies are thus absorbed on a smaller distance from the source of the sound wave.

Substituting $k = k_r + \mathrm{i}k_i = k_r(1 + \mathrm{i}\varkappa)$ in the dispersion relation (VI.18) and writing the identity of the real and imaginary parts, one obtains the system

$$\begin{cases} 2\varkappa = \omega\tau_\nu(1 - \varkappa^2) \\ \omega^2 = c_s^2 k_r^2(1 + 2\omega\tau_\nu\varkappa - \varkappa^2) \end{cases}$$

The first equation is a quadratic equation in $\varkappa$ that admits one positive and one negative solution: the latter can be rejected, while the former is $\varkappa \simeq \omega\tau_\nu/2 + \mathcal{O}\big((\omega\tau_\nu)^2\big)$. Inserting it in the second equation leads to the wanted results. □

An exact solution of the system of equations exists, yes it is neither enlightening mathematically, nor relevant from the physical point of view in the general case, as discussed in the next remark.

One may naturally also analyze the general case in which both ω and k are complex numbers. In any case, the phase velocity is given by $c_\varphi \equiv \omega/k_r$, although it is more difficult to recognize the physical content of the mathematical relations.

$\ast$ For air or water, the reduced kinetic viscosity $\bar{\nu}$ is of order 10^{-6}–10^{-5} $\mathrm{m}^2 \cdot \mathrm{s}^{-1}$. With speeds of sound $c_s \simeq 300$–$1500\ \mathrm{m} \cdot \mathrm{s}^{-1}$, this yields typical time scales τ_ν of order 10^{-12}–10^{-10} s. That is, the change in the speed of sound (VI.21), or equivalently deviations from the assumption $\omega\tau_\nu \ll 1$ underlying the attenuation coefficient (VI.22), become relevant for sound waves in the gigahertz/terahertz regime(!). This explains why measuring the bulk viscosity is a non-trivial task.

The wavelengths $c_s\tau_\nu$ corresponding to the above frequencies τ_ν^{-1} are of order 10^{-9}–10^{-7} m. This is actually not far from the value of the mean free path in classical fluids, so that the whole description as a continuous medium starts being questionable.