V.1.3 Plane Poiseuille flow

Let us now consider the flow of a Newtonian fluid between two motionless plane plates with a finite length along the x direction—yet still infinitely extended along the z direction—, as illustrated in Fig. V.2. The pressure is assumed to be different at both ends of the plates in the x direction, leading to the presence of a pressure gradient along x.

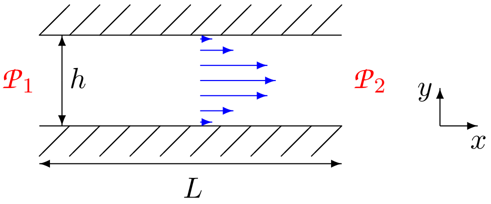

Figure V.2 – Flow between two motionless plates for $\mathcal{P}_1 > \mathcal{P}_2$, i.e. $\delta\mathcal{P} > 0$.

Assuming for the flow velocity $\vec{\mathsf{v}}(\vec{r})$ the same form $\mathsf{v}(y)\,\vec{\mathrm{e}}_x$, independent of x, as in the case of the plane Couette flow, the equations of motion governing $\mathsf{v}(y)$ and pressure $\mathcal{P}(\vec{r})$ are the same as in the previous § V.1.2, namely Eqs. (V.3)–(V.4). The boundary conditions are however different. Thus, $\mathcal{P}_1 \neq \mathcal{P}_2$ results in a finite constant pressure gradient along x, $\alpha = \partial\mathcal{P}(\vec{r})/\partial x = -\delta\mathcal{P}/L \neq 0$, with $\delta\mathcal{P} \equiv \mathcal{P}_1 - \mathcal{P}_2$ the pressure drop. Equation (V.4) then leads to

$$\mathsf{v}(y) = -\frac{1}{2\eta}\frac{\delta\mathcal{P}}{L}y^2 + \gamma y + \delta,$$

with γ and δ two new constants.

The "no-slip" boundary conditions for the velocity at the two plates read

$$\mathsf{v}(y\!=\!0) = 0, \qquad \mathsf{v}(y\!=\!h) = 0,$$

which leads to $\delta = 0$ and $\gamma = \dfrac{1}{2\eta}\dfrac{\delta\mathcal{P}}{L}h$. The flow velocity thus has the parabolic profile

$$\mathsf{v}(y) = \frac{1}{2\eta}\frac{\delta\mathcal{P}}{L}\big[y(h-y)\big] \quad \text{for } 0 \leq y \leq h, \tag{V.5}$$

directed along the direction of the pressure gradient.

Remark: The flow velocity (V.5) becomes clearly problematic in the limit $\eta \to 0$! Tracing the problem back to its source, the equations of motion (V.3) cannot hold with a finite pressure gradient along the x direction and a vanishing viscosity. One quickly checks that the only possibility in the case of a perfect fluid is to drop one of the assumptions, either incompressibility or laminarity.

V.1.4 Hagen–Poiseuille flow

The previous two examples involved plates with an infinite length in at least one direction, thus were idealized constructions. In contrast, an experimentally realizable fluid motion is that of the *Hagen*(aw)*–Poiseuille flow*, in which a Newtonian fluid flows under the influence of a pressure gradient in a cylindrical tube with finite length L and radius a (Fig. V.3). Again, the motion is assumed to be steady, incompressible and laminar.

(aw) G. Hagen, 1797–1884

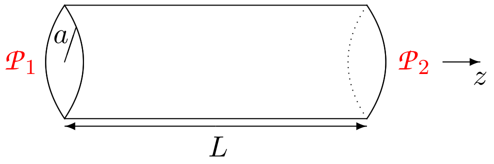

Figure V.3 – Setup of the Hagen–Poiseuille flow.

Using cylindrical coordinates, the ansatz $\vec{\mathsf{v}}(\vec{r}) = \mathsf{v}(r)\,\vec{\mathrm{e}}_z$ with $r = \sqrt{x^2+y^2}$ satisfies the continuity equation $\vec{\nabla}\cdot\vec{\mathsf{v}}(\vec{r}) = 0$ and gives for the incompressible Navier–Stokes equation

$$\vec{\nabla}\mathcal{P}(\vec{r}) = \eta\triangle\vec{\mathsf{v}}(\vec{r}) \quad\Leftrightarrow\quad \begin{cases} \dfrac{\partial\mathcal{P}(\vec{r})}{\partial x} = \dfrac{\partial\mathcal{P}(\vec{r})}{\partial y} = 0 \\ \dfrac{\partial\mathcal{P}(\vec{r})}{\partial z} = \eta\left[\dfrac{\partial^2\mathsf{v}(r)}{\partial x^2} + \dfrac{\partial^2\mathsf{v}(r))}{\partial y^2}\right] = \eta\left[\dfrac{\mathrm{d}^2\mathsf{v}(r)}{\mathrm{d}r^2} + \dfrac{1}{r}\dfrac{\mathrm{d}\mathsf{v}(r)}{\mathrm{d}r}\right]. \end{cases} \tag{V.6}$$

The right member of the equation in the second line is independent of z, implying that the pressure gradient along the z direction is constant. Using the boundary conditions yields

$$\frac{\partial\mathcal{P}(\vec{r})}{\partial z} = -\frac{\delta\mathcal{P}}{L},$$

with $\delta\mathcal{P} \equiv \mathcal{P}_1 - \mathcal{P}_2$. The z component of the Navier–Stokes equation (V.6) thus becomes

$$\frac{\mathrm{d}^2\mathsf{v}(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\mathsf{v}}{\mathrm{d}r} = -\frac{\delta\mathcal{P}}{\eta L}. \tag{V.7}$$

As always, this linear differential equation is solved in two successive steps, starting with the associated homogeneous equation. To find the general solution of the latter, one may introduce $\chi(r) \equiv \mathrm{d}\mathsf{v}(r)/\mathrm{d}r$, which obeys the simpler equation

$$\frac{\mathrm{d}\chi(r)}{\mathrm{d}r} + \frac{\chi(r)}{r} = 0.$$

The generic solution is $\ln\chi(r) = -\ln r + \text{const.}$, i.e. $\chi(r) = A/r$ with A a constant. This then leads to $\mathsf{v}(r) = A\ln r + B$ with B an additional constant.

A particular solution of the inhomogeneous equation (V.7) is $\mathsf{v}(r) = Cr^2$ with $C = -\delta\mathcal{P}/4\eta L$. The general solution of Eq. (V.7) is then given by

$$\mathsf{v}(r) = A\ln r + B - \frac{\delta\mathcal{P}}{4\eta L}r^2,$$

where the two integration constants still need to be determined.

To have a regular flow velocity at $r = 0$, the constant A should vanish. In turn, the boundary condition at the tube wall, $\mathsf{v}(r{=}a) = 0$, determines the value of the constant $B = (\delta\mathcal{P}/4\eta L)a^2$. All in all, the velocity profile thus reads

$$\mathsf{v}(r) = \frac{\delta\mathcal{P}}{4\eta L}\left(a^2 - r^2\right) \quad \text{for } r \leq a. \tag{V.8}$$

This is again parabolic, with $\vec{\mathsf{v}}$ pointing in the same direction as the pressure drop.

The mass flow rate across the tube cross section follows from a straightforward integration:

$$Q = \int_0^a \rho\mathsf{v}(r)\,2\pi r\,\mathrm{d}r = 2\pi\rho\frac{\delta\mathcal{P}}{4\eta L}\int_0^a \left(a^2 r - r^3\right)\mathrm{d}r = 2\pi\rho\frac{\delta\mathcal{P}}{4\eta L}\frac{a^4}{4} = \frac{\pi\rho a^4}{8\eta}\frac{\delta\mathcal{P}}{L}. \tag{V.9}$$

This result, known as *Hagen–Poiseuille law* (or equation), shows that the mass flow rate is proportional to the pressure drop per unit length.

Remarks:

$*$ The Hagen–Poiseuille law only holds under the assumption that the flow velocity vanishes at the tube walls. The experimental confirmation of the law—which was actually deduced from experiment by Hagen (1839) and Poiseuille (1840)—is thus a proof of the validity of the no-slip assumption for the boundary condition.

$\ast$ The mass flow rate across the tube cross section may be used to define the average flow velocity such that $Q = \pi a^2 \rho \langle \mathsf{v} \rangle$ with

$$\langle \mathsf{v} \rangle \equiv \frac{1}{\pi a^2} \int_0^a \mathsf{v}(r)\, 2\pi r \,\mathrm{d}r = \frac{1}{2}\mathsf{v}(r\!=\!0).$$

The Hagen–Poiseuille law then expresses a proportionality between the pressure drop per unit length and $\langle \mathsf{v} \rangle$ in a laminar flow.

> Viewing $\delta\mathcal{P}/L$ as the "generalized force" driving the motion, the corresponding "response" $\langle \mathsf{v} \rangle$ of the fluid is thus linear.

The relation is quite different in the case of a *turbulent* flow with the same geometry: for instance, measurements by Reynolds [21] gave $\delta\mathcal{P}/L \propto \langle \mathsf{v} \rangle^{1.722}$.