V.1 Statics and steady laminar flows of a Newtonian fluid

In this Section, we first write down the equations governing the statics of a Newtonian fluid (§ V.1.1), then we investigate a few idealized stationary laminar fluid motions, in which the velocity field is entirely driven by the no-slip condition at boundaries (§ V.1.2–V.1.4).

V.1.1 Static Newtonian fluid

Consider a motionless $[\vec{\mathsf{v}}(t,\vec{r}) = \vec{0}]$ Newtonian fluid in an external gravitational potential $\Phi(\vec{r})$—or more generally, submitted to conservative volume forces such that $\vec{f}_V(t,\vec{r}) = -\rho(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r})$. The three coupled equations (III.12), (III.35) and (III.40) respectively simplify to

$$\frac{\partial\rho(t,\vec{r})}{\partial t} = 0, \tag{V.1a}$$

which means that the mass density $\rho(t,\vec{r})$ is time independent,

$$\vec{\nabla}\mathcal{P}(t,\vec{r}) = -\rho(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}), \tag{V.1b}$$

identical to the fundamental equation (IV.2) governing the hydrostatics of a perfect fluid, and

$$\frac{\partial e(t,\vec{r})}{\partial t} = \vec{\nabla}\cdot\big[\kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r})\big], \tag{V.1c}$$

which describes the transport of energy without macroscopic fluid motion, i.e. non-convectively, thanks to *heat conduction*.

Given an equation of state relating the internal energy density to the temperature, Eq. (V.1c) can become an equation for $T(t,\vec{r})$ only, in particular if the various thermodynamic and transport coefficients involved are assumed to be uniform across the fluid.

V.1.2 Plane Couette flow

In the example of this Section and the next two ones (§ V.1.3–V.1.4), we consider steady, incompressible, laminar flows, in absence of significant volume forces. Since the mass density ρ is fixed, thus known, only four equations are needed to determine the flow velocity $\vec{\mathsf{v}}(\vec{r})$ and pressure $\mathcal{P}(\vec{r})$, the simplest possibility being to use the continuity and Navier–Stokes equations. In the stationary and incompressible regime, these become

$$\vec{\nabla}\cdot\vec{\mathsf{v}}(\vec{r}) = 0 \tag{V.2a}$$

$$\big[\vec{\mathsf{v}}(\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(\vec{r}) = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}(\vec{r}) + \nu\triangle\vec{\mathsf{v}}(\vec{r}), \tag{V.2b}$$

with ν the kinematic shear viscosity, assumed to be the same throughout the fluid.

The so-called (plane) *Couette*[as] *flow* is, in its idealized version, the motion of a viscous fluid between two infinitely extended plane plates, as represented in Fig. V.1, where the lower plate is at rest, while the upper one moves in its own plane with a constant velocity $\vec{\mathsf{u}}$. It will be assumed that the same pressure $\mathcal{P}_\infty$ holds "at infinity" in any direction in the (x,z)-plane.

[as] M. Couette, 1858–1943

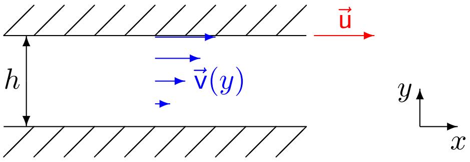

Figure V.1 – Setup of the plane Couette flow.

Since the flow is assumed to be laminar, the geometry of the problem is invariant under arbitrary translations in the (x,z)-plane. This is automatically accounted for by the ansatz $\vec{\mathsf{v}}(\vec{r}) = \mathsf{v}(y)\,\vec{\mathrm{e}}_x$ for the flow velocity. Inserting this form in Eqs. (V.2) yields

$$\frac{\partial\mathsf{v}(y)}{\partial x} = 0, \tag{V.3a}$$

$$\mathsf{v}(y)\frac{\partial\mathsf{v}(y)}{\partial x}\,\vec{\mathrm{e}}_x = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}(\vec{r}) + \nu\frac{\mathrm{d}^2\mathsf{v}(y)}{\mathrm{d}y^2}\,\vec{\mathrm{e}}_x. \tag{V.3b}$$

With the ansatz for $\vec{\mathsf{v}}(\vec{r})$, the first equation is automatically fulfilled, while the term on the left hand side of the second equation vanishes. Projecting the latter on the y and z directions thus yields $\partial\mathcal{P}(\vec{r})/\partial y = 0$—expressing the assumed absence of sizable effects from gravity—and $\partial\mathcal{P}(\vec{r})/\partial z = 0$—since the problem is independent of z. Along the x direction, one finds

$$\frac{\partial\mathcal{P}(\vec{r})}{\partial x} = \eta\frac{\mathrm{d}^2\mathsf{v}(y)}{\mathrm{d}y^2}. \tag{V.4}$$

Since the right member of this equation is independent of x and z, a straightforward integration gives $\mathcal{P}(\vec{r}) = \alpha(y)x + \beta(y)$, where the functions α, β only depend on y. These functions are determined by the boundary conditions: from $\mathcal{P}(x{=}-\infty) = \mathcal{P}(x{=}\infty) = \mathcal{P}_\infty$ follow $\alpha(y) = 0$, $\beta(y) = \mathcal{P}_\infty$, and Eq. (V.4) eventually simplifies to

$$\frac{\mathrm{d}^2\mathsf{v}(y)}{\mathrm{d}y^2} = 0.$$

This yields $\mathsf{v}(y) = \gamma y + \delta$, with two integration constants γ and δ that are again fixed by the boundary conditions. At each plate, the relative velocity of the fluid with respect to the plate must vanish:

$$\vec{\mathsf{v}}(y{=}0) = 0, \qquad \vec{\mathsf{v}}(y{=}h) = \vec{\mathsf{u}},$$

leading to $\delta = 0$ and $\gamma = |\vec{u}|/h$. All in all, the velocity thus depends linearly on y

$$\vec{\mathsf{v}}(\vec{r}) = \frac{y}{h}\,\vec{\mathsf{u}} \quad \text{for } 0 \leq y \leq h.$$

Consider now a surface element $\mathrm{d}^2\mathcal{S}$. The contact force $\mathrm{d}^2\vec{F}_s$ exerted on it by the fluid follows from the Cauchy stress tensor, whose Cartesian components (III.31c) here read

$$\sigma^{ij}(\vec{r}) = -\mathcal{P}(\vec{r})\delta^{ij} + \eta\left[\frac{\partial\mathsf{v}^i(\vec{r})}{\partial x_j} + \frac{\partial\mathsf{v}^j(\vec{r})}{\partial x_i}\right] \cong \begin{pmatrix} -\mathcal{P}_\infty & \eta\frac{|\vec{\mathsf{u}}|}{h} & 0 \\ \eta\frac{|\vec{\mathsf{u}}|}{h} & -\mathcal{P}_\infty & 0 \\ 0 & 0 & -\mathcal{P}_\infty \end{pmatrix}.$$

The force per unit area on the motionless plate at $y = 0$, corresponding to a unit normal vector $\vec{\mathrm{e}}_n(\vec{r}) = \vec{\mathrm{e}}_y$, is

$$\frac{\mathrm{d}^2\vec{F}_s(\vec{r})}{\mathrm{d}^2\mathcal{S}} = \vec{\mathcal{T}}_s(\vec{r}) = \left[\sum_{i,j=1}^{3}\sigma^{ij}(\vec{r})\,\vec{\mathrm{e}}_i\otimes\vec{\mathrm{e}}_j\right]\cdot\vec{\mathrm{e}}_y = \sum_{i,j=1}^{3}\sigma^{ij}(\vec{r})\big(\vec{\mathrm{e}}_j\cdot\vec{\mathrm{e}}_y\big)\vec{\mathrm{e}}_i = \begin{pmatrix} \eta\frac{|\vec{\mathsf{u}}|}{h} \\ -\mathcal{P}_\infty \\ 0 \end{pmatrix}.$$

Due to the friction exerted by the fluid, the lower plate is dragged by the flow in the (positive) x direction. In turn, there is also the normal stress $-\mathcal{P}_\infty\vec{\mathrm{e}}_y$ due to pressure, oriented towards the negative y-direction as could be anticipated.

Remark: The tangential stress on the lower plate is $\eta\vec{\mathsf{u}}/h$, proportional to the shear viscosity: measuring the tangential stress with known $|\vec{\mathsf{u}}|$ and h provides a measurement of the dynamical shear viscosity η. In practice, this measurement rather involves the more realistic setup with the fluid in the space between two concentric cylinders, one of which is at rest while the other is rotating, leading to the so-called *cylindrical Couette flow* (or *Taylor*(at)*–Couette flow*).

(at) G. I. Taylor, 1886–1975