

III.4.2 Energy conservation in Newtonian fluids

In a real fluid, there exist further mechanisms for transporting energy besides the convective transport due to the fluid motion, namely *diffusion*, either of momentum or of energy:

- On the one hand, the viscous friction forces in the fluid, which already lead to the transport of momentum between neighboring fluid layers moving with different velocities, exert some work in the motion, which induces a diffusive transport of energy. This is accounted for by a contribution $\boldsymbol{\pi} \cdot \vec{v}$ to the energy flux density—component-wise, a contribution $\sum_j \pi_j^i v^j$ to the i -th component of the flux density—, with $\boldsymbol{\pi}$ the viscous stress tensor, given in the case of a Newtonian fluid by Eq. (III.30c).

- On the other hand, there is also *heat conduction* from the regions with higher temperatures towards those with lower temperatures. This transport is described by the introduction in the energy flux density of a heat current ^(xliv) $\vec{j}_Q(t, \vec{r}) = -\kappa(t, \vec{r}) \vec{\nabla} T(t, \vec{r})$ —in accordance with the local formulation of *Fourier's* ^(ab) *law*, see e.g. Sec. 1.2.1 in Ref. [2]—, with κ the *heat conductivity* ^(xlv) of the fluid.

Taking into account these additional contributions, the local formulation of energy conservation in a Newtonian fluid reads

$$\begin{aligned}
 & \frac{\partial}{\partial t} \left[\frac{1}{2} \rho(t, \vec{r}) \vec{v}(t, \vec{r})^2 + e(t, \vec{r}) + \rho(t, \vec{r}) \Phi(t, \vec{r}) \right] \\
 & + \vec{\nabla} \cdot \left\{ \left[\frac{1}{2} \rho(t, \vec{r}) \vec{v}(t, \vec{r})^2 + e(t, \vec{r}) + \mathcal{P}(t, \vec{r}) + \rho(t, \vec{r}) \Phi(t, \vec{r}) \right] \vec{v}(t, \vec{r}) \right. \\
 & \quad \left. - \eta(t, \vec{r}) \left[\left(\vec{v}(t, \vec{r}) \cdot \vec{\nabla} \right) \vec{v}(t, \vec{r}) + \vec{\nabla} \left(\frac{\vec{v}(t, \vec{r})^2}{2} \right) \right] \right. \\
 & \quad \left. - \left[\zeta(t, \vec{r}) - \frac{2\eta(t, \vec{r})}{3} \right] \left[\vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right] \vec{v}(t, \vec{r}) - \kappa(t, \vec{r}) \vec{\nabla} T(t, \vec{r}) \right\} = 0.
 \end{aligned} \tag{III.40}$$

with Φ the potential energy per unit mass due to conservative external volume forces. If the three transport coefficients η , ζ and κ vanish, this equation simplifies to that for perfect fluids, Eq. (III.38).

Remark: The energy flux density can be read off Eq. (III.40), since it represents the term between curly brackets. Dropping the external potential Φ , one can check that it can also be written as

$$\begin{aligned}
 & \left[\frac{1}{2} \rho(t, \vec{r}) \vec{v}(t, \vec{r})^2 + e(t, \vec{r}) + \mathcal{P}(t, \vec{r}) \right] \vec{v}(t, \vec{r}) - 2\eta(t, \vec{r}) \mathbf{S}(t, \vec{r}) \cdot \vec{v}(t, \vec{r}) \\
 & - \zeta(t, \vec{r}) \left[\vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right] \vec{v}(t, \vec{r}) - \kappa(t, \vec{r}) \vec{\nabla} T(t, \vec{r}), \tag{III.41}
 \end{aligned}$$

with $\mathbf{S}(t, \vec{r})$ the traceless symmetric rate-of-shear tensor. One recognizes the various physical sources of energy transport.

III.4.3 Entropy balance in Newtonian fluids

In a real fluid, with viscous friction forces and heat conductivity, one can expect a priori that the transformation of mechanical energy into heat will lead in general to an increase in entropy, at least as long as one considers a closed and isolated system.

Consider a volume \mathcal{V} of flowing Newtonian fluid, delimited by a surface \mathcal{S} at each point \vec{r} of which the boundary conditions

$$\vec{v}(t, \vec{r}) \cdot \vec{e}_n(\vec{r}) = 0 \quad \text{and} \quad \vec{j}_Q(t, \vec{r}) \cdot \vec{e}_n(\vec{r}) = 0$$

hold, where $\vec{e}_n(\vec{r})$ denotes the unit normal vector to \mathcal{S} at \vec{r} . Physically, these boundary conditions mean that neither matter nor heat flows across the surface \mathcal{S} , so that the system inside \mathcal{S} is closed and isolated. To completely exclude energy exchanges with the exterior of \mathcal{S} , it is also assumed that there are no external volume forces acting on the fluid inside volume \mathcal{V} . We shall investigate the implications of the continuity equation (III.13), the Navier–Stokes equation (III.35), and the energy conservation equation (III.40) for the total entropy S of the fluid inside \mathcal{V} . For the sake of brevity, the variables t, \vec{r} will be omitted in the remainder of this Section.

^(xliv) Wärmestromvektor ^(xlv) Wärmeleitfähigkeit

^(ab) J. FOURIER, 1768–1830

Starting with the energy conservation equation (III.40), the contribution

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho \vec{v}^2 \right) + \vec{\nabla} \cdot \left[\left(\frac{1}{2} \rho \vec{v}^2 \right) \vec{v} \right]$$

in its first two lines can be replaced by

$$\rho \vec{v} \cdot \frac{\partial \vec{v}}{\partial t} + \frac{1}{2} \frac{\partial \rho}{\partial t} \vec{v}^2 + \frac{1}{2} \left[\vec{\nabla} \cdot (\rho \vec{v}) \right] \vec{v}^2 + \sum_{i=1}^3 \rho v_i (\vec{v} \cdot \vec{\nabla}) v^i = \sum_{i=1}^3 \rho v_i \left[\frac{\partial v^i}{\partial t} + (\vec{v} \cdot \vec{\nabla}) v^i \right], \quad (\text{III.42a})$$

where the continuity equation (III.12) was used.

As recalled in Appendix ??, the fundamental thermodynamic relation $U = TS - \mathcal{P}\mathcal{V} + \mu N$ gives on the one hand $e + \mathcal{P} = Ts + \mu n$, which leads to

$$\vec{\nabla} \cdot [(e + \mathcal{P}) \vec{v}] = T \vec{\nabla} \cdot (s \vec{v}) + \mu \vec{\nabla} \cdot (n \vec{v}) + \vec{v} \cdot (s \vec{\nabla} T + n \vec{\nabla} \mu) = T \vec{\nabla} \cdot (s \vec{v}) + \mu \vec{\nabla} \cdot (n \vec{v}) + \vec{v} \cdot \vec{\nabla} \mathcal{P}, \quad (\text{III.42b})$$

where the second identity follows from the Gibbs–Duhem relation $d\mathcal{P} = s dT + n d\mu$. On the other hand, it leads to $de = T ds + \mu dn$, which under consideration of the continuity equation for particle number yields

$$\frac{\partial e}{\partial t} = T \frac{\partial s}{\partial t} + \mu \frac{\partial n}{\partial t} = T \frac{\partial s}{\partial t} - \mu \vec{\nabla} \cdot (n \vec{v}). \quad (\text{III.42c})$$

With the help of relations (III.42a)–(III.42c), the energy conservation equation (III.40) can be rewritten as

$$\begin{aligned} & \sum_{i=1}^3 \rho v_i \left[\frac{\partial v^i}{\partial t} + (\vec{v} \cdot \vec{\nabla}) v^i \right] + T \frac{\partial s}{\partial t} + T \vec{\nabla} \cdot (s \vec{v}) + \vec{v} \cdot \vec{\nabla} \mathcal{P} = \\ & \sum_{i,j=1}^3 \frac{\partial}{\partial x^j} \left[\eta \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) v_i \right] + \sum_{i=1}^3 \frac{\partial}{\partial x_i} [\zeta (\vec{\nabla} \cdot \vec{v}) v_i] + \vec{\nabla} \cdot (\kappa \vec{\nabla} T). \end{aligned} \quad (\text{III.42d})$$

Multiplying the i -th component of Eq. (III.34b) by v_i gives

$$\rho v_i \left[\frac{\partial v^i}{\partial t} + (\vec{v} \cdot \vec{\nabla}) v^i \right] + v_i \frac{\partial \mathcal{P}}{\partial x_i} = \sum_{j=1}^3 v_i \frac{\partial}{\partial x^j} \left[\eta \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \right] + v_i \frac{\partial}{\partial x^i} (\zeta \vec{\nabla} \cdot \vec{v}).$$

Subtracting this identity, summed over $i = 1, 2, 3$, from Eq. (III.42d), yields

$$T \frac{\partial s}{\partial t} + T \vec{\nabla} \cdot (s \vec{v}) = \eta \sum_{i,j=1}^3 \frac{\partial v_i}{\partial x^j} \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) + \zeta (\vec{\nabla} \cdot \vec{v})^2 + \vec{\nabla} \cdot (\kappa \vec{\nabla} T). \quad (\text{III.43})$$

On the right hand side of this equation, one may use the identity

$$\frac{1}{2} \sum_{i,j=1}^3 \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} - \frac{2}{3} g_{ij} \vec{\nabla} \cdot \vec{v} \right) = \sum_{i,j=1}^3 \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \frac{\partial v_j}{\partial x^i}, \quad (\text{III.44a})$$

which follows from the fact that both symmetric terms $\partial v_i / \partial x^j$ and $\partial v_j / \partial x^i$ from the left member give the same contribution, while the term in g_{ij} yields a zero contribution, since it multiplies a traceless term.

Additionally, one has

$$\vec{\nabla} \cdot (\kappa \vec{\nabla} T) = T \vec{\nabla} \cdot \left(\frac{\kappa \vec{\nabla} T}{T} \right) + \frac{\kappa}{T} (\vec{\nabla} T)^2. \quad (\text{III.44b})$$

All in all, Eqs. (III.43) and (III.44) lead to

$$\begin{aligned} \frac{\partial s}{\partial t} + \vec{\nabla} \cdot (s \vec{v}) - \vec{\nabla} \cdot \left(\frac{\kappa \vec{\nabla} T}{T} \right) &= \frac{\eta}{2T} \sum_{i,j=1}^3 \left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} - \frac{2}{3} g_{ij} \vec{\nabla} \cdot \vec{v} \right) \\ &+ \frac{\zeta}{T} (\vec{\nabla} \cdot \vec{v})^2 + \kappa \frac{(\vec{\nabla} T)^2}{T^2}. \end{aligned} \quad (\text{III.45a})$$

This may still be recast in the slightly more compact form

$$\frac{\partial s(t, \vec{r})}{\partial t} + \vec{\nabla} \cdot \left[s(t, \vec{r}) \vec{v}(t, \vec{r}) - \kappa(t, \vec{r}) \frac{\vec{\nabla} T(t, \vec{r})}{T(t, \vec{r})} \right] = \frac{1}{T(t, \vec{r})} \left\{ 2\eta(t, \vec{r}) \mathbf{S}(t, \vec{r}) : \mathbf{S}(t, \vec{r}) + \zeta(t, \vec{r}) [\vec{\nabla} \cdot \vec{v}(t, \vec{r})]^2 + \kappa(t, \vec{r}) \frac{[\vec{\nabla} T(t, \vec{r})]^2}{T(t, \vec{r})} \right\} \quad (\text{III.45b})$$

with $\mathbf{S} : \mathbf{S} \equiv \mathbf{S}_{ij} \mathbf{S}^{ij}$ the scalar obtained by doubly contracting the rate-of-shear tensor with itself.

This equation can then be integrated over the \mathcal{V} occupied by the fluid:

- When computing the integral of the divergence term on the left hand side with the Stokes theorem, it vanishes thanks to the boundary conditions imposed at the surface \mathcal{S} ;
- the remaining term in the left member is simply the time derivative dS/dt of the total entropy of the closed system;
- if all three transport coefficients η , ζ , κ are *positive*, then it is also the case of the three terms on the right hand side.

One thus finds $\frac{dS}{dt} \geq 0$, in agreement with the second law of thermodynamics.

Remarks:

- * The previous derivation may be seen as a proof that the transport coefficients must be positive to ensure that the second law of thermodynamics holds.
- * If all three transport coefficients η , ζ , κ vanish, i.e. in the case of a non-dissipative fluid, Eq. (III.45) simply reduces to the entropy conservation equation in perfect fluids (III.39).