

IV.3 Vortex dynamics in perfect fluids

We now turn back to the case of an arbitrary flow $\vec{v}(t, \vec{r})$, still in the case of a perfect fluid. The vorticity vector field, defined as the curl of the flow velocity field, was introduced in § [II.1.3](#), together with the vorticity lines. Modulo a few assumptions on the fluid equation of state and the volume forces, one can show that vorticity is “frozen” in the flow of a perfect fluid, in the sense that the

flux of vorticity across a material surface remains constant as the latter is being transported. This behavior will be investigated and formulated both at the integral level (§ IV.3.1) and differentially (§ IV.3.2).

IV.3.1 Circulation of the flow velocity. Kelvin's theorem

Definition: Let $\vec{\gamma}(t, \lambda)$ be a closed curve, parametrized by a real number $\lambda \in [0, 1]$, which is being swept along by the fluid in its motion. The integral

$$\Gamma_{\vec{\gamma}}(t) \equiv \oint_{\vec{\gamma}} \vec{v}(t, \vec{\gamma}(t, \lambda)) \cdot d\vec{\ell} \quad (\text{IV.13})$$

is called the *circulation* around the curve of the velocity field.

Remark: According to Stokes' theorem,⁽²⁰⁾ if the area bounded by the contour $\vec{\gamma}(t, \lambda)$ is simply connected, $\Gamma_{\vec{\gamma}}(t)$ equals the surface integral—the “flux”—of the vorticity field over every surface $\mathcal{S}_{\vec{\gamma}}(t)$ delimited by $\vec{\gamma}$:

$$\Gamma_{\vec{\gamma}}(t) = \int_{\mathcal{S}_{\vec{\gamma}}} [\vec{\nabla} \times \vec{v}(t, \vec{r})] \cdot d^2\vec{\mathcal{S}} = \int_{\mathcal{S}_{\vec{\gamma}}} \vec{\omega}(t, \vec{r}) \cdot d^2\vec{\mathcal{S}}. \quad (\text{IV.14})$$

Stated differently, the vorticity field is the flux density of the circulation of the velocity.

This relationship between circulation of $\vec{v}(t, \vec{r})$ and vorticity will now be further exploited: we shall first establish and formulate results at the integral level, namely for the circulation, which will then be expressed at the differential level, i.e. in terms of the vorticity, in § IV.3.2.

Many results take a simpler form in a so-called *barotropic fluid*,⁽¹⁾ in which the pressure can be expressed as function of only the mass density: $\mathcal{P} = \mathcal{P}(\rho)$, irrespective of whether the fluid is otherwise perfect or dissipative. An example of such a result is

Kelvin's circulation theorem^(al)

In a perfect barotropic fluid with conservative volume forces, the circulation of the flow velocity around a closed curve (enclosing a simply connected region) comoving with the fluid is conserved. (IV.15a)

Denoting by $\vec{\gamma}(t, \lambda)$ the closed contour in the theorem,

$$\frac{D\Gamma_{\vec{\gamma}}(t)}{Dt} = 0. \quad (\text{IV.15b})$$

This result is also sometimes called *Thomson's theorem*.

Proof: For the sake of brevity, the arguments of the fields are omitted in case it is not necessary to specify them. Differentiating definition (IV.13) first gives

$$\frac{D\Gamma_{\vec{\gamma}}}{Dt} = \frac{D}{Dt} \int_0^1 \frac{\partial \vec{\gamma}(t, \lambda)}{\partial \lambda} \cdot \vec{v}(t, \vec{\gamma}(t, \lambda)) d\lambda = \int_0^1 \left[\frac{\partial^2 \vec{\gamma}}{\partial \lambda \partial t} \cdot \vec{v} + \frac{\partial \vec{\gamma}}{\partial \lambda} \cdot \left(\frac{\partial \vec{v}}{\partial t} + \sum_i \frac{\partial \vec{v}}{\partial x^i} \frac{\partial \gamma^i}{\partial t} \right) \right] d\lambda.$$

Since the contour $\vec{\gamma}(t, \lambda)$ flows with the fluid, $\frac{\partial \vec{\gamma}(t, \lambda)}{\partial t} = \vec{v}(t, \vec{\gamma}(t, \lambda))$, which leads to

$$\frac{D\Gamma_{\vec{\gamma}}}{Dt} = \int_0^1 \left\{ \frac{\partial \vec{v}}{\partial \lambda} \cdot \vec{v} + \frac{\partial \vec{\gamma}}{\partial \lambda} \cdot \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right] \right\} d\lambda.$$

⁽²⁰⁾ which in its classical form used here is also known as Kelvin–Stokes theorem...

⁽¹⁾ *barotropes Fluid*

^(al) W. THOMSON, Baron KELVIN, 1824–1907

The first term in the curly brackets is clearly the derivative with respect to λ of $\vec{v}^2/2$, so that its integral along a closed curve vanishes. The second term involves the material derivative of \vec{v} , as given by the Euler equation. Using Eq. (III.22) with $\vec{a}_V = -\vec{\nabla}\Phi$ leads to

$$\frac{D\Gamma_{\vec{\gamma}}}{Dt} = \int_0^1 \left(-\frac{\vec{\nabla}\mathcal{P}}{\rho} - \vec{\nabla}\Phi \right) \cdot \frac{\partial\vec{\gamma}}{\partial\lambda} d\lambda.$$

Again, the circulation of the gradient $\vec{\nabla}\Phi$ around a closed contour vanishes, leaving

$$\frac{D\Gamma_{\vec{\gamma}}(t)}{Dt} = - \oint_{\vec{\gamma}} \frac{\vec{\nabla}\mathcal{P}(t, \vec{r})}{\rho(t, \vec{r})} \cdot d\vec{\ell}, \quad (\text{IV.16})$$

which constitutes the general case of Kelvin's circulation theorem for perfect fluids with conservative volume forces.

Transforming the contour integral with Stokes' theorem yields the surface integral of

$$\vec{\nabla} \times \left(\frac{\vec{\nabla}\mathcal{P}}{\rho} \right) = \frac{\vec{\nabla} \times \vec{\nabla}\mathcal{P}}{\rho} + \frac{\vec{\nabla}\mathcal{P} \times \vec{\nabla}\rho}{\rho^2} = \frac{\vec{\nabla}\mathcal{P} \times \vec{\nabla}\rho}{\rho^2}. \quad (\text{IV.17})$$

In a barotropic fluid, the rightmost term of this identity vanishes since $\vec{\nabla}\mathcal{P}$ and $\vec{\nabla}\rho$ are collinear, which yields relation (IV.15). \square

Remark: Using relation (IV.14) and the fact that the area $S_{\vec{\gamma}}(t)$ bounded by the curve $\vec{\gamma}$ at time t defines a material surface, which will be transported in the fluid motion, Kelvin's theorem (IV.15) can be restated as

In a perfect barotropic fluid with conservative volume forces, the flux of the vorticity across a material surface is conserved. (IV.18)

Kelvin's theorem leads to two trivial corollaries, namely

Helmholtz's theorem ^(am)

In the flow of a perfect barotropic fluid with conservative volume forces, the vorticity lines and vorticity tubes move with the fluid. (IV.19)

Similar to the definition of stream tubes in § I.3.3, a vorticity tube is defined as the surface formed by the vorticity lines tangent to a given closed geometrical curve.

In the case of vanishing vorticity $\vec{\omega} = \vec{0}$, one has

Lagrange's theorem:

In a perfect barotropic fluid with conservative volume forces, if the flow is irrotational at a given instant t , it remains irrotational at later times. (IV.20)

Kelvin's circulation theorem (IV.15) and its corollaries imply that vorticity cannot be created nor destroyed in the flow of a perfect barotropic fluid with conservative volume forces. However, the more general form (IV.16) already show that in a non-barotropic fluid, there is a "source" for vorticity, which leads to the non-conservation of the circulation of the flow velocity. Similarly, non-conservative forces—for instance the Coriolis force in a rotating reference frame—may contribute a non-vanishing term in Eq. (IV.16) leading to a change in the circulation. We shall see in Sec. ?? that viscous stresses also affect the transport of vorticity in a fluid.

^(am)H. VON HELMHOLTZ, 1821–1894

IV.3.2 Vorticity transport equation in perfect fluids

Consider the Euler equation (III.23), in the case of conservative volume forces, $\vec{a}_V = -\vec{\nabla}\Phi$. Taking the rotational curl of both sides yields after some straightforward algebra

$$\frac{\partial \vec{\omega}(t, \vec{r})}{\partial t} - \vec{\nabla} \times [\vec{v}(t, \vec{r}) \times \vec{\omega}(t, \vec{r})] = -\frac{\vec{\nabla}\mathcal{P}(t, \vec{r}) \times \vec{\nabla}\rho(t, \vec{r})}{\rho(t, \vec{r})^2}. \quad (\text{IV.21})$$

This relation can be further transformed using the identity (we omit the variables)

$$\vec{\nabla} \times (\vec{v} \times \vec{\omega}) = (\vec{\omega} \cdot \vec{\nabla})\vec{v} + (\vec{\nabla} \cdot \vec{\omega})\vec{v} - (\vec{v} \cdot \vec{\nabla})\vec{\omega} - (\vec{\nabla} \cdot \vec{v})\vec{\omega}.$$

Since the divergence of the vorticity field $\vec{\nabla} \cdot \vec{\omega}(t, \vec{r})$ vanishes, the previous two equations yield

$$\frac{\partial \vec{\omega}(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}]\vec{\omega}(t, \vec{r}) - [\vec{\omega}(t, \vec{r}) \cdot \vec{\nabla}]\vec{v}(t, \vec{r}) = -[\vec{\nabla} \cdot \vec{v}(t, \vec{r})]\vec{\omega}(t, \vec{r}) - \frac{\vec{\nabla}\mathcal{P}(t, \vec{r}) \times \vec{\nabla}\rho(t, \vec{r})}{\rho(t, \vec{r})^2}. \quad (\text{IV.22})$$

While it is tempting to introduce the material derivative $D\vec{\omega}/Dt$ on the left hand side of this equation, for the first two terms, we rather define the whole left member to be a new derivative

$$\frac{\mathcal{D}_{\vec{v}}\vec{\omega}(t, \vec{r})}{Dt} \equiv \frac{\partial \vec{\omega}(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}]\vec{\omega}(t, \vec{r}) - [\vec{\omega}(t, \vec{r}) \cdot \vec{\nabla}]\vec{v}(t, \vec{r}) \quad (\text{IV.23a})$$

or equivalently

$$\frac{\mathcal{D}_{\vec{v}}\vec{\omega}(t, \vec{r})}{Dt} \equiv \frac{D\vec{\omega}(t, \vec{r})}{Dt} - [\vec{\omega}(t, \vec{r}) \cdot \vec{\nabla}]\vec{v}(t, \vec{r}). \quad (\text{IV.23b})$$

We shall refer to $\mathcal{D}_{\vec{v}}/Dt$ as the *comoving time derivative*, for reasons that will be explained at the end of this Section.

Using this definition, Eq. (IV.22) reads

$$\frac{\mathcal{D}_{\vec{v}}\vec{\omega}(t, \vec{r})}{Dt} = -[\vec{\nabla} \cdot \vec{v}(t, \vec{r})]\vec{\omega}(t, \vec{r}) - \frac{\vec{\nabla}\mathcal{P}(t, \vec{r}) \times \vec{\nabla}\rho(t, \vec{r})}{\rho(t, \vec{r})^2}. \quad (\text{IV.24})$$

In the particular case of a barotropic fluid—recall that we also assumed that it is ideal and only has conservative volume forces—this becomes

$$\frac{\mathcal{D}_{\vec{v}}\vec{\omega}(t, \vec{r})}{Dt} = -[\vec{\nabla} \cdot \vec{v}(t, \vec{r})]\vec{\omega}(t, \vec{r}). \quad (\text{IV.25})$$

Thus, the comoving time-derivative of the vorticity is parallel to itself.

From Eq. (IV.25), one deduces at once that if $\vec{\omega}(t, \vec{r})$ vanishes at some time t , it remains zero—which is the differential formulation of corollary (IV.20).

Invoking the continuity equation (III.12), the volume expansion rate $\vec{\nabla} \cdot \vec{v}$ on the right hand side of Eq. (IV.25) can be replaced by $-(1/\rho)D\rho/Dt$. For scalar fields, material derivative and comoving time-derivative coincide, which leads to the compact form

$$\frac{\mathcal{D}_{\vec{v}}}{Dt} \left[\frac{\vec{\omega}(t, \vec{r})}{\rho(t, \vec{r})} \right] = \vec{0} \quad (\text{IV.26})$$

for perfect barotropic fluids with conservative volume forces. That is, anticipating on the discussion of the comoving time derivative hereafter, $\vec{\omega}/\rho$ evolves in the fluid flow in the same way as the separation between two material neighboring points: the ratio is “frozen” in the fluid evolution.

Comoving time derivative

To understand the meaning of the comoving time derivative $\mathcal{D}_{\vec{v}}/Dt$, let us come back to Fig. II.1 depicting the positions at successive times t and $t + \delta t$ of a small material vector $\delta\vec{\ell}(t)$. By definition

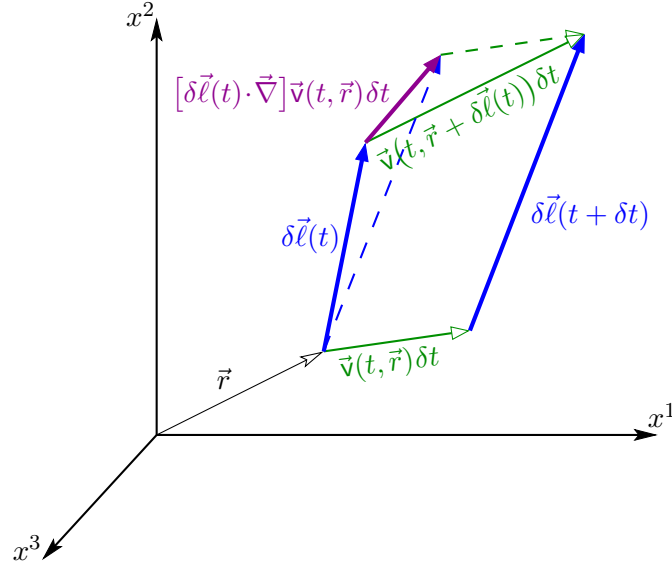


Figure IV.6 – Positions of a material line element $\delta\vec{\ell}$ at successive times t and $t + \delta t$.

of the material derivative, the change in $\delta\vec{\ell}$ between these two instants—as given by the trajectories of the two material points which are at \vec{r} resp. $\vec{r} + \delta\vec{\ell}(t)$ at time t —is

$$\delta\vec{\ell}(t + \delta t) - \delta\vec{\ell}(t) = \frac{D\delta\vec{\ell}(t)}{Dt} \delta t.$$

On the other hand, displacing the origin of $\delta\vec{\ell}(t + \delta t)$ to let it coincide with that of $\delta\vec{\ell}(t)$, one sees on Fig. [IV.6](#) that this change equals

$$\delta\vec{\ell}(t + \delta t) - \delta\vec{\ell}(t) = [\delta\vec{\ell}(t) \cdot \vec{\nabla}] \vec{v}(t, \vec{r}) \delta t.$$

Equating both results and dividing by δt , one finds $\frac{D\delta\vec{\ell}(t)}{Dt} = [\delta\vec{\ell}(t) \cdot \vec{\nabla}] \vec{v}(t, \vec{r})$, i.e. precisely

$$\frac{\mathcal{D}_{\vec{v}}\delta\vec{\ell}(t)}{Dt} = \vec{0}. \quad (\text{IV.27})$$

Thus, the comoving time derivative of a material vector, which moves with the fluid, vanishes. In turn, the comoving time derivative at a given instant t of an arbitrary vector measures its rate of change with respect to a material vector with which it coincides at time t .

This interpretation suggests—this can be proven more rigorously—what the action of the comoving time derivative on a scalar field should be. In that case, $\mathcal{D}_{\vec{v}}/Dt$ should coincide with the material derivative, which already accounts for all changes—due to non-stationarity and convective transport—affecting material points in their motion. This justifies a posteriori our using $\mathcal{D}_{\vec{v}}\rho/Dt = D\rho/Dt$ above.

More generally, the comoving time derivative introduced in Eq. [\(IV.23a\)](#) may be rewritten as

$$\frac{\mathcal{D}_{\vec{v}}}{Dt}(\cdot) \equiv \frac{\partial}{\partial t}(\cdot) + \mathcal{L}_{\vec{v}}(\cdot), \quad (\text{IV.28})$$

where $\mathcal{L}_{\vec{v}}$ denotes the *Lie derivative* along the velocity field $\vec{v}(\vec{r})$, whose action on an arbitrary vector field $\vec{\omega}(\vec{r})$ is precisely (time plays no role here)

$$\mathcal{L}_{\vec{v}}\vec{\omega}(\vec{r}) \equiv [\vec{v}(\vec{r}) \cdot \vec{\nabla}]\vec{\omega}(\vec{r}) - [\vec{\omega}(\vec{r}) \cdot \vec{\nabla}]\vec{v}(\vec{r}),$$

while it operates on an arbitrary scalar field $\rho(\vec{r})$ according to

$$\mathcal{L}_{\vec{v}}\rho(\vec{r}) \equiv [\vec{v}(\vec{r}) \cdot \vec{\nabla}]\rho(\vec{r}).$$

More information on the Lie derivative, including its operation on 1-forms or more generally on $\binom{m}{n}$ -tensors—from which the action of the comoving time derivative follows—, can be found e.g. in Ref. [\[17\]](#) Chap. 3.1–3.5].

IV.4 Potential flows

According to Lagrange's theorem (IV.20), every flow of a perfect barotropic fluid with conservative volume forces which is everywhere irrotational at a given instant remains irrotational at every time.

Focusing accordingly on the *incompressible* and irrotational motion of an ideal fluid with conservative volume forces, which is also referred to as a *potential flow*^(li), the dynamical equations can be recast such that the main one is a linear partial differential equation for the *velocity potential* (§ IV.4.1), for which there exist mathematical results (§ IV.4.2). Two-dimensional potential flows are especially interesting, since one may then introduce a complex velocity potential—and the corresponding complex velocity—, which is a holomorphic function (§ IV.4.3). This allows one to use the full power of complex analysis so as to devise flows around obstacles with various geometries by combining “elementary” solutions and deforming them.

IV.4.1 Equations of motion in potential flows

Using a known result from vector analysis, a vector field whose curl vanishes everywhere on a simply connected domain of \mathbb{R}^3 can be written as the gradient of a scalar field. Thus, in the case of an irrotational flow $\vec{\nabla} \times \vec{v}(t, \vec{r}) = \vec{0}$, the velocity field can be expressed as

$$\vec{v}(t, \vec{r}) = -\vec{\nabla}\varphi(t, \vec{r}) \quad (\text{IV.29})$$

with $\varphi(t, \vec{r})$ the so-called *velocity potential*^(lii).

Remarks:

* The minus sign in definition (IV.29) is purely conventional. While the choice adopted here is not universal, it has the advantage of being directly analogous to the convention in electrostatics ($\vec{E} = -\vec{\nabla}\Phi_{\text{Coul.}}$) or Newtonian gravitation physics ($\vec{g} = -\vec{\nabla}\Phi_{\text{Newt.}}$).

* Since Lagrange's theorem does not hold in a dissipative fluid, in which vorticity can be created or annihilated (Sec. ??), the rationale behind the definition of the velocity potential disappears.

Using the velocity potential (IV.29) and the relation $\vec{a}_V = -\vec{\nabla}\Phi$ expressing that the volume forces are conservative, the Euler equation (III.23) reads

$$-\frac{\partial \vec{\nabla}\varphi(t, \vec{r})}{\partial t} + \vec{\nabla} \left\{ \frac{[\vec{\nabla}\varphi(t, \vec{r})]^2}{2} + \Phi(t, \vec{r}) \right\} = -\frac{1}{\rho(t, \vec{r})} \vec{\nabla}\mathcal{P}(t, \vec{r}).$$

Assuming that the flow is also incompressible, and thus ρ constant, this becomes

$$-\frac{\partial \vec{\nabla}\varphi(t, \vec{r})}{\partial t} + \vec{\nabla} \left\{ \frac{[\vec{\nabla}\varphi(t, \vec{r})]^2}{2} + \frac{\mathcal{P}(t, \vec{r})}{\rho} + \Phi(t, \vec{r}) \right\} = \vec{0}. \quad (\text{IV.30})$$

or equivalently

$$-\frac{\partial \varphi(t, \vec{r})}{\partial t} + \frac{[\vec{\nabla}\varphi(t, \vec{r})]^2}{2} + \frac{\mathcal{P}(t, \vec{r})}{\rho} + \Phi(t, \vec{r}) = C(t), \quad (\text{IV.31})$$

where $C(t)$ denotes a function of time only.

Eventually, expressing the incompressibility condition [cf. Eq. (II.16)] $\vec{\nabla} \cdot \vec{v}(t, \vec{r}) = 0$ leads to the *Laplace*^(an) equation for the velocity potential φ

$$\Delta\varphi(t, \vec{r}) = 0. \quad (\text{IV.32})$$

^(li) *Potentialströmung* ^(lii) *Geschwindigkeitspotential*

^(an) P.-S. LAPLACE, 1749–1827

Equations (IV.29), (IV.31) and (IV.32) are the three equations of motion governing potential flows. Since the Laplace equation is partial differential, it is still necessary to specify the corresponding boundary conditions.

In agreement with the discussion in § III.3.2c, there are two types of conditions: at walls or obstacles, which are impermeable to the fluid; and “at infinity”—for a flow in an unbounded domain of space—, where the fluid flow is generally assumed to be uniform. Choosing a proper reference frame \mathcal{R} , this uniform motion of the fluid may be turned into having a fluid at rest. Denoting by $\mathcal{S}(t)$ the material surface associated with walls or obstacles, which are assumed to be moving with velocity $\vec{v}_{\text{obs.}}$ with respect to \mathcal{R} , and by $\vec{e}_n(t, \vec{r})$ the unit normal vector to $\mathcal{S}(t)$ at a given point \vec{r} , the condition of vanishing relative normal velocity reads

$$-\vec{e}_n(t, \vec{r}) \cdot \vec{\nabla} \varphi(t, \vec{r}) = \vec{e}_n(t, \vec{r}) \cdot \vec{v}_{\text{obs.}}(t, \vec{r}) \quad \text{on } \mathcal{S}(t). \quad (\text{IV.33a})$$

In turn, the condition of rest at infinity reads

$$\varphi(t, \vec{r}) \underset{|\vec{r}| \rightarrow \infty}{\sim} K(t), \quad (\text{IV.33b})$$

where in practice the scalar function $K(t)$ will be given.

Remarks:

* Since the Laplace equation (IV.32) is linear—the non-linearity of the Euler equation is in Eq. (IV.31), which becomes trivial once the spatial dependence of the velocity potential has been determined—, it will be possible to *superpose* the solutions of “simple” problems to obtain the solution for a more complicated geometry.

* In potential flows, the dependences on time and space are somewhat separated: The Laplace equation (IV.32) governs the spatial dependence of φ and thus \vec{v} ; meanwhile, time enters the boundary conditions (IV.33), and is thus used to fix the amplitude of the solution of the Laplace equation. In turn, when φ is known, relation (IV.31) gives the pressure field, where the integration “constant” $C(t)$ will also be fixed by boundary conditions.

IV.4.2 Mathematical results on potential flows

The *boundary value problem* consisting of the Laplace differential equation (IV.32) together with the boundary conditions on normal derivatives (IV.33) is called a *Neumann^(ao) problem* or boundary value problem of the second kind. For such problems, results on the existence and unicity of solutions have been established, which we shall now state without further proof.⁽²¹⁾

IV.4.2a Potential flows in simply connected regions

The simplest case is that of a potential flow on a simply connected domain \mathcal{D} of space. \mathcal{D} may be unbounded, provided the condition at infinity is that the fluid should be at rest, Eq. (IV.33b).

On a simply connected domain, the Neumann problem (IV.32)–(IV.33) for the velocity potential admits a solution $\varphi(t, \vec{r})$, which is unique up to an additive constant. In turn, the flow velocity field $\vec{v}(t, \vec{r})$ given by relation (IV.29) is unique. (IV.34)

For a flow on a simply connected region, the relation (IV.29) between the flow velocity and its potential is “easily” invertible: fixing some reference position \vec{r}_0 in the domain, one may write

$$\varphi(t, \vec{r}) = \varphi(t, \vec{r}_0) - \int_{\vec{\gamma}} \vec{v}(t, \vec{r}') \cdot d\vec{\ell}(\vec{r}') \quad (\text{IV.35})$$

where the line integral is taken along *any* path $\vec{\gamma}$ on \mathcal{D} connecting the positions \vec{r}_0 and \vec{r} .

⁽²¹⁾The Laplace differential equation is dealt with in many textbooks, as e.g. in Ref. [18, Chapters 7–9], [19, Chapter 4], or [20, Chapter VII].

^(ao)C. NEUMANN, 1832–1925

That the line integral only depends on the path extremities \vec{r}_0, \vec{r} , not on the path itself, is clearly equivalent to Stokes' theorem stating that the circulation of velocity along *any* closed contour in the domain \mathcal{D} is zero—it equals the flux of the vorticity, which is everywhere zero, through a surface delimited by the contour and entirely contained in \mathcal{D} .

Thus, $\varphi(t, \vec{r})$ is uniquely defined once the value $\varphi(t, \vec{r}_0)$, which is the arbitrary additive constant mentioned above, has been fixed.

This reasoning no longer holds in a multiply connected domain, as we now further discuss.