IV.2.2 Applications of the Bernoulli equation

Throughout this Section, we assume that the flow is incompressible, i.e. the mass density is uniform, and rely on Eq. (IV.11). Of course, one may release this assumption, in which case one should replace pressure by enthalpy density everywhere below.[16]

IV.2.2 a Drainage of a vessel. Torricelli's law

Consider a liquid contained in a vessel with a small hole at its bottom, through which the liquid can flow (Fig. IV.2).

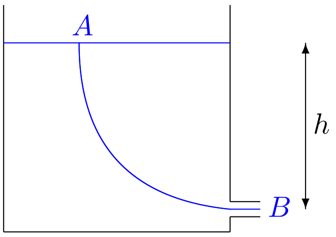

Figure IV.2

At points A and B that lie on the same streamline, the pressure in the liquid equals the atmospheric pressure[17] $\mathcal{P}_A = \mathcal{P}_B = \mathcal{P}_0$. The Bernoulli equation (at constant pressure) then yields

$$\frac{\mathsf{v}_A^2}{2} + gz_A = \frac{\mathsf{v}_B^2}{2} + gz_B,$$

with z_A resp. z_B the height of point A resp. B, i.e.

$$\mathsf{v}_B^2 = \mathsf{v}_A^2 + 2gh.$$

If the velocity at point A vanishes, one finds *Torricelli's*[ah] *law*[xlix]

$$\mathsf{v}_B = \sqrt{2gh}.$$

That is, the speed of efflux is the same as that acquired by a body *in free fall* from the same height h in the same gravity field.

Remark: To be allowed to apply the Bernoulli equation, one should first show that the liquid flows steadily. If the horizontal cross section of the vessel is much larger than the aperture of the hole and h large enough, this holds to a good approximation.

IV.2.2 b Venturi effect

Consider now the incompressible flow of a fluid inside the geometry illustrated in Fig. IV.3. As we shall only be interested in the average velocity or pressure of the fluid across a cross section of the tube, the flow is effectively one-dimensional.

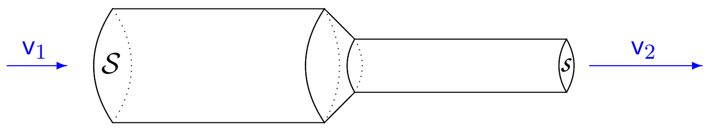

Figure IV.3

The conservation of the mass flow rate in the tube, which represents the integral formulation of the continuity equation (III.12), leads to $\rho\mathcal{S}\mathsf{v}_1 = \rho s\mathsf{v}_2$, i.e. $\mathsf{v}_2 = (\mathcal{S}/s)\mathsf{v}_1 > \mathsf{v}_1$, with $\mathcal{S}$ resp. s the area of the tube cross section in its broad resp. narrow section.

On the other hand, the Bernoulli equation at constant height, and thus potential energy, gives

$$\frac{\mathsf{v}_1^2}{2} + \frac{\mathcal{P}_1}{\rho} = \frac{\mathsf{v}_2^2}{2} + \frac{\mathcal{P}_2}{\rho}. \tag{IV.12}$$

All in all, the pressure in the narrow section is thus smaller than in the broad section, $\mathcal{P}_2 < \mathcal{P}_1$, which constitutes the *Venturi effect*.[ai]

[16] The author confesses that he has a better physical intuition of pressure than of enthalpy, hence his parti pris.

[17] One can show that the pressure in the liquid at point B equals the atmospheric pressure provided the local streamlines are parallel to each other—that is, if the flow is laminar.

[xlix] *Torricellis Theorem*

[ah] E. Torricelli, 1608–1647 [ai] G. B. Venturi, 1746–1822

Remarks:

∗ Using mass conservation and the Bernoulli equation, one can express v_1 or v_2 in terms of the tube cross section areas and the pressure difference. For instance, the mass flow rate reads

$$\rho\mathcal{S}\left[2\frac{\mathcal{P}_1-\mathcal{P}_2}{\rho}\bigg/\left(\frac{\mathcal{S}^2}{s^2}-1\right)\right]^{1/2}.$$

∗ From Eq. (IV.12) and the relation between the velocities, one sees that is the ratio $\mathcal{S}/s$, and thus v_2, is large enough, then $\mathcal{P}_2$ may be negative—which seems somewhat unsettling. Physically, the pressure does not truly become negative, but instead the assumed model breaks down. More precisely, when $\mathcal{P}_2$ reaches the saturated vapor pressure of the liquid, then the thermodynamically stable state is locally the gas phase, so that vapor bubbles will appear. This phenomenon is referred to as (hydrodynamic) *cavitation*.

IV.2.2 c Pitot tube

Figure IV.4 represents schematically the flow of a fluid around a (static) *Pitot*[(aj)] *tube*, which is a device used to estimate a flow velocity through the measurement of a pressure difference. Three streamlines are shown, starting far away from the Pitot tube, where the flow is (approximately) uniform and has velocity $\vec{\mathsf{v}}$, which one wants to measure. The flow is assumed to be incompressible.

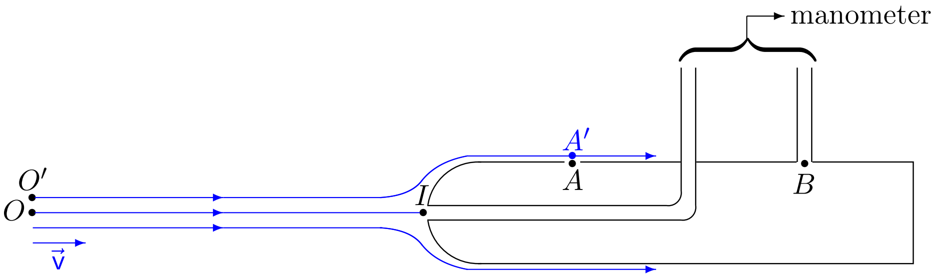

Figure IV.4 – Flow around a Pitot tube.

The Pitot tube consists of two long thin concentric tubes:

- Despite the presence of the hole at the end point I, the flow does not penetrate in the inner tube, so that $\vec{\mathsf{v}}_I = \vec{0}$ to a good approximation.
- In the broader tube, there is a hole at a point A, which is far enough from I to ensure that the fluid flow in the vicinity of A is no longer perturbed by the extremity of the tube: $\vec{\mathsf{v}}_A = \vec{\mathsf{v}}_{A'} \simeq \vec{\mathsf{v}}$, where the second identity holds thanks to the thinness of the tube, which thus perturbs the flow properties minimally. In addition, the pressure in the broader tube is uniform, so that $\mathcal{P}_A = \mathcal{P}_B$.

If one neglects the height differences—which is a posteriori justified by the numerical values we shall find—the (incompressible) Bernoulli equation gives first

$$\mathcal{P}_O + \rho\frac{\vec{\mathsf{v}}^2}{2} = \mathcal{P}_I$$

along the streamline OI, and

$$\mathcal{P}_{O'} + \rho\frac{\vec{\mathsf{v}}^2}{2} = \mathcal{P}_{A'} + \rho\frac{\vec{\mathsf{v}}_{A'}^2}{2}$$

along the streamline $O'A'$. Using $\mathcal{P}_{O'} \simeq \mathcal{P}_O$, $\mathcal{P}_{A'} \simeq \mathcal{P}_A$ and $\vec{\mathsf{v}}_{A'} \simeq \vec{\mathsf{v}}$, the latter identity leads to

(aj) H. Pitot, 1695–1771

$\mathcal{P}_O \simeq \mathcal{P}_A = \mathcal{P}_B$. One thus finds

$$\mathcal{P}_I - \mathcal{P}_B = \rho\frac{\vec{v}^2}{2},$$

so that a measurement of $\mathcal{P}_I - \mathcal{P}_B$ gives an estimate of $|\vec{v}|$.

For instance, in air ($\rho \sim 1.3\ \mathrm{kg\cdot m^{-3}}$) a velocity of $100\ \mathrm{m\cdot s^{-1}}$ results in a pressure difference of 6.5×10^3 Pa $= 6.5\times10^{-2}$ atm. With a height difference h of a few centimeters between O and A', the neglected term $\rho g h$ is of order 0.1–1 Pa.

Remarks:

∗ The flow of a fluid with velocity $\vec{v}$ around a motionless Pitot tube is equivalent to the motion of a Pitot tube with velocity $-\vec{v}$ in a fluid at rest. Thus Pitot tubes are used to measure the speed of airplanes.[18]

∗ Is the flow of air really incompressible at velocities of $100\ \mathrm{m\cdot s^{-1}}$ or higher? Not really, since the Mach number (II.19) becomes larger than 0.3. In practice, one thus rather uses the "compressible" Bernoulli equation (IV.10), yet the basic principles presented above remain valid.

IV.2.2 d Magnus effect

Consider an initially uniform and steady flow with velocity $\vec{v}_0$. One introduces in it a cylinder that rotates about its axis with angular velocity $\vec{\omega}_C$ perpendicular to the flow velocity (Fig. IV.5).

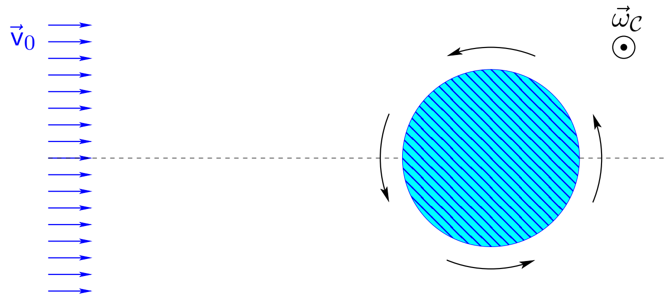

Figure IV.5 – Fluid flow around a rotating cylinder.

Intuitively, one can expect that the cylinder will drag the neighboring fluid layers along in its rotation.[19] In that case, the fluid velocity due to that rotation will add up to resp. be subtracted from the initial flow velocity in the lower resp. upper region close to the cylinder in Fig. IV.5.

Invoking now the Bernoulli equation—in which the height difference between both sides of the cylinder is neglected—, the pressure will be larger above the cylinder than below it. Accordingly, the cylinder will experience a resulting force directed "downwards" (for the setup of the Figure), which constitutes the *Magnus*[ak] *effect*. More precisely, the force is proportional to $\vec{v}_0 \times \vec{\omega}_C$.

[18] When he introduced the idea in 1732, Pitot rather had the velocity of ships in his mind.

[19] Strictly speaking, this is not true in perfect fluids, only in real fluids with friction! Nevertheless, the tangential forces due to viscosity in the latter may be small enough that the Bernoulli equation remains approximately valid, as is assumed here.

[ak] G. Magnus, 1802–1870