Chapter IV

Non-relativistic flows of perfect fluids

In the previous Chapter, we have introduced the coupled dynamical equations that govern the flows of perfect fluids in the non-relativistic regime, namely the continuity (III.12), Euler (III.21), and energy conservation (III.38) equations for the mass density $\rho(t,\vec{r})$, fluid velocity $\vec{v}(t,\vec{r})$ and pressure $\mathcal{P}(t,\vec{r})$. The present Chapter discusses solutions of that system of equations, i.e. possible motions of perfect fluids,[(10)] obtained when using various assumptions to simplify the problem so as to render the equations tractable analytically.

In the simplest possible case, there is simply no motion at all in the fluid; yet the volume forces acting at each point still drive the behavior of the pressure and local mass density throughout the medium (Sec. IV.1). Steady flows, in which there is by definition no real dynamics, are also easily dealt with: both the Euler and energy conservation equations yield the Bernoulli equation, which can be further simplified by kinematic assumptions on the flow (Sec. IV.2).

Section IV.3 deals with the dynamics of vortices, i.e. of the vorticity vector field, in the motion of a perfect fluid. In such fluids, in case the pressure only depends on the mass density, there exists a quantity, related to vorticity, that remains conserved if the volume forces at play are conservative.

The latter assumption is also necessary to define potential flows (Sec. IV.4), in which the further hypothesis of an incompressible motion leads to simplified equations of motion, for which a number of exact mathematical results are known, especially in the case of two-dimensional flows.

Throughout the Chapter, it is assumed that the body forces in the fluid, whose volume density was denoted by $\vec{f}_V$ in Chapter III, are conservative, so that they derive from a potential. More specifically, anticipating the fact that these volume forces are proportional to the amount of mass

[(10)] . . . at least in an idealized world. Yet the reader is encouraged to relate the results to observations of her everyday life—beyond the few illustrative examples provided by the author—, and to wonder how a small set of seemingly "simple" mathematical equations can describe a wide variety of physical phenomena.

they act upon, we introduce the potential energy per unit mass Φ, such that

$$\vec{f}_V(t,\vec{r}) = -\rho(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}). \tag{IV.1}$$

IV.1 Hydrostatics of a perfect fluid

The simplest possibility is that of *static* solutions of the system of equations governing the dynamics of perfect fluids, namely those with $\vec{v} = \vec{0}$ everywhere—in an appropriate global reference frame—and additionally $\partial/\partial t = 0$. Accordingly, there is strictly speaking no fluid flow: this is the regime of *hydrostatics*, for which the only[11] non-trivial equation—following from the Euler equation (III.21)—reads

$$\frac{1}{\rho(\vec{r})}\vec{\nabla}\mathcal{P}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}). \tag{IV.2}$$

Throughout this Section, we adopt a fixed system of Cartesian coordinates $(x^1, x^2, x^3) = (x, y, z)$, with the basis vector $\vec{e}_3$ oriented vertically and pointing upwards. In the following examples, we shall consider the case of fluids in a homogeneous gravity field, leading to $\Phi(\vec{r}) = gz$, with $g = 9.8\ \mathrm{m\cdot s^{-2}}$.

Remark: If the stationarity condition is relaxed, the continuity equation still leads to $\partial\rho/\partial t = 0$, i.e. to a time-independent mass density. Whether time derivatives vanish or not makes no change in the Euler equation when $\vec{v} = \vec{0}$. Eventually, energy conservation requires that the internal energy density e—and thereby the pressure—follow the same time evolution as the "external" potential energy Φ. Thus, there is a non-stationary hydrostatics, but in which the time evolution decouples from the spatial problem.

IV.1.1 Archimedes' principle

Consider first a fluid, or a system of several fluids, at rest, occupying some region of space. Let $\mathcal{S}$ be a closed control surface inside that region, as depicted in Fig. IV.1 (left), and $\mathcal{V}$ be the volume delimited by $\mathcal{S}$. The fluid inside $\mathcal{S}$ will be denoted by Σ, and that outside by Σ'.

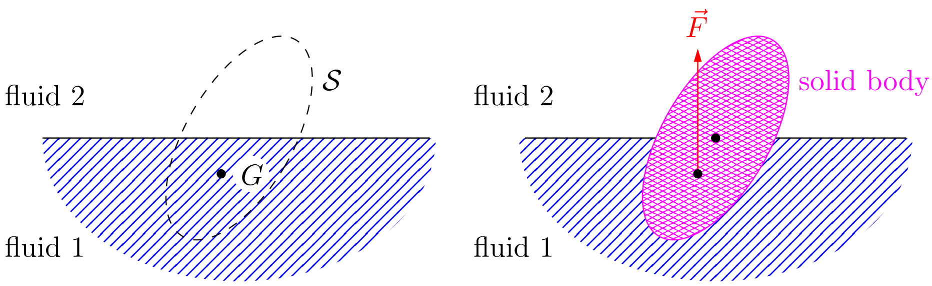

Figure IV.1 – Gedankenexperiment to illustrate Archimedes' principle.

The system Σ is in mechanical equilibrium, i.e. the sum of the gravity forces acting at each point of the volume $\mathcal{V}$ and the pressure forces exerted at each point of $\mathcal{S}$ by the fluid Σ' must vanish:

- The gravity forces at each point result in a single force $\vec{F}_G$, applied at the center of mass G of Σ, whose direction and magnitude are those of the weight of the system Σ.
- According to the equilibrium condition, the resultant of the pressure forces must equal $-\vec{F}_G$:

$$\oint_{\mathcal{S}} \mathcal{P}(\vec{r})\,\mathrm{d}^2\vec{\mathcal{S}} = -\vec{F}_G.$$

[11] This is true only in the case of perfect fluids; for dissipative ones, there emerge new possibilities, see § ??.

If one now replaces the fluid system Σ by a rigid body $\mathcal{B}$, while keeping the fluids Σ' outside $\mathcal{S}$ in the same equilibrium state, the mechanical stresses inside Σ' remain unchanged. Thus, the resultant of the contact forces exerted by Σ' on $\mathcal{B}$ is still given by $\vec{F} = -\vec{F}_G$, which still applies at the center of mass G of the fluid system Σ. This constitutes the celebrated *Archimedes*[(ab)] *principle*:

$$\text{\textit{Any object, wholly or partially immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object.}} \tag{IV.3}$$

In addition, we have obtained the point of application of the resultant force ("buoyancy"[(xlvi)]) from the fluid.

Remark: If the center of mass G of the "displaced" fluid system does not coincide with the center of mass of the body $\mathcal{B}$, the latter will be submitted to a torque, since $\vec{F}$ and its weight are applied at two different points. This is e.g. the case in Fig. IV.1, which describes a mechanically unstable situation.

[(xlvi)] *statischer Auftrieb*

[(ab)] Archimedes (Ἀρχιμήδης), c.287–c.212 BC