III.3.2 e The Euler equation as a balance equation

The Euler equation can be rewritten in the generic form for a balance equation, namely as the identity of the sum of the time derivative of a density and the divergence of a flux density with a source term—which vanishes if the quantity under consideration is conserved. Accordingly, we first introduce two

Definitions: One associates with the i-th component in a given coordinate system of the momentum of a material system its

- *density*[xxxix] $\rho(t,\vec{r})\,\mathsf{v}^i(t,\vec{r})$ and (III.24a)
- *flux density*[xl] (in direction j) $\mathbf{T}^{ij}(t,\vec{r}) \equiv \rho(t,\vec{r})\,\mathsf{v}^i(t,\vec{r})\,\mathsf{v}^j(t,\vec{r}) + \mathcal{P}(t,\vec{r})\,g^{ij}(t,\vec{r}),$ (III.24b)

with g^{ij} the components of the inverse metric tensor $\mathbf{g}^{-1}$.

Physically, $\mathbf{T}^{ij}$ represents the amount of momentum along $\vec{e}_i$ transported per unit time through a unit surface[5] perpendicular to the direction of $\vec{e}_j$—i.e. transported in direction j. That is, it is the i-th component of the force upon a test unit surface with normal unit vector $\vec{e}_j$.

The first contribution to $\mathbf{T}^{ij}$—namely the transported momentum density multiplied by the velocity—is precisely the ij-component of the tensor in the integrand of the surface integral in Eq. (III.15): this is (a component of) the *convective flux density*, which arises from the convective transport caused by the macroscopic motion. In turn, the second term in Eq. (III.24b), involving pressure, represents the momentum transport due to the thermal, random motion of the atoms of the fluid.

Remarks:

∗ As thermal motion is random and (statistically) isotropic, it does not contribute to the momentum density $\rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})$, only to the momentum flux density.

∗ In tensor notation, the momentum flux density (III.24b), viewed as a $\binom{2}{0}$-tensor, is given by

$$\mathbf{T}(t,\vec{r}) = \rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r}) \otimes \vec{\mathsf{v}}(t,\vec{r}) + \mathcal{P}(t,\vec{r})\,\mathbf{g}^{-1}(t,\vec{r}) \quad \text{for a perfect fluid.} \tag{III.25}$$

[5]... which must be at rest in the reference frame in which the fluid has the velocity $\vec{\mathsf{v}}$ entering definition (III.24b).

[xxxix] *Impulsdichte* [xl] *Impulsstromdichte*

∗ Given its physical meaning, the momentum flux (density) tensor $\mathbf{T}$ is obviously related to the Cauchy stress tensor $\boldsymbol{\sigma}$. More precisely, $\mathbf{T}$ represents the forces exerted by a material point on its neighbors, as measured in an arbitrary reference frame, in which the fluid velocity is $\vec{\mathsf{v}}(t,\vec{r})$. In turn $\boldsymbol{\sigma}$ stands for the stresses acting upon the material point due to its neighbors, as measured in the reference frame in which the material point is at rest, such that $\vec{\mathsf{v}}(t,\vec{r}) = \vec{0}$. In the latter reference frame, invoking Newton's third law—which in continuum mechanics is referred to as *Cauchy's fundamental lemma*—, the two tensors are simply opposite to each other. Coming back to an arbitrary reference frame, one may write

$$\mathbf{T}(t,\vec{r}) = \rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})\otimes\vec{\mathsf{v}}(t,\vec{r}) - \boldsymbol{\sigma}(t,\vec{r}) \qquad \text{(III.26)}$$

where $\boldsymbol{\sigma}(t,\vec{r})$ is given by Eq. (III.19c) for a perfect fluid.

∗ Building on the previous remark, the absence of shear stress defining a perfect fluid can be reformulated as a condition of the momentum flux tensor:

> *A perfect fluid is a fluid at each point of which one can find a local velocity, such that for an observer moving with that velocity the fluid is locally isotropic. The momentum flux tensor is thus diagonal in the observer's reference frame.* (III.27)

The observer traveling with the same velocity as the local fluid velocity will be hereafter referred to as *comoving observer*, and the reference frame in which she is sitting as *local rest frame*.

Using definitions (III.24), one easily checks that the Euler equation (III.21) is equivalent to the balance equations (for $j = 1,2,3$)

$$\frac{\partial}{\partial t}\big[\rho(t,\vec{r})\,\mathsf{v}^j(t,\vec{r})\big] + \sum_{i=1}^{3}\frac{\mathrm{d}\mathbf{T}^{ij}(t,\vec{r})}{\mathrm{d}x^i} = f_V^j(t,\vec{r}). \qquad \text{(III.28a)}$$

with f_V^j the j-th component of the volumetric force density and $\mathrm{d}/\mathrm{d}x^i$ the covariant derivatives (see Appendix ??), that coincide with the partial derivatives in Cartesian coordinates.

Proof: For the sake of brevity, the $(t,\vec{r})$-dependence of the various fields will not be specified. With the product rule and the definition of $\mathbf{T}^{ij}$, one finds

$$\frac{\partial(\rho\mathsf{v}^j)}{\partial t} + \sum_{i=1}^{3}\frac{\mathrm{d}\mathbf{T}^{ij}}{\mathrm{d}x^i} = \frac{\partial\rho}{\partial t}\mathsf{v}^j + \rho\frac{\partial\mathsf{v}^j}{\partial t} + \sum_{i=1}^{3}\mathsf{v}^j\frac{\mathrm{d}(\rho\mathsf{v}^i)}{\mathrm{d}x^i} + \sum_{i=1}^{3}\rho\mathsf{v}^i\frac{\mathrm{d}\mathsf{v}^j}{\mathrm{d}x^i} + \sum_{i=1}^{3}g^{ij}\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}x^i}$$
$$= \mathsf{v}^j\left[\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{\mathsf{v}})\right] + \rho\left[\frac{\partial\mathsf{v}^j}{\partial t} + (\vec{\mathsf{v}}\cdot\vec{\nabla})\mathsf{v}^j\right] + \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}x_j},$$

where we have used $\sum_i g^{ij}\mathrm{d}/\mathrm{d}x^j = \mathrm{d}/\mathrm{d}x_j$. The first term between square brackets vanishes thanks to the continuity equation (III.12). In turn, the second term is precisely the j-th component of the left member of the Euler equation (III.21), i.e. it equals the j-th component of $\vec{f}_V$ minus the third term, which represents the j-th component of $\vec{\nabla}\mathcal{P}$. □

In tensor notation, Eq. (III.28a) reads

$$\frac{\partial}{\partial t}\big[\rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})\big] + \vec{\nabla}\cdot\mathbf{T}(t,\vec{r}) = \vec{f}_V(t,\vec{r}), \qquad \text{(III.28b)}$$

where the action of the divergence on a $\binom{2}{0}$-tensor is defined through its components, which is to be read off Eq. (III.28a).

III.4 Energy conservation, entropy balance

The conservation of mass and Newton's second law for linear momentum lead to four partial differential equations, one scalar—continuity equation (III.12)—and one vectorial—Euler (III.21) or Navier–Stokes (III.35)—, describing the coupled evolutions of five fields: $\rho(t,\vec{r})$, the three components of $\vec{\mathrm{v}}(t,\vec{r})$ and $\mathcal{P}(t,\vec{r})$.[(7)] To fully determine the latter, a fifth equation is needed. For this last constraint, there are several possibilities.

A first alternative is if some of the *kinematic* properties of the fluid flow are imposed a priori. Thus, requiring that the motion should be steady or irrotational or incompressible... might suffice to fully constrain the fluid flow for the geometry under consideration: we shall see several examples in the next three Chapters.

A second possibility, which will also be illustrated in Chap. IV–??, is that of a *thermodynamic* constraint: isothermal flow, isentropic flow... For instance, one sees in thermodynamics that in an adiabatic process for an ideal gas, the pressure and volume of the latter obey the relation $\mathcal{P}\mathcal{V}^\gamma$ = constant, where γ denotes the ratio of the heat capacities at constant pressure ($C_\mathcal{P}$) and constant volume ($C_\mathcal{V}$). Since $\mathcal{V}$ is proportional to $1/\rho$, this so-called "adiabatic equation of state" provides the needed constraint relating pressure and mass density.

Eventually, one may argue that non-relativistic physics automatically implies a further conservation law besides those for mass and linear momentum, namely energy conservation. Thus, using the reasoning adopted in Secs. (III.2) and (III.3), the rate of change of the total energy—internal, kinetic and potential—of the matter inside a given volume equals the negative of the flow of energy through the surface delimiting this volume. In agreement with the first law of thermodynamics, one must take into account in the energy exchanged with the exterior of the volume not only the convective transport of internal, kinetic and potential energies, but also the exchange of the mechanical work of contact forces and—for dissipative fluids—of heat.

III.4.1 Energy and entropy conservation in perfect fluids

In non-dissipative non-relativistic fluids, energy is either transported convectively—as it accompanies some flowing mass of fluid—or exchanged in the form of the mechanical work of the pressure forces between neighboring regions. Mathematically, this is expressed at the local level by the equation

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})^2 + e(t,\vec{r}) + \rho(t,\vec{r})\,\Phi(t,\vec{r})\right] + \vec{\nabla}\cdot\left\{\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})^2 + e(t,\vec{r}) + \mathcal{P}(t,\vec{r}) + \rho(t,\vec{r})\,\Phi(t,\vec{r})\right]\vec{\mathrm{v}}(t,\vec{r})\right\} = 0, \qquad \text{(III.38)}$$

where e denotes the local density of internal energy and Φ the potential energy per unit mass of volume forces—assumed to be conservative—such that the acceleration $\vec{a}_V$ present in Eq. (III.22) equals $-\vec{\nabla}\Phi$.

[(7)] The density of volume forces $\vec{f}_V$ or equivalently the corresponding potential energy per unit mass Φ, which stand for gravity or inertial forces, are given "from the outside" and not counted as a degree of freedom.

Equation (III.38) will not be proven here—we shall see later in § ?? that it emerges as low-velocity limit of one of the equations of non-dissipative relativistic fluid dynamics. It is however clearly of the usual form for a conservation equation, involving

- the total energy density, consisting of the kinetic ($\frac{1}{2}\rho\vec{v}^2$), internal (e) and potential ($\rho\Phi$) energy densities; and
- the total energy flux density, which involves the previous three forms of energy, as well as that exchanged as mechanical work of the pressure forces.[8]

Remarks:

∗ The presence of pressure in the flux density, however not in the density, is reminiscent of the same property in definitions (III.24).

∗ The assumption that the volume forces are conservative is of course not innocuous. For instance, it does not hold for Coriolis forces, which means that one must be careful when working in a rotating reference frame.

∗ The careful reader will have noticed that energy conservation (III.38) constitutes a fifth equation complementing the continuity and Euler equations (III.12) and (III.21), yet at the cost of introducing a new scalar field, the energy density, so that now a sixth equation is needed. The latter is provided by the thermal equation of state of the fluid, which relates its energy density, mass density and pressure.[9] In contrast to the other equations, this equation of state is not "dynamical", i.e. for instance it does not involve time or spatial derivatives, but is purely algebraic.

One can show—again, this will be done in the relativistic case (§ ??), and can also be seen as special case of the result obtained for Newtonian fluids in § III.4.3—that in a perfect, non-dissipative fluid, the relation (III.38) expressing energy conservation locally, together with thermodynamic relations, lead to the local conservation of entropy, expressed as

$$\frac{\partial s(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\big[s(t,\vec{r})\,\vec{v}(t,\vec{r})\big] = 0, \tag{III.39}$$

where $s(t,\vec{r})$ is the entropy density, while $s(t,\vec{r})\,\vec{v}(t,\vec{r})$ represents the entropy flux density. The motion of a perfect fluid is thus automatically *isentropic*.

This equation, together with a thermodynamic relation, is sometimes more practical than the energy conservation equation (III.38), to which it is however totally equivalent.

[8] Remember that when a system with pressure $\mathcal{P}$ increases its volume by an amount $\mathrm{d}\mathcal{V}$, it exerts a mechanical work $\mathcal{P}\,\mathrm{d}\mathcal{V}$, "provided" to its exterior.

[9] This is where the assumption of local thermodynamic equilibrium (§ I.1.3) plays a crucial role.