

## III.2 Mass and particle number conservation: continuity equation

The mass  $M$  and the particle number  $N$  of a closed non-relativistic system  $\Sigma$  remain constant in its motion:

$$\frac{DM(t)}{Dt} = 0, \quad \frac{DN(t)}{Dt} = 0. \quad (\text{III.9})$$

These conservation laws lead with the help of Reynolds' transport theorem to partial differential equations for some of the local fields that characterize a fluid flow.

**Remark:** Considering “particle number” (and its conservation) in a fluid-based description is somewhat paradoxical.  $N(t)$  is to be understood as standing for any quantum number, carried at the microscopic level by particles, which is conserved in the fluid motion, like electric charge.

### III.2.1 Integral formulation

For an arbitrary control volume  $\mathcal{V}$  delimited by surface  $\partial\mathcal{V}$ , the Reynolds transport theorem (III.8) with  $\mathcal{G}(t) = M$ , to which corresponds the volumetric density  $g(t, \vec{r}) = \rho(t, \vec{r})$ , reads

$$\frac{DM(t)}{Dt} = \int_{\mathcal{V}} \frac{\partial \rho(t, \vec{r})}{\partial t} d^3\vec{r} + \oint_{\partial\mathcal{V}} [\rho(t, \vec{r}) \vec{v}(t, \vec{r})] \cdot d^2\vec{S} = 0. \quad (\text{III.10})$$

That is, the time derivative of the mass contained in  $\mathcal{V}$  equals the negative of the mass flow rate through  $\partial\mathcal{V}$ . In the integrand of the surface integral  $\rho(t, \vec{r}) \vec{v}(t, \vec{r})$  is the *mass flux density*, (xxxvi) while its integral is the *mass flow rate*. (xxxvii)

Taking now  $\mathcal{G}(t) = N$ , the associated volumetric density is the particle-number density  $n(t, \vec{r})$ , so that Reynolds' theorem (III.8) leads to

$$\frac{DN(t)}{Dt} = \int_{\mathcal{V}} \frac{\partial n(t, \vec{r})}{\partial t} d^3\vec{r} + \oint_{\partial\mathcal{V}} [n(t, \vec{r}) \vec{v}(t, \vec{r})] \cdot d^2\vec{S} = 0, \quad (\text{III.11})$$

where  $n(t, \vec{r}) \vec{v}(t, \vec{r})$  represents the *particle number flux density*. (xxxviii)

Equation (III.10) resp. (III.11) constitutes the *integral formulation* of mass resp. particle number conservation.

(xxxvi) *Massenstromdichte* (xxxvii) *Massenstrom* (xxxviii) *Teilchenstromdichte*

**Remark:** In the case of a steady motion, Eq. (III.10) shows that the net mass flow rate through an arbitrary closed geometrical surface  $\partial\mathcal{V}$  vanishes. That is, the entrance of some amount of fluid into a (control) volume  $\mathcal{V}$  must be compensated by the simultaneous departure of an equal mass from the volume.

### III.2.2 Local formulation

Transforming the surface integral in Eq. (III.10) resp. (III.11) with the help of Gauss's<sup>(v)</sup> divergence theorem into a volume integral yields

$$\int_{\mathcal{V}} \left\{ \frac{\partial \rho(t, \vec{r})}{\partial t} + \vec{\nabla} \cdot [\rho(t, \vec{r}) \vec{v}(t, \vec{r})] \right\} d^3\vec{r} = 0,$$

resp.

$$\int_{\mathcal{V}} \left\{ \frac{\partial n(t, \vec{r})}{\partial t} + \vec{\nabla} \cdot [n(t, \vec{r}) \vec{v}(t, \vec{r})] \right\} d^3\vec{r} = 0.$$

These identities hold for an arbitrary integration volume  $\mathcal{V}$ . Using the continuity of the respective integrands, one deduces the following so-called *continuity equations*:

$$\frac{\partial \rho(t, \vec{r})}{\partial t} + \vec{\nabla} \cdot [\rho(t, \vec{r}) \vec{v}(t, \vec{r})] = 0 \quad (\text{III.12})$$

resp.

$$\frac{\partial n(t, \vec{r})}{\partial t} + \vec{\nabla} \cdot [n(t, \vec{r}) \vec{v}(t, \vec{r})] = 0. \quad (\text{III.13})$$

Equation (III.12) represents the first of five dynamical (partial differential) equations which govern the evolution of a non-relativistic fluid flow.

#### Remarks:

\* The form of the continuity equation (III.12) does not depend on the properties of the flowing medium, as for instance whether dissipative effects play a significant role or not. This should be contrasted with the findings of the next two Sections.

\* In the case of a steady fluid flow, Eq. (III.12) gives  $\vec{\nabla} \cdot [\rho(t, \vec{r}) \vec{v}(t, \vec{r})] = 0$ , i.e.

$$\rho(t, \vec{r}) \vec{\nabla} \cdot \vec{v}(t, \vec{r}) + \vec{v}(t, \vec{r}) \cdot \vec{\nabla} \rho(t, \vec{r}) = 0.$$

Thus, the stationary flow of a *homogeneous* fluid, i.e. for which  $\rho(t, \vec{r})$  is position independent, will be incompressible [ $\vec{\nabla} \cdot \vec{v}(t, \vec{r}) = 0$ , cf. Eq. (II.16)].

## III.3 Momentum balance: Euler and Navier–Stokes equations

For a closed system  $\Sigma$  with total linear momentum  $\vec{P}$  with respect to a given reference frame  $\mathcal{R}$ , Newton's second law reads

$$\frac{D\vec{P}(t)}{Dt} = \vec{F}(t), \quad (\text{III.14})$$

with  $\vec{F}$  the sum of the “external” forces acting on  $\Sigma$ .

The left hand side of this equation can be transformed with the help of Reynolds' transport theorem (III.8), irrespective of any assumption on the fluid under consideration (§ III.3.1). In contrast, the forces acting on a fluid element, more precisely the forces exerted by the neighboring elements, do depend on the (assumed) properties of the fluid. The two most widespread models

<sup>(v)</sup>C. F. GAUSS, 1777–1855

used for fluids are that of a perfect fluid, which leads to the Euler equation (§ III.3.2), and of a Newtonian fluid, for which Newton’s second law (III.14) translates into the Navier<sup>(w)</sup>–Stokes<sup>(x)</sup> equation (§ III.3.3).

Throughout this Section, we use the shorter designation “momentum” instead of the more accurate “linear momentum”.

### III.3.1 Material derivative of momentum

As already noted below Eq. (III.2), the volumetric density associated with the momentum  $\vec{P}(t)$  is simply the product of the mass density with the flow velocity,  $\rho(t, \vec{r}) \vec{v}(t, \vec{r})$ . Applying Reynolds’ theorem (III.8) to the momentum of the material system contained at time  $t$  inside a control volume  $\mathcal{V}$ , the material derivative on the left hand side of Newton’s law (III.14) can be recast as

$$\frac{D\vec{P}(t)}{Dt} = \int_{\mathcal{V}} \frac{\partial}{\partial t} [\rho(t, \vec{r}) \vec{v}(t, \vec{r})] d^3\vec{r} + \oint_{\partial\mathcal{V}} \vec{v}(t, \vec{r}) \rho(t, \vec{r}) \vec{v}(t, \vec{r}) \cdot d^2\vec{S}. \quad (\text{III.15})$$

Both terms on the right hand side can be transformed to yield more tractable expressions. On the one hand, the action of the time derivative on  $\rho(t, \vec{r}) \vec{v}(t, \vec{r})$  in the integrand of the volume integral is given by the usual product rule. On the other hand, one can show the identity

$$\oint_{\partial\mathcal{V}} \vec{v}(t, \vec{r}) \rho(t, \vec{r}) \vec{v}(t, \vec{r}) \cdot d^2\vec{S} = \int_{\mathcal{V}} \left\{ -\vec{v}(t, \vec{r}) \frac{\partial \rho(t, \vec{r})}{\partial t} + \rho(t, \vec{r}) [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}] \vec{v}(t, \vec{r}) \right\} d^3\vec{r}. \quad (\text{III.16})$$

All in all, one thus obtains

$$\frac{D\vec{P}(t)}{Dt} = \int_{\mathcal{V}} \rho(t, \vec{r}) \left\{ \frac{\partial \vec{v}(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}] \vec{v}(t, \vec{r}) \right\} d^3\vec{r} = \int_{\mathcal{V}} \rho(t, \vec{r}) \frac{D\vec{v}(t, \vec{r})}{Dt} d^3\vec{r}. \quad (\text{III.17})$$

Proof of relation (III.16): consider the  $i$ -th component of the vector defined by the surface integral on the left hand side of that identity. Gauss’s divergence theorem gives

$$\oint_{\partial\mathcal{V}} [v^i(t, \vec{r}) \rho(t, \vec{r}) \vec{v}(t, \vec{r})] \cdot d^2\vec{S} = \int_{\mathcal{V}} \vec{\nabla} \cdot [v^i(t, \vec{r}) \rho(t, \vec{r}) \vec{v}(t, \vec{r})] d^3\vec{r}.$$

The action of the differential operator yields  $v^i(t, \vec{r}) \vec{\nabla} \cdot [\rho(t, \vec{r}) \vec{v}(t, \vec{r})] + \rho(t, \vec{r}) \vec{v}(t, \vec{r}) \cdot \vec{\nabla} v^i(t, \vec{r})$ : the divergence in the first term can be recast using the continuity equation (III.12) as the negative of the time derivative of the mass density, leading to

$$\vec{\nabla} \cdot [v^i(t, \vec{r}) \rho(t, \vec{r}) \vec{v}(t, \vec{r})] = -v^i(t, \vec{r}) \frac{\partial \rho(t, \vec{r})}{\partial t} + \rho(t, \vec{r}) [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}] v^i(t, \vec{r}).$$

This relation holds for all three components, from where Eq. (III.16) follows.  $\square$

**Remark:** The derivation of Eq. (III.17) relies on purely algebraic transformations, either as encoded in Reynolds’ transport theorem, or when going from relation (III.15) to (III.17). That is, it does not imply any model—apart from that of a continuous medium—for the fluid properties. In particular, whether or not dissipative effects are important in the fluid did not play any role here.

### III.3.2 Perfect fluid: Euler equation

In this section, we first introduce the notion, or rather the model, of a *perfect fluid*, which is defined by the choice of a specific ansatz for the stress tensor that encodes the contact forces between neighboring fluid elements (§ III.3.2 a). Using that ansatz and the results of the previous paragraph, Newton’s second law (III.14) is shown in § III.3.2 b to be equivalent to a local formulation, the so-called *Euler equation*. Eventually, the latter is recast in the generic form for a local conservation or balance equation, involving the time derivative of a local density and the divergence of the corresponding flux density (§ III.3.2 e).

<sup>(w)</sup>C.-L. NAVIER, 1785–1836 <sup>(x)</sup>G. G. STOKES, 1819–1903

### III.3.2a Forces in a perfect fluid

The forces in a fluid were already discussed on a general level in § I.4.1. Thus, the total force on the right hand side of Eq. (III.14) consists of volume and surface forces, which can respectively be expressed as a volume or a surface integral

$$\vec{F}(t) = \int_{\mathcal{V}} \vec{f}_V(t, \vec{r}) d^3\vec{r} + \oint_{\mathcal{S}} \vec{T}_s(t, \vec{r}) d^2\mathcal{S}, \quad (\text{III.18})$$

where  $\vec{f}_V$  denotes the local (volumetric) density of body forces, while  $\vec{T}_s$  is the mechanical stress vector introduced in Eq. (I.20). The latter will now allow us to introduce various models of fluids.

The first, simplest model is that of a *perfect fluid*, or *ideal fluid*:

A perfect fluid is a fluid in which there are no shear stresses nor heat conduction.

(III.19a)

Stated differently, at every point of a perfect fluid the stress vector  $\vec{T}_s$  on a (test) surface element  $d^2\mathcal{S}$  moving with the fluid is normal to  $d^2\mathcal{S}$ , irrespective of whether the fluid is at rest or in motion. That is, introducing the normal unit vector  $\vec{e}_n(\vec{r})$  to  $d^2\mathcal{S}$  oriented towards the exterior of the material region acted upon<sup>(3)</sup> one may write

$$\vec{T}_s(t, \vec{r}) = -\mathcal{P}(t, \vec{r}) \vec{e}_n(\vec{r}), \quad (\text{III.19b})$$

with  $\mathcal{P}(t, \vec{r})$  the pressure at position  $\vec{r}$ . Accordingly, the mechanical stress tensor in a perfect fluid in a reference frame  $\mathcal{R}$  which is moving with the fluid is given by

$$\boldsymbol{\sigma}(t, \vec{r}) = -\mathcal{P}(t, \vec{r}) \mathbf{g}^{-1}(t, \vec{r}), \quad (\text{III.19c})$$

with  $\mathbf{g}^{-1}$  the inverse metric tensor, just like in a fluid at rest [Eq. (I.23)]. In a given coordinate system in  $\mathcal{R}$ , the  $\binom{0}{0}$ -components of  $\boldsymbol{\sigma}$  thus simply read

$$\boldsymbol{\sigma}^{ij}(t, \vec{r}) = -\mathcal{P}(t, \vec{r}) g^{ij}(t, \vec{r}), \quad (\text{III.19d})$$

i.e. the  $\binom{1}{1}$ -components are  $\boldsymbol{\sigma}^i_j(t, \vec{r}) = -\mathcal{P}(t, \vec{r}) \delta^i_j$ .

Using relation (III.19b), the total surface forces in Eq. (III.18) can be transformed into a volume integral:

$$\oint_{\mathcal{S}} \vec{T}_s(t, \vec{r}) d^2\mathcal{S} = -\oint_{\mathcal{S}} \mathcal{P}(t, \vec{r}) \vec{e}_n(\vec{r}) d^2\mathcal{S} = -\oint_{\mathcal{S}} \mathcal{P}(t, \vec{r}) d^2\vec{\mathcal{S}} = -\int_{\mathcal{V}} \vec{\nabla} \mathcal{P}(t, \vec{r}) d^3\vec{r}, \quad (\text{III.20})$$

where the last identity follows from a corollary of the usual divergence theorem.

**Remark:** Although this might not be intuitive at first, the pressure  $\mathcal{P}(t, \vec{r})$  entering Eqs. (III.19b)–(III.19d) is actually the hydrostatic pressure already introduced in the definition of the mechanical stress in a fluid at rest, see Eq. (I.23). One heuristic justification is that the stresses are defined as the forces per unit area exerted by a piece of fluid situated on one side of a surface on the fluid situated on the other side. Even if the fluid is moving, the two fluid elements on both sides of the surface—as well as the comoving test surface—have the same velocity<sup>(4)</sup> i.e. their relative velocity vanishes, just like in a fluid at rest.

### III.3.2b Euler equation

Gathering Eqs. (III.14), (III.17), (III.18) and (III.20) yields

$$\int_{\mathcal{V}} \rho(t, \vec{r}) \left\{ \frac{\partial \vec{v}(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}] \vec{v}(t, \vec{r}) \right\} d^3\vec{r} = \int_{\mathcal{V}} \left[ -\vec{\nabla} \mathcal{P}(t, \vec{r}) + \vec{f}_V(t, \vec{r}) \right] d^3\vec{r}.$$

<sup>(3)</sup> Cf. the discussion between Eqs. (I.21a)–(I.21c).

<sup>(4)</sup> ... thanks to the usual continuity assumption: this no longer holds at a discontinuity!

Since this identity must hold irrespective of the control volume  $\mathcal{V}$ , the integrands on both sides must be equal. That is, the various fields they involve obey the *Euler equation*

$$\rho(t, \vec{r}) \left\{ \frac{\partial \vec{v}(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}] \vec{v}(t, \vec{r}) \right\} = -\vec{\nabla} \mathcal{P}(t, \vec{r}) + \vec{f}_V(t, \vec{r}). \quad (\text{III.21})$$

**Remarks:**

- \* The term in curly brackets on the left hand side is exactly the acceleration (I.17) of a material point, as in Newton’s second law.
- \* Due to the convective term  $(\vec{v} \cdot \vec{\nabla})\vec{v}$ , the Euler equation is a *nonlinear* partial differential equation.
- \* Besides Newton’s second law for linear momentum, one could also think of investigating the consequence of its analogue for angular momentum. Since we have assumed that the material points do not have any intrinsic spin, the conservation of angular momentum, apart from leading to the necessary symmetry of the stress tensor—which is realized in a perfect fluid, see Eq. (III.19c) or (III.19d), and will also hold in a Newtonian fluid, see Eq. (III.30)—does not bring any new dynamical equation.

### III.3.2c Boundary conditions

To fully formulate the mathematical problem representing a given fluid flow, one must also specify *boundary conditions* for the various partial differential equations. These conditions reflect the geometry of the problem under consideration.

- Far from an obstacle or from walls, one may specify a given pattern for the flow velocity field. For instance, one may require that the flow be uniform “at infinity”, as e.g. for the motion far from the rotating cylinder in Fig. ?? illustrating the geometry of the Magnus effect.
- At an obstacle, in particular at a wall, the component of velocity *perpendicular* to the obstacle should vanish: that is, the fluid cannot penetrate the obstacle or wall, which makes sense and will be hereafter often referred to as *impermeability* condition. In case the obstacle is itself in motion, one should consider the (normal component of the) *relative* velocity of the fluid with respect to the obstacle.

On the other hand, the model of a perfect fluid, in which there is by definition no friction, does not specify the value of the tangential component of the fluid relative velocity at an obstacle.