CHAPTER III

Fundamental equations of non-relativistic fluid dynamics

Some of the most fundamental laws of physics are conservation equations for various quantities: energy, linear and angular momentum, electric charge, and so on. When applying these laws to many-body systems, in particular to continuous media like moving fluids, care must be taken to consider isolated and closed systems, to ensure their validity. At the very least, the amount of quantity exchanged with the exterior of the system—for example the change in linear momentum per unit time due to external forces, as given by Newton's second law, or the change in energy due to the mechanical work of these forces—must be quantifiable.

When this is the case, it is possible to re-express global conservation laws or more generally balance equations—given in terms of macroscopic quantities like total mass, total energy, total momentum, etc.—in a local form involving densities, using the generic recipe provided by Reynolds'(u) transport theorem (Sec. III.1). In the framework of a non-relativistic theory, in which the mass or equivalently the particle number of a closed system is conserved, one may thus derive a general continuity equation, holding at every point of the fluid (Sec. III.2).

The same approach may be followed to derive equations expressing the time evolution of momentum or energy under the influence of external forces acting at every point of the fluid. In either case, it is however necessary to account for the possibility that several physical phenomena may contribute to the transport of momentum and energy. One defines various fluid-dynamical models according to the level of approximation at which the possible forms of transport are incorporated in the description. To those different models correspond distinct sets of equations for the local expression of Newton's second law (Sec. III.3) or of energy balance (Sec. III.4).

(u) O. REYNOLDS, 1842–1912

III.1 Reynolds transport theorem

The material derivative of a quantity was already introduced in § I.3.4, where its action on a local function of both time t and position vector $\vec{r}$ was defined. In this Section, we shall derive a formula for the substantial derivative of an extensive physical quantity carried by a "macroscopic" material system. This formula will in the remainder of the Chapter represent the key relation that will allow us to express the usual (conservation) laws of Newtonian mechanics, which hold for closed systems, in terms of Eulerian variables.

III.1.1 Closed system, open system

Consider the motion of a continuous medium, in particular a flowing fluid, described in a reference frame $\mathcal{R}$. Let $\mathcal{S}$ be an arbitrary closed and connected geometrical surface that remains fixed in $\mathcal{R}$. This surface will hereafter be referred to as *control surface*, and the geometrical volume $\mathcal{V}$ it encloses as *control volume*. Due to the macroscopic transport of matter in the flowing medium, the fluid contained inside a given control surface represents an *open system*, which can exchange matter and energy with its exterior as time elapses.

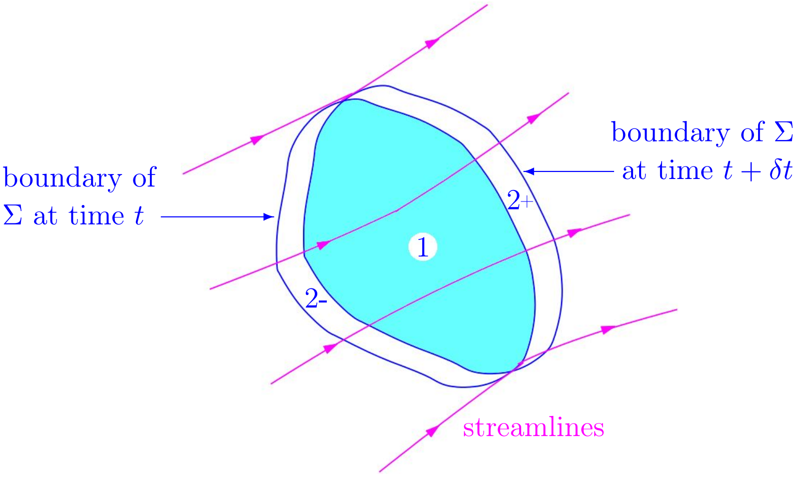


Figure III.1 – Time evolution of a closed material system transported in the motion of a continuous medium.

In contrast, let Σ be a *closed* system, whose constituent material points occupy at some given time t the volume $\mathcal{V}_\Sigma(t)$. At a shortly later time $t+\delta t$, the material system Σ has moved and now occupies a new volume $\mathcal{V}_\Sigma(t+\delta t)$. On Fig. III.1, one can distinguish between three regions in position space:

- (1), which is common to the successive positions of Σ at t and $t+\delta t$; the corresponding volume will be denoted $\mathcal{V}$ and its surface $\partial\mathcal{V}$;
- (2−), which is left behind by Σ between t and $t+\delta t$;
- (2+), into which Σ penetrates between t and $t+\delta t$.

Note already that in the limit $\delta t \to 0$, the volumes $\mathcal{V}_\Sigma(t)$, $\mathcal{V}_\Sigma(t+\delta t)$ and $\mathcal{V}$ coincide; more precisely, their difference is of order δt.

III.1.2 Material derivative of an extensive quantity

Let $\mathcal{G}(t)$ be one of the extensive quantities that characterize a macroscopic physical property of the closed material system Σ. To this extensive quantity, one can associate at every point $\vec{r}$ the corresponding intensive *volumetric density* $\mathscr{g}(t,\vec{r})$, defined as the local amount of $\mathcal{G}$ per unit volume.

Denoting by $\mathrm{d}\mathcal{V}(t,\vec{r})$ a small material volume at position $\vec{r}$ at time t, and by $\mathrm{d}\mathcal{G}(t,\vec{r})$ the amount of $\mathcal{G}$ of that material volume, one can symbolically write

$$\mathcal{g}(t,\vec{r}) = \frac{\mathrm{d}\mathcal{G}(t,\vec{r})}{\mathrm{d}\mathcal{V}(t,\vec{r})}, \tag{III.1}$$

where the notation with differentials is used to suggest that the identity holds in the limit of a vanishing material volume $\mathrm{d}\mathcal{V}(t,\vec{r}) \to 0$. In particular, if $\mathcal{G}$ is the mass (M), then the associated volumetric density is the usual *mass density*(xxxii) (or more shortly *density*) $\rho(t,\vec{r})$:

$$\rho(t,\vec{r}) = \frac{\mathrm{d}M(t,\vec{r})}{\mathrm{d}\mathcal{V}(t,\vec{r})}, \tag{III.2}$$

For instance, the linear momentum and the kinetic energy of a mass $\mathrm{d}M$ of (non-relativistic) fluid moving with velocity $\vec{\mathsf{v}}$ are $\mathrm{d}\vec{P} = (\mathrm{d}M)\vec{\mathsf{v}}$ and $\mathrm{d}K = \frac{1}{2}(\mathrm{d}M)\vec{\mathsf{v}}^2$, respectively. Invoking Eq. (III.2), the associated volumetric densities are $\mathrm{d}\vec{P}/\mathrm{d}\mathcal{V} = \rho\vec{\mathsf{v}}$ resp. $\mathrm{d}K/\mathrm{d}\mathcal{V} = \frac{1}{2}\rho\vec{\mathsf{v}}^2$.

For a material system Σ occupying a volume $\mathcal{V}_\Sigma(t)$ at time t, Eq. (III.1) leads to

$$\mathcal{G}(t) = \int_{\mathcal{V}_\Sigma(t)} \mathcal{g}(t,\vec{r})\,\mathrm{d}^3\vec{r} \tag{III.3}$$

for the value of $\mathcal{G}$ of the system.

Remarks:

∗ The above examples illustrate the fact that the tensorial nature—scalar, vector, tensor of higher order—of the (mathematical) function associated with the physical quantity $\mathcal{G}$ can be arbitrary. This naturally also holds for the associated volumetric density $\mathcal{g}$.

∗ Instead of the volume density $\mathcal{g}(t,\vec{r})$, one can also use the *specific density* $\mathcal{g}_{\mathrm{m}}(t,\vec{r})$, i.e. the local amount of $\mathcal{G}$ per unit mass of matter. Denoting by $\mathrm{d}M(t,\vec{r})$ the amount of mass inside a small material volume at position $\vec{r}$ at time t, one has similar to Eq. (III.1)

$$\mathcal{g}_{\mathrm{m}}(t,\vec{r}) = \frac{\mathrm{d}\mathcal{G}(t,\vec{r})}{\mathrm{d}M(t,\vec{r})}. \tag{III.4}$$

With the help of the (volumetric) mass density (III.2), one readily finds $\mathcal{g}_{\mathrm{m}}(t,\vec{r}) = \mathcal{g}(t,\vec{r})/\rho(t,\vec{r})$.

Let us now assume that the material system Σ is moving as part of a larger, flowing continuous medium. To find the substantial derivative $\mathrm{D}\mathcal{G}(t)/\mathrm{D}t$ of $\mathcal{G}(t)$, we shall first compute the variation $\delta\mathcal{G}$ of $\mathcal{G}$ for the material system Σ—followed in its motion—between times t and $t+\delta t$, assuming δt to be small. At the end of the calculation, we shall take the limit $\delta t \to 0$.

Going back to the regions (1), (2−), (2+) defined in Fig. III.1, one can write

$$\delta\mathcal{G} = \big(\mathcal{G}_1 + \mathcal{G}_{2+}\big)_{t+\delta t} - \big(\mathcal{G}_1 + \mathcal{G}_{2-}\big)_t = \delta\mathcal{G}_1 + \delta\mathcal{G}_2,$$

where the various indices denote the respective spatial domains and instants, and

$$\delta\mathcal{G}_1 \equiv \big(\mathcal{G}_1\big)_{t+\delta t} - \big(\mathcal{G}_1\big)_t, \qquad \delta\mathcal{G}_2 \equiv \big(\mathcal{G}_{2+}\big)_{t+\delta t} - \big(\mathcal{G}_{2-}\big)_t.$$

- $\delta\mathcal{G}_1$ represents the variation of $\mathcal{G}$ inside region (1) due to the non-stationarity of the flow:

  $$\delta\mathcal{G}_1 = \int_{\mathcal{V}} \big[\mathcal{g}(t+\delta t,\vec{r}) - \mathcal{g}(t,\vec{r})\big]\,\mathrm{d}^3\vec{r}.$$

  Performing a Taylor expansion of $\mathcal{g}(t+\delta t,\vec{r})$ to linear order in δt, one may write

  $$\delta\mathcal{G}_1 \simeq \int_{\mathcal{V}} \frac{\partial\mathcal{g}(t,\vec{r})}{\partial t}\,\mathrm{d}^3\vec{r}\;\delta t, \tag{III.5}$$

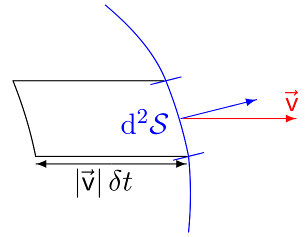
  valid up to terms of order $(\delta t)^2$.

(xxxii) Massendichte

- $\delta\mathcal{G}_2$ represents the net amount of $\mathcal{G}$ traversing the control surface $\partial\mathcal{V}$ between t and $t+\delta t$ because of the fluid motion, either leaving (region 2+) or entering (region 2−) the volume $\mathcal{V}$, where in the latter case the amount is counted negatively. This is precisely the product by the duration δt of the time interval of the *convective flux*(xxxiii) of $\mathcal{G}$ through the surface $\partial\mathcal{V}$, oriented towards the exterior.(2) In turn, the convective flux through a finite surface is the integral of an appropriate *convective flux density*,(xxxiv) defined as the convective flux through a unit surface. Denoting by $\vec{\mathcal{J}}_{\mathcal{G}}^{\text{conv.}}(t,\vec{r})$ the convective flux density (at time t and position $\vec{r}$) of the quantity $\mathcal{G}$, one thus has

  $$\delta\mathcal{G}_2 = \oint_{\partial\mathcal{V}} \vec{\mathcal{J}}_{\mathcal{G}}^{\text{conv.}}(t,\vec{r})\cdot \mathrm{d}^2\vec{\mathcal{S}}\;\delta t. \tag{III.6}$$

  There is a simple relationship between the convective flux density and the volumetric density $g(t,\vec{r})$ of the associated quantity.



Indeed, let $\vec{\mathrm{v}}(t,\vec{r})$ denote the fluid velocity at position $\vec{r}$ at time t. The amount of quantity $\mathcal{G}$ that traverses in δt a small surface element $\mathrm{d}^2\mathcal{S}$ situated at $\vec{r}$ equals the amount inside an elementary cylinder with base $\mathrm{d}^2\mathcal{S}$ and height $|\vec{\mathrm{v}}(t,\vec{r})|\,\delta t$, i.e. $\mathrm{d}^3\mathcal{G} = g(t,\vec{r})\,\mathrm{d}^3\mathcal{V}$, with $\mathrm{d}^3\mathcal{V} = |\mathrm{d}^2\vec{\mathcal{S}}\cdot\vec{\mathrm{v}}(t,\vec{r})|\,\delta t$, where the vector $\mathrm{d}^2\vec{\mathcal{S}}$ is normal to the surface element. From that, one deduces

$$\vec{\mathcal{J}}_{\mathcal{G}}^{\text{conv.}}(t,\vec{r}) = g(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r}). \tag{III.7}$$

All in all, Eqs. (III.5)–(III.6) yield after dividing by δt and taking the limit $\delta t\to 0$ the so-called *Reynolds transport theorem*:(xxxv)

$$\boxed{\frac{\mathrm{D}\mathcal{G}(t)}{\mathrm{D}t} = \int_{\mathcal{V}} \frac{\partial g(t,\vec{r})}{\partial t}\,\mathrm{d}^3\vec{r} + \oint_{\partial\mathcal{V}} g(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\cdot\mathrm{d}^2\vec{\mathcal{S}}.} \tag{III.8}$$

The first term on the right hand side of this relation represents a *local* time derivative, similar to the first term in Eq. (I.18), for which reason it is often denoted $\partial\mathcal{G}/\partial t$. In contrast, the second term is of *convective* type, i.e. directly caused by the motion of matter, and represents the (macroscopic) transport of $\mathcal{G}$.

Anticipating on the remainder of the Chapter, this theorem will help us as follows. The "usual" laws of dynamics are valid for closed, material systems Σ, rather than for open ones. Accordingly, these laws involve time derivatives "following the system in its motion", which is precisely what the material derivative $\mathrm{D}/\mathrm{D}t$ describes. Reynolds' transport theorem (III.8) expresses the latter, for extensive quantities $\mathcal{G}(t)$, in terms of local densities attached to fixed spatial positions, i.e. in Eulerian variables.

Remarks:

$*$ Paralleling the remark following Eq. (III.2), the actual tensorial nature of the (convective) flux density depends on that of the transported quantity: If $\mathcal{G}$ is a scalar, $\vec{\mathcal{J}}_{\mathcal{G}}^{\text{conv}}$ is a vector. If the transported quantity is represented by a vector $\vec{\mathcal{G}}$, then the corresponding flux density is a tensor of order 2—corresponding to having a vector for each component of $\mathcal{G}$ in a given basis. This will be illustrated in Secs. (III.2)–(III.4).

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(2) The *flux* of a quantity through a surface is by definition the amount of the quantity that flows through the surface per unit time. The convective (or *advective*) flux is the fraction of the total flux which is due to the macroscopic motion.

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∗ A flux, or a flux density, are defined as the amounts of quantity traversing per unit time a surface at rest, which automatically selects a given reference frame—the rest frame of the surface. This in particular holds for the closed surface $\partial\mathcal{V}$ (and thus for the volume it encloses) in Reynolds' transport theorem.

∗ Since relation (III.8) is traditionally referred to as a theorem, one may wonder what are its assumptions. Obviously, the derivation of the result relies on the assumption that the density $\mathscr{g}(t,\vec{r})$ and the velocity field $\vec{\mathrm{v}}(t,\vec{r})$ are both continuous and differentiable, in agreement with the generic hypotheses in § I.2.2. Figure III.1 actually also embodies the hidden, but necessary assumption that the motion is continuous, which leads to the smooth evolution of the connected system of material points which are together inside the control surface $\partial\mathcal{V}$ at time t. Again, this follows from suitable properties of $\vec{\mathrm{v}}$.

∗ Accordingly, the Reynolds transport theorem (III.8) does not hold if the velocity field, or the volumetric density $\mathscr{g}$, is discontinuous. As was already mentioned in § I.2.2, such discontinuities are however necessary to account for some phenomena (shock waves, boundary between two immiscible fluids…). In such cases, it will be necessary to reformulate the transport theorem to take into account the discontinuities.