Using the tensors **D** and **R** we just introduced, whose physical meaning will be discussed at length in § II.1.3–II.1.4, relation (II.2b) can be recast as

$$\vec{\mathsf{v}}\big(t,\vec{r}+\delta\vec{r}\big) = \vec{\mathsf{v}}\big(t,\vec{r}\big) + \mathbf{D}(t,\vec{r})\cdot\delta\vec{r} + \mathbf{R}(t,\vec{r})\cdot\delta\vec{r} + \mathcal{O}\big(|\delta\vec{r}|^2\big) \tag{II.5}$$

where, as stated at the beginning, every field is considered at the same time.

Under consideration of relation (II.5) with $\delta\vec{r} = \delta\vec{\ell}(t)$, Eq. (II.1) for the time evolution of the material line element becomes

$$\delta\vec{\ell}(t+\delta t) = \delta\vec{\ell}(t) + \big[\mathbf{D}(t,\vec{r})\cdot\delta\vec{\ell}(t) + \mathbf{R}(t,\vec{r})\cdot\delta\vec{\ell}(t)\big]\delta t + \mathcal{O}\big(\delta t^2\big). \tag{II.6}$$

Subtracting $\delta\vec{\ell}(t)$ from both sides, dividing by δt and taking the limit $\delta t \to 0$, one finds for the rate of change of the material vector:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{\ell}(t) = \mathbf{D}(t,\vec{r})\cdot\delta\vec{\ell}(t) + \mathbf{R}(t,\vec{r})\cdot\delta\vec{\ell}(t) \tag{II.7}$$

In the following two subsections, we shall investigate the physical content of each of the tensors $\mathbf{R}(t,\vec{r})$ and $\mathbf{D}(t,\vec{r})$.

II.1.3 Rotation rate tensor and vorticity vector

The tensor $\mathbf{R}(t,\vec{r})$ defined by Eq. (II.3b) is called, for reasons that will become clearer below, *rotation rate tensor*.(xxiii)

By construction, this tensor is antisymmetric. Accordingly, one can naturally associate with it a dual (pseudo)-vector $\vec{\Omega}(t,\vec{r})$, such that for any vector $\vec{V}$

$$\mathbf{R}\big(t,\vec{r}\big)\cdot\vec{V} = \vec{\Omega}\big(t,\vec{r}\big)\times\vec{V} \quad \forall\vec{V}\in\mathbb{R}^3. \tag{II.8}$$

In Cartesian coordinates, the components of $\vec{\Omega}(t,\vec{r})$ are related to those of the rotation rate tensor by

$$\Omega^i(t,\vec{r}) \equiv -\frac{1}{2}\sum_{j,k=1}^{3}\epsilon^{ijk}\,\mathbf{R}_{jk}(t,\vec{r}) \tag{II.9a}$$

with ϵ^{ijk} the totally antisymmetric Levi-Civita(s) symbol. Using the antisymmetry of $\mathbf{R}\big(t,\vec{r}\big)$, this equivalently reads

$$\Omega^1(t,\vec{r}) \equiv -\mathbf{R}_{23}(t,\vec{r}), \quad \Omega^2(t,\vec{r}) \equiv -\mathbf{R}_{31}(t,\vec{r}), \quad \Omega^3(t,\vec{r}) \equiv -\mathbf{R}_{12}(t,\vec{r}). \tag{II.9b}$$

Comparing with Eq. (II.3c), one finds

$$\vec{\Omega}(t,\vec{r}) = \frac{1}{2}\vec{\nabla}\times\vec{\mathsf{v}}(t,\vec{r}). \tag{II.10}$$

Proof of Eqs. (II.8), (II.10): introducing the Cartesian components $\{V^j\}$ of $\vec{V}$ and dropping for brevity the $(t,\vec{r})$-dependence of fields, the i-th component of $\mathbf{R}\cdot\vec{V}$ reads

$$\mathbf{R}_{ij}V^j = \frac{1}{2}\big(\partial_j\mathsf{v}_i - \partial_i\mathsf{v}_j\big)V^j,$$

where we used the summation convention over the repeated index j and the shorthand notation ∂_i for the partial derivative with respect to x^i. This may further be rewritten as

$$\mathbf{R}_{ij}V^j = -\frac{1}{2}\big(\delta_i^k\delta_j^l - \delta_j^k\delta_i^l\big)(\partial_k\mathsf{v}_l)V^j,$$

which now involves three sums. The term with the four Kronecker symbols is in fact the sum

(xxiii) *Wirbeltensor*

(s) T. Levi-Civita, 1873–1941

(over a fifth index m) of the product $\epsilon_{ijm}\epsilon^{mkl}$ of two Levi-Civita symbols:

$$\mathsf{R}_{ij}V^j = -\frac{1}{2}\epsilon_{ijm}\epsilon^{mkl}(\partial_k\mathsf{v}_l)V^j.$$

On the right hand side of this identity, $\epsilon^{mkl}\partial_k\mathsf{v}_l$ is the m-th component of the curl $\vec{\nabla}\times\vec{\mathsf{v}}$, i.e. using definition (II.10):

$$\mathsf{R}_{ij}V^j = -\epsilon_{ijm}\Omega^m V^j = \epsilon_{imj}\Omega_m V^j,$$

which is precisely the i-th component of $\Omega\times\vec{\mathsf{v}}$. □

Let us now rewrite relation (II.7) with the help of the vector $\vec{\Omega}(t,\vec{r})$, assuming that $\mathsf{D}(t,\vec{r})$ vanishes so as to isolate the effect of the remaining term. Under this assumption, the rate of change of the material vector between two neighboring points reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{\ell}(t) = \mathsf{R}(t,\vec{r})\cdot\delta\vec{\ell}(t) = \vec{\Omega}(t,\vec{r})\times\delta\vec{\ell}(t). \tag{II.11}$$

The term on the right hand side is then exactly the rate of rotation of a vector $\delta\vec{\ell}(t)$ in the motion of a rigid body with instantaneous angular velocity $\vec{\Omega}(t,\vec{r})$. Accordingly, the pseudovector $\vec{\Omega}(t,\vec{r})$ is referred to as *local angular velocity*.[(xxiv)] This a posteriori justifies the denomination *rotation rate tensor* for the antisymmetric tensor $\mathsf{R}(t,\vec{r})$.

Remarks:

∗ Besides the local angular velocity $\vec{\Omega}(t,\vec{r})$, one also defines the *vorticity vector*[(xxv)] as the curl of the velocity field

$$\boxed{\vec{\omega}(t,\vec{r}) \equiv \vec{\nabla}\times\vec{\mathsf{v}}(t,\vec{r})} = 2\vec{\Omega}(t,\vec{r}). \tag{II.12}$$

In fluid mechanics, the vorticity is actually more often used than the local angular velocity.

∗ The local angular velocity $\vec{\Omega}(t,\vec{r})$ or equivalently the vorticity vector $\vec{\omega}(t,\vec{r})$ define, at fixed t, divergence-free (pseudo)vector fields, since obviously $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\mathsf{v}}) = 0$. The corresponding field lines are called *vorticity lines*[(xxvi)] and are given by [cf. Eq. (I.15)]

$$\mathrm{d}\vec{x}\times\vec{\omega}(t,\vec{r}) = \vec{0} \tag{II.13a}$$

or equivalently, at a point where none of the components of the vorticity vector vanishes,

$$\frac{\mathrm{d}x_1}{\omega^1(t,\vec{r})} = \frac{\mathrm{d}x_2}{\omega^2(t,\vec{r})} = \frac{\mathrm{d}x_3}{\omega^3(t,\vec{r})}. \tag{II.13b}$$

II.1.4 Strain rate tensor

According to the previous subsection, the local rotational motion of a material vector is governed by the (local and instantaneous) rotation rate tensor $\mathsf{R}(t,\vec{r})$. In turn, the translational motion is simply the displacement—which must be described in an affine space, not a vector one—of one of the endpoints of $\delta\vec{\ell}$ by an amount given by the product of velocity and length of time interval. That is, both components of the motion of a rigid body are already accounted for without invoking the symmetric tensor $\mathsf{D}(t,\vec{r})$.

In other words, the tensor $\mathsf{D}(t,\vec{r})$ characterizes the local deviation between the velocity fields in a deformable body, in particular a fluid, and in a rigid body rotating with angular velocity $\vec{\Omega}(t,\vec{r})$. Accordingly, it is called *strain rate tensor* or *deformation rate tensor*.[(xxvii)]

As we shall now see, the diagonal and off-diagonal components of $\mathsf{D}(t,\vec{r})$ actually describe the rates of change of different kinds of deformation. For simplicity, we assume throughout this subsection that $\vec{\Omega}(t,\vec{r}) = \vec{0}$, or equivalently all components $\mathsf{R}_{ij}(t,\vec{r}) = 0$.

[(xxiv)] *Wirbelvektor* [(xxv)] *Wirbligkeit* [(xxvi)] *Wirbellinien* [(xxvii)] *Verzerrungsgeschwindigkeitstensor, Deformationsgeschwindigkeitstensor*

II.1.4 a Diagonal components

We first assume that all off-diagonal terms in the strain rate tensor vanish: $\mathbf{D}_{ij}(t,\vec{r}) = 0$ for $i \neq j$, so as to isolate the meaning of the diagonal components.

Going back to Eq. (II.6), let us simply project it along one of the axes of the coordinate system, say along direction i. Denoting by $\delta\ell^i$ the i-th component of $\delta\vec{\ell}$, one thus finds

$$\delta\ell^i(t+\delta t) - \delta\ell^i(t) \simeq \sum_{j=1}^{3} \frac{\partial \mathsf{v}^i(t,\vec{r})}{\partial x^j}\, \delta\ell^j(t)\, \delta t,$$

up to terms of higher order in $|\delta\vec{\ell}(t)|$ or δt. Since we have assumed that all components $\mathbf{R}_{ij}(t,\vec{r})$ of the rotation rate tensor and the off-diagonal $\mathbf{D}_{ij}(t,\vec{r})$ with $i \neq j$ are zero, one checks that the partial derivative $\partial\mathsf{v}^i(t,\vec{r})/\partial x^j$ vanishes for $i \neq j$. That is, the only non-zero term in the sum is that with $j = i$, so that the equation simplifies to

$$\delta\ell^i(t+\delta t) - \delta\ell^i(t) \simeq \frac{\partial \mathsf{v}^i(t,\vec{r})}{\partial x^i}\, \delta\ell^i(t)\, \delta t = \mathbf{D}^i{}_i(t,\vec{r})\, \delta\ell^i(t)\, \delta t.$$

Thus, the relative elongation in δt of the i-th component—remember that there is no local rotation, so that the change in $\delta\ell^i$ is entirely due to a variation of the length of the material vector—is given by

$$\frac{\delta\ell^i(t+\delta t) - \delta\ell^i(t)}{\delta\ell^i(t)} = \mathbf{D}^i{}_i(t,\vec{r})\, \delta t \tag{II.14}$$

or alternatively, taking the limit $\delta t \to 0$

$$\frac{1}{\delta\ell^i(t)} \frac{\mathrm{d}}{\mathrm{d}t} \delta\ell^i(t) = \mathbf{D}^i{}_i(t,\vec{r}). \tag{II.15}$$

This equation means that the diagonal component $\mathbf{D}^i{}_i(t,\vec{r})$ represents the local rate of linear elongation in direction i.

Volume expansion rate

Instead of considering a one-dimensional material vector, one can study the evolution of a small "material rectangular parallelepiped" of continuous medium, situated at time t at position $\vec{r}$ with instantaneous edge lengths $\delta L^1(t)$, $\delta L^2(t)$, $\delta L^3(t)$—where for simplicity the coordinate axes are taken along the parallelepiped edges—, so that its volume at time t is simply $\delta\mathcal{V}(t) = \delta L^1(t)\, \delta L^2(t)\, \delta L^3(t)$.

Taking into account Eq. (II.14) for the relative elongation of each edge length, one finds that the relative change in volume between t and $t + \delta t$ is

$$\begin{aligned}\frac{\delta\mathcal{V}(t+\delta t) - \delta\mathcal{V}(t)}{\delta\mathcal{V}(t)} &= \frac{\delta L^1(t+\delta t) - \delta L^1(t)}{\delta L^1(t)} + \frac{\delta L^2(t+\delta t) - \delta L^2(t)}{\delta L^2(t)} + \frac{\delta L^3(t+\delta t) - \delta L^3(t)}{\delta L^3(t)} \\ &= \left[\mathbf{D}^1{}_1(t,\vec{r}) + \mathbf{D}^2{}_2(t,\vec{r}) + \mathbf{D}^3{}_3(t,\vec{r})\right] \delta t.\end{aligned}$$

In the second line, one recognizes the *trace* of the tensor $\mathbf{D}(t,\vec{r})$, which going back to the definition of the latter is equal to the divergence of the velocity fluid:

$$\mathbf{D}^1{}_1(t,\vec{r}) + \mathbf{D}^2{}_2(t,\vec{r}) + \mathbf{D}^3{}_3(t,\vec{r}) = \frac{\partial \mathsf{v}^1(t,\vec{r})}{\partial x^1} + \frac{\partial \mathsf{v}^2(t,\vec{r})}{\partial x^2} + \frac{\partial \mathsf{v}^3(t,\vec{r})}{\partial x^3} = \vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}).$$

That is, this divergence represents the local and instantaneous *volume expansion rate* of the continuous medium. Accordingly, the flow of a fluid is referred to as *incompressible* in some region when the velocity field in that region is divergence-free:

$$\boxed{\text{incompressible flow} \quad \Leftrightarrow \quad \vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}) = 0.} \tag{II.16}$$

We shall comment on this definition in § II.2.3.

II.1.4 b Off-diagonal components

Let us now assume that $\mathbf{D}_{12}(t,\vec{r})$, and thereby automatically $\mathbf{D}_{21}(t,\vec{r})$, is the only non-vanishing component of the strain rate tensor. To see the influence of that component, we need to consider the time evolution of a different object than a material vector, since anything that can affect the latter—translation, rotation, dilatation—has already been described above.

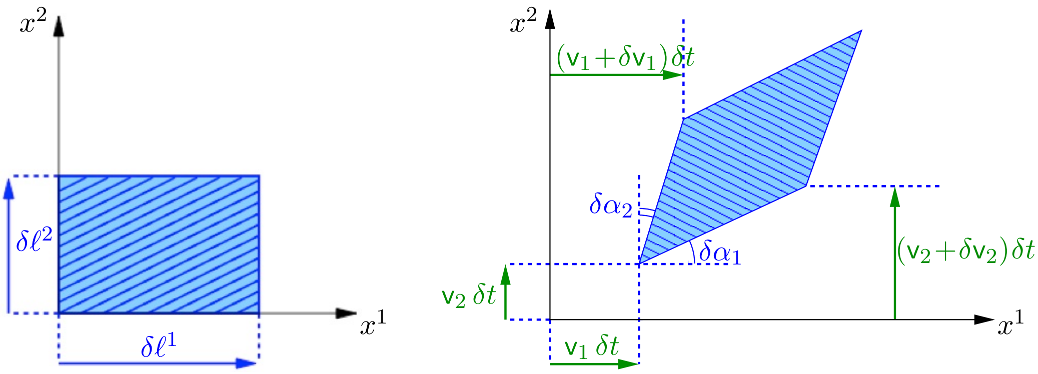

Figure II.2 – Evolution of a material rectangle caught in the motion of a continuous medium between times t (left) and $t+\delta t$ (right).

Accordingly, we now look at the change between successive instants t and $t+\delta t$ of an elementary "material rectangle", as pictured in Fig. II.2. Since it was assumed that only $\mathbf{D}_{12}(t,\vec{r}) = \mathbf{D}_{21}(t,\vec{r})$ is non-zero, the diagonal components of $\mathbf{D}$ vanish, which means that the area of the parallelogram on the right hand side is that of the original rectangle. We denote by $\vec{\mathrm{v}}$ resp. $\vec{\mathrm{v}}+\delta\vec{\mathrm{v}}$ the velocity at time t at the lower left resp. upper right corner of the rectangle. Taylor expansions give for the Cartesian components of the shift $\delta\vec{\mathrm{v}}$

$$\delta\mathrm{v}_1 \simeq \frac{\partial\mathrm{v}_1(t,\vec{r})}{\partial x^2}\,\delta\ell^2, \qquad \delta\mathrm{v}_2 \simeq \frac{\partial\mathrm{v}_2(t,\vec{r})}{\partial x^1}\,\mathrm{d}\ell^1.$$

Figure II.2 shows that what at time t is a right angle becomes an angle $\pi/2-\delta\alpha$ at $t+\delta t$ with $\delta\alpha = \delta\alpha_1 - \delta\alpha_2$, where both $\delta\alpha_1$ and $\delta\alpha_2$ are counted positive in the counterclockwise direction, yet with $\delta\alpha_1$ being measured from the x^1-axis and $\delta\alpha_2$ from the x^2-axis (so that it is negative in the figure). In the limit of small δt, both $\delta\alpha_1$ and $\delta\alpha_2$ will be small and thus approximately equal to their respective tangents. Using the fact that the parallelogram still has the same area, the projection of any side of the deformed rectangle at time $t+\delta t$ on its original direction at time t keeps approximately the same length, up to corrections of order δt. One thus finds for the oriented angles

$$\delta\alpha_1 \simeq \frac{\delta\mathrm{v}_2\,\delta t}{\delta\ell^1} \quad\text{and}\quad \delta\alpha_2 \simeq -\frac{\delta\mathrm{v}_1\,\delta t}{\delta\ell^2}.$$

With the expressions for $\delta\mathrm{v}_1$ and $\delta\mathrm{v}_2$ given above, this leads to

$$\delta\alpha_1 \simeq \frac{\partial\mathrm{v}_2(t,\vec{r})}{\partial x^1}\,\delta t, \qquad \delta\alpha_2 \simeq -\frac{\partial\mathrm{v}_1(t,\vec{r})}{\partial x^2}\,\delta t.$$

Gathering all pieces, one finds

$$\frac{\delta\alpha}{\delta t} \simeq \frac{\partial\mathrm{v}_2(t,\vec{r})}{\partial x^1} + \frac{\partial\mathrm{v}_1(t,\vec{r})}{\partial x^2} = 2\,\mathbf{D}_{12}(t,\vec{r}). \tag{II.17}$$

In the limit $\delta t\to 0$, one sees that the off-diagonal component $\mathbf{D}_{12}(t,\vec{r})$ represents half the local velocity of the "angular deformation"—the *shear*—around direction x^3.

Remark: To separate the two physical effects present in the strain rate tensor, the latter is often written as the sum of a diagonal *rate-of-expansion tensor* proportional to the identity $\mathbb{1}$—which is in fact the $\binom{1}{1}$-form of the metric tensor $\mathbf{g}$ of Cartesian coordinates—and a traceless *rate-of-shear tensor* $\mathbf{S}$:

$$\mathbf{D}(t,\vec{r}) = \frac{1}{3}\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\mathbb{1} + \mathbf{S}(t,\vec{r}) \tag{II.18a}$$

with

$$\mathbf{S}(t,\vec{r}) \equiv \frac{1}{2}\bigg(\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r}) + \big[\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}} - \frac{2}{3}\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\mathbb{1}\bigg). \tag{II.18b}$$

Component-wise, and generalizing to curvilinear coordinates, this reads

$$\mathbf{D}_{ij}(t,\vec{r}) = \frac{1}{3}\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]g_{ij}(t,\vec{r}) + \mathbf{S}_{ij}(t,\vec{r}) \tag{II.18c}$$

with [cf. Eq. (II.4a)]

$$\mathbf{S}_{ij}(t,\vec{r}) \equiv \frac{1}{2}\bigg[g^k_{\ i}(t,\vec{r})g^l_{\ j}(t,\vec{r})\bigg(\frac{\mathrm{d}\mathsf{v}_k(t,\vec{r})}{\mathrm{d}x^l} + \frac{\mathrm{d}\mathsf{v}_l(t,\vec{r})}{\mathrm{d}x^k}\bigg) - \frac{2}{3}\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]g_{ij}(t,\vec{r})\bigg]. \tag{II.18d}$$

Summary

Gathering the findings of this Section, the most general motion of a material volume element inside a continuous medium, in particular in a fluid, can be decomposed in four elements:

- a translation;
- a rotation, with a local angular velocity $\vec{\Omega}(t,\vec{r})$ given by Eq. (II.10)—i.e. related to the antisymmetric part $\mathbf{R}(t,\vec{r})$ of the velocity gradient—and equal to twice the (local) vorticity vector $\vec{\omega}(t,\vec{r})$;
- a local dilatation or contraction, in which the geometric form of the material volume element remains unchanged, whose rate is given by the divergence of the velocity field $\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})$, i.e. encoded in the diagonal elements of the strain rate tensor $\mathbf{D}(t,\vec{r})$;
- a change of shape ("deformation") of the material volume element at constant volume, controlled by the rate-of-shear tensor $\mathbf{S}(t,\vec{r})$ [Eqs. (II.18b), (II.18d)], obtained by taking the traceless symmetric part of the velocity gradient.

Remark: In the case of a uniform motion, all spatial derivatives are by definition zero, so that the vorticity $\vec{\omega}(t,\vec{r})$, the expansion rate $\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})$ and the rate-of-shear tensor $\mathbf{S}(t,\vec{r})$ actually vanish everywhere in the flow. Accordingly, the motion of a material element in that case is simply a pure translation, without deformation or rotation.