Chapter II

Kinematics of a continuous medium

The goal of fluid dynamics is to investigate the motion of fluids under consideration of the forces at play, as well as to study the mechanical stresses exerted by moving fluids on bodies with which they are in contact. The description of the motion itself, irrespective of the forces, is the object of *kinematics*.

The possibilities for the motion of a deformable continuous medium, in particular of a fluid, are richer than for a mere point particle or a rigid body: besides translations and global rotations, a deformable medium may also rotate locally and undergo… deformations! The latter term actually encompasses two different yet non-exclusive possibilities, namely either a change of shape or a variation of the volume. All these various types of motion are encoded in the local properties of the velocity field at each instant (Sec. II.1). Generic fluid motions are then classified according to several criteria, especially taking into account kinematics (Sec. II.2).

For the sake of reference, the characterization of deformations themselves, complementing that of their rate of change, is briefly presented in Sec. II.A. That formalism is not needed within fluid dynamics, but rather for the study of deformable solids, like elastic ones.

II.1 Generic motion of a continuous medium

Let $\vec{\mathsf{v}}(t,\vec{r})$ denote the velocity field in a continuous medium, measured with respect to some reference frame $\mathcal{R}$. Throughout this section, we assume that $\vec{\mathsf{v}}(t,\vec{r})$ is continuously differentiable ($\mathscr{C}^1$) with respect to its variables (see § I.3.2), so that it does not wildly vary from one position to a neighboring one, or from one instant to the next.

To illustrate (some of) the possible motions that occur in a deformable body, Fig. II.1 shows the positions at successive instants t and $t+\delta t$ of a small "material vector" $\delta\vec{\ell}(t)$, that is, a continuous set of material points distributed along the straight line element stretching between two neighboring geometrical points. Let $\vec{r}$ and $\vec{r}+\delta\vec{\ell}(t)$ denote the geometrical endpoints of this material vector at time t.

Thanks to the continuity of the mappings $\vec{R}\mapsto\vec{r}(t,\vec{r})$ and its inverse $\vec{r}\mapsto\vec{R}(t,\vec{r})$, the material vector defined at instant t remains a connected set of material points as time evolves, in particular

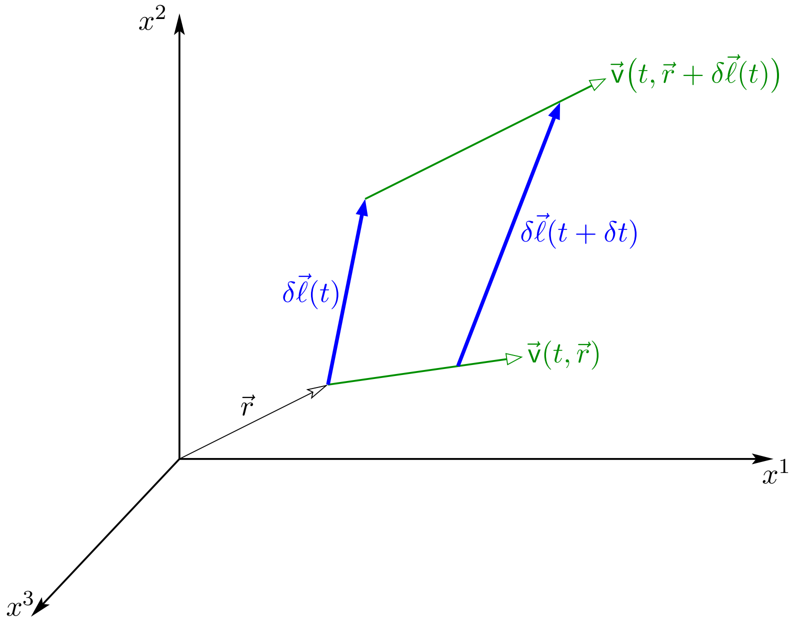

Figure II.1 – Positions of a material line element $\delta\vec{\ell}$ at successive times t and $t+\delta t$.

at $t+\delta t$. Assuming that both the initial length $|\delta\vec{\ell}(t)|$ as well as δt are small enough, the evolved set at $t+\delta t$ remains approximately along a straight line, and constitutes a new material vector, denoted $\delta\vec{\ell}(t+\mathrm{d}t)$. The position vectors of its endpoints simply follow from the initial positions of the corresponding material points: $\vec{r}$ resp. $\vec{r}+\delta\vec{\ell}(t)$, to which should be added the respective displacement vectors between t and t+δt, namely the product by δt of the initial velocity $\vec{\mathsf{v}}(t,\vec{r})$ resp. $\vec{\mathsf{v}}(t,\vec{r}+\delta\vec{\ell}(t))$. That is, one finds

$$\delta\vec{\ell}(t+\delta t)=\delta\vec{\ell}(t)+\left[\vec{\mathsf{v}}\big(t,\vec{r}+\delta\vec{\ell}(t)\big)-\vec{\mathsf{v}}\big(t,\vec{r}\big)\right]\delta t+\mathcal{O}\big(\delta t^2\big). \tag{II.1}$$

Figure II.1 already suggests that the motion of the material vector consists not only of a translation, but also of a rotation, as well as an "expansion"—the change in length of the vector.

II.1.1 Translation motion

Consider one endpoint of the material vector $\delta\vec{\ell}$, for instance the lower one in Fig. II.1. At time t it is at position $\vec{r}$ with the velocity $\vec{\mathsf{v}}(t,\vec{r})$. At time $t+\delta t$, it is then approximately at $\vec{r}+\vec{\mathsf{v}}(t,\vec{r})\,\delta t$.

Assuming that $\vec{\mathsf{v}}(t,\vec{r})$ is non-zero in the reference frame used for the study, then the velocity at time t of every point of $\delta\vec{\ell}$ is also approximately $\vec{\mathsf{v}}(t,\vec{r})$: more accurately, thanks to the assumed $\mathscr{C}^1$-character of $\vec{\mathsf{v}}$, the departure from that value is (at least) linear in $\delta\vec{\ell}(t)$, that is, "small" if the material vector is infinitesimal. In that case, Eq. (II.1) shows that $\delta\vec{\ell}$ is unchanged between t and $t+\delta t$, i.e. it is not rotated and its length remains constant.

Accordingly, when neglecting terms of order $|\delta\vec{\ell}|$ or higher, the motion of $\delta\vec{\ell}$ between t and $t+\delta t$ is a simple translation, by a displacement $\vec{\mathsf{v}}(t,\vec{r})\,\delta t$.

As we shall see presently, deviations from this translation motion—rotations or deformations in a wide sense—are (at least) proportional to $\delta\vec{\ell}(t)$ (and to δt).

II.1.2 Local distribution of velocities in a continuous medium

Considering first a fixed time t, let $\vec{\mathsf{v}}(t,\vec{r})$ resp. $\vec{\mathsf{v}}(t,\vec{r})+\delta\vec{\mathsf{v}}$ be the velocity at the geometric point situated at position $\vec{r}$ resp. at $\vec{r}+\delta\vec{r}$ in the reference frame $\mathcal{R}$. For simplicity, we introduce a system of Cartesian coordinates (x^1,x^2,x^3) in $\mathcal{R}$.

The Taylor expansion of the i-th component $\mathsf{v}^i(t,\vec{r}+\delta\vec{r})$ of the velocity field at $\vec{r}+\delta\vec{r}$ gives, to first order in the components $\{\delta x^j\}$ of $\delta\vec{r}$

$$\delta\mathsf{v}^i \simeq \sum_{j=1}^{3} \frac{\partial\mathsf{v}^i(t,\vec{r})}{\partial x^j}\,\delta x^j. \tag{II.2a}$$

for the corresponding component of $\delta\vec{\mathsf{v}}$. Introducing the $\binom{1}{1}$-tensor $\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})$ whose components in the coordinate system used here are the partial derivatives $\partial\mathsf{v}^i(t,\vec{r})/\partial x^j$, the above relation can be recast in the coordinate-independent form

$$\delta\vec{\mathsf{v}} \simeq \vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\cdot\delta\vec{r}. \tag{II.2b}$$

Like every tensor of order 2, the *velocity gradient tensor*[(xxii)] $\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})$ at time t and position $\vec{r}$ can be decomposed into the sum of the symmetric and an antisymmetric part:

$$\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r}) = \mathbf{D}(t,\vec{r}) + \mathbf{R}(t,\vec{r}), \tag{II.3a}$$

where one conventionally writes

$$\mathbf{D}(t,\vec{r}) \equiv \frac{1}{2}\Big(\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r}) + \big[\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}}\Big), \qquad \mathbf{R}(t,\vec{r}) \equiv \frac{1}{2}\Big(\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r}) - \big[\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}}\Big) \tag{II.3b}$$

with $\big[\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}}$ the transposed tensor to $\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})$. These definitions are to be understood as follows: Using the same Cartesian coordinate system as above, the components of the two tensors $\mathbf{D}$, $\mathbf{R}$, viewed for simplicity as $\binom{0}{2}$-tensors, respectively read

$$\mathbf{D}_{ij}(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial\mathsf{v}_i(t,\vec{r})}{\partial x^j} + \frac{\partial\mathsf{v}_j(t,\vec{r})}{\partial x^i}\right], \qquad \mathbf{R}_{ij}(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial\mathsf{v}_i(t,\vec{r})}{\partial x^j} - \frac{\partial\mathsf{v}_j(t,\vec{r})}{\partial x^i}\right]. \tag{II.3c}$$

Note that here we have silently used the fact that for Cartesian coordinates, the position—subscript or superscript—of the index does not change the value of the component, i.e. numerically $\mathsf{v}_i = \mathsf{v}^i$ for every $i \in \{1,2,3\}$.

Relations (II.3c) clearly represent the desired symmetric and antisymmetric parts. However, one sees that the definitions would not appear to fulfill their task if the indices were not both either up or down, as e.g.

$$\mathbf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial\mathsf{v}^i(t,\vec{r})}{\partial x^j} + \frac{\partial\mathsf{v}_j(t,\vec{r})}{\partial x_i}\right],$$

in which the symmetry is no longer obvious. The trick is to rewrite the previous identity as

$$\mathbf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}\delta^{ik}\delta^l{}_j\left[\frac{\partial\mathsf{v}_k(t,\vec{r})}{\partial x^l} + \frac{\partial\mathsf{v}_l(t,\vec{r})}{\partial x^k}\right] = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\partial\mathsf{v}_k(t,\vec{r})}{\partial x^l} + \frac{\partial\mathsf{v}_l(t,\vec{r})}{\partial x^k}\right],$$

where we have used the fact that the metric tensor of Cartesian coordinates coincides with the Kronecker symbol (and is position-independent). To fully generalize to curvilinear coordinates, the partial derivatives in the rightmost term should be replaced by the covariant derivatives discussed in Appendix **??**, leading eventually to

$$\mathbf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}_k(t,\vec{r})}{\mathrm{d}x^l} + \frac{\mathrm{d}\mathsf{v}_l(t,\vec{r})}{\mathrm{d}x^k}\right] \tag{II.4a}$$

$$\mathbf{R}^i{}_j(t,\vec{r}) = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}_k(t,\vec{r})}{\mathrm{d}x^l} - \frac{\mathrm{d}\mathsf{v}_l(t,\vec{r})}{\mathrm{d}x^k}\right] \tag{II.4b}$$

With these new forms, which are valid in any coordinate system, the raising or lowering of indices does not affect the visual symmetric or antisymmetric aspect of the tensor.

[(xxii)] *Geschwindigkeitsgradient(tensor)*