Chapter II

Kinematics of a continuous medium

The goal of fluid dynamics is to investigate the motion of fluids under consideration of the forces at play, as well as to study the mechanical stresses exerted by moving fluids on bodies with which they are in contact. The description of the motion itself, irrespective of the forces, is the object of *kinematics*.

The possibilities for the motion of a deformable continuous medium, in particular of a fluid, are richer than for a mere point particle or a rigid body: besides translations and global rotations, a deformable medium may also rotate locally and undergo... deformations! The latter term actually encompasses two different yet non-exclusive possibilities, namely either a change of shape or a variation of the volume. All these various types of motion are encoded in the local properties of the velocity field at each instant (Sec. II.1). Generic fluid motions are then classified according to several criteria, especially taking into account kinematics (Sec. II.2).

For the sake of reference, the characterization of deformations themselves, complementing that of their rate of change, is briefly presented in Sec. II.A. That formalism is not needed within fluid dynamics, but rather for the study of deformable solids, like elastic ones.

II.1 Generic motion of a continuous medium

Let $\vec{\mathsf{v}}(t,\vec{r})$ denote the velocity field in a continuous medium, measured with respect to some reference frame $\mathcal{R}$. To illustrate (some of) the possible motions that occur in a deformable body, Fig. II.1 shows the positions at successive instants t and $t+\delta t$ of a small "material vector" $\delta\vec{\ell}(t)$, that is, a continuous set of material points distributed along the straight line element stretching between two neighboring geometrical points. Let $\vec{r}$ and $\vec{r}+\delta\vec{\ell}(t)$ denote the geometrical endpoints of this material vector at time t.

Thanks to the continuity of the mappings $\vec{R}\mapsto\vec{r}(t,\vec{r})$ and its inverse $\vec{r}\mapsto\vec{R}(t,\vec{r})$, the material vector defined at instant t remains a connected set of material points as time evolves, in particular at $t+\delta t$. Assuming that both the initial length $|\delta\vec{\ell}(t)|$ as well as δt are small enough, the evolved set at $t+\delta t$ remains approximately along a straight line, and constitutes a new material vector, denoted $\delta\vec{\ell}(t+\mathrm{d}t)$. The position vectors of its endpoints simply follow from the initial positions of the corresponding material points: $\vec{r}$ resp. $\vec{r}+\delta\vec{\ell}(t)$, to which should be added the respective

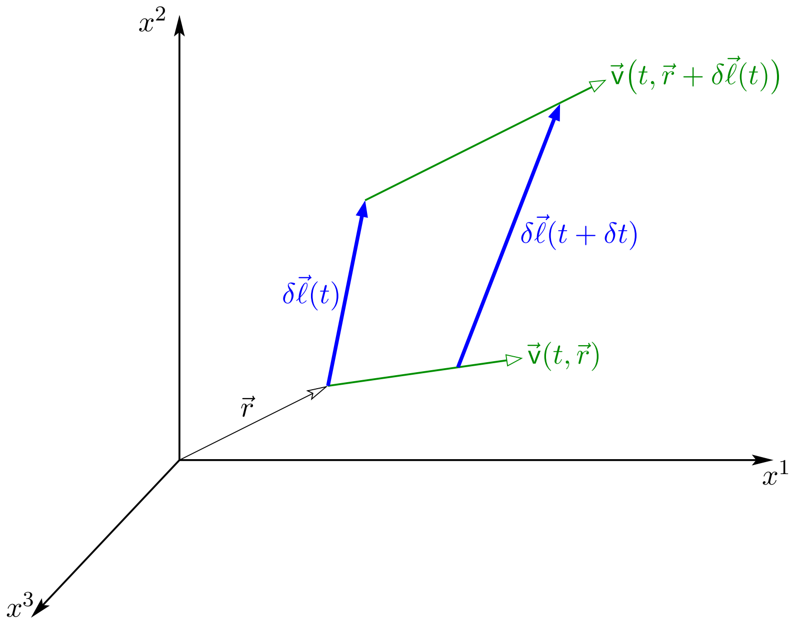

Figure II.1 – Positions of a material line element $\delta\vec{\ell}$ at successive times t and $t+\delta t$.

displacement vectors between t and $t+\delta t$, namely the product by δt of the initial velocity $\vec{\mathsf{v}}(t,\vec{r})$ resp. $\vec{\mathsf{v}}(t,\vec{r}+\delta\vec{\ell}(t))$. That is, one finds

$$\delta\,\vec{\ell}(t+\delta t) = \delta\vec{\ell}(t) + \left[\vec{\mathsf{v}}\big(t,\vec{r}+\delta\vec{\ell}(t)\big) - \vec{\mathsf{v}}\big(t,\vec{r}\big)\right]\delta t + \mathcal{O}\big(\delta t^2\big). \tag{II.1}$$

Figure II.1 already suggests that the motion of the material vector consists not only of a translation, but also of a rotation, as well as an "expansion"—the change in length of the vector.

II.1.1 Local distribution of velocities in a continuous medium

Considering first a fixed time t, let $\vec{\mathsf{v}}(t,\vec{r})$ resp. $\vec{\mathsf{v}}(t,\vec{r})+\delta\vec{\mathsf{v}}$ be the velocity at the geometric point situated at position $\vec{r}$ resp. at $\vec{r}+\delta\vec{r}$ in $\mathcal{R}$.

Introducing for simplicity a system of Cartesian coordinates (x^1,x^2,x^3) in $\mathcal{R}$, the Taylor expansion of the i-th component of the velocity field—which is at least piecewise $\mathscr{C}^1$ in its variables, see § I.3.2—gives to first order

$$\delta\mathsf{v}^i \simeq \sum_{j=1}^{3} \frac{\partial \mathsf{v}^i(t,\vec{r})}{\partial x^j}\,\delta x^j, \tag{II.2a}$$

where $\{\delta x^j\}$ denote the components of $\delta\vec{r}$. Introducing the $\binom{1}{1}$-tensor $\vec{\nabla}\vec{\mathbf{v}}(t,\vec{r})$ whose components in the coordinate system used here are the partial derivatives $\partial\mathsf{v}^i(t,\vec{r})/\partial x^j$, the above relation can be recast in the coordinate-independent form

$$\delta\vec{\mathsf{v}} \simeq \vec{\nabla}\vec{\mathbf{v}}(t,\vec{r})\cdot\delta\vec{r}. \tag{II.2b}$$

Like every rank 2 tensor, the *velocity gradient tensor* $\vec{\nabla}\vec{\mathbf{v}}(t,\vec{r})$ at time t and position $\vec{r}$ can be decomposed into the sum of the symmetric and an antisymmetric part:

$$\vec{\nabla}\vec{\mathbf{v}}(t,\vec{r}) = \mathbf{D}(t,\vec{r}) + \mathbf{R}(t,\vec{r}), \tag{II.3a}$$

where one conventionally writes

$$\mathbf{D}(t,\vec{r}) \equiv \frac{1}{2}\Big(\vec{\nabla}\vec{\mathbf{v}}(t,\vec{r}) + \big[\vec{\nabla}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}}\Big), \qquad \mathbf{R}(t,\vec{r}) \equiv \frac{1}{2}\Big(\vec{\nabla}\vec{\mathbf{v}}(t,\vec{r}) - \big[\vec{\nabla}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}}\Big) \tag{II.3b}$$

with $\big[\vec{\nabla}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}}$ the transposed tensor to $\vec{\nabla}\vec{\mathbf{v}}(t,\vec{r})$. These definitions are to be understood as follows:

Using the same Cartesian coordinate system as above, the components of the two tensors **D**, **R**, viewed for simplicity as $\binom{0}{2}$-tensors, respectively read

$$\mathbf{D}_{ij}(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial \mathsf{v}_i(t,\vec{r})}{\partial x^j} + \frac{\partial \mathsf{v}_j(t,\vec{r})}{\partial x^i}\right], \qquad \mathbf{R}_{ij}(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial \mathsf{v}_i(t,\vec{r})}{\partial x^j} - \frac{\partial \mathsf{v}_j(t,\vec{r})}{\partial x^i}\right]. \tag{II.3c}$$

Note that here we have silently used the fact that for Cartesian coordinates, the position—subscript or superscript—of the index does not change the value of the component, i.e. numerically $\mathsf{v}_i = \mathsf{v}^i$ for every $i \in \{1,2,3\}$.

Relations (II.3c) clearly represent the desired symmetric and antisymmetric parts. However, one sees that the definitions would not appear to fulfill their task if the indices were not both either up or down, as e.g.

$$\mathbf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial \mathsf{v}^i(t,\vec{r})}{\partial x^j} + \frac{\partial \mathsf{v}_j(t,\vec{r})}{\partial x_i}\right],$$

in which the symmetry is no longer obvious. The trick is to rewrite the previous identity as

$$\mathbf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}\delta^{ik}\delta^l{}_j\left[\frac{\partial \mathsf{v}_k(t,\vec{r})}{\partial x^l} + \frac{\partial \mathsf{v}_l(t,\vec{r})}{\partial x^k}\right] = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\partial \mathsf{v}_k(t,\vec{r})}{\partial x^l} + \frac{\partial \mathsf{v}_l(t,\vec{r})}{\partial x^k}\right],$$

where we have used the fact that the metric tensor of Cartesian coordinates coincides with the Kronecker symbol. To fully generalize to curvilinear coordinates, the partial derivatives in the rightmost term should be replaced by the covariant derivatives discussed in Appendix **??**, leading eventually to

$$\mathbf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}_k(t,\vec{r})}{\mathrm{d}x^l} + \frac{\mathrm{d}\mathsf{v}_l(t,\vec{r})}{\mathrm{d}x^k}\right] \tag{II.4a}$$

$$\mathbf{R}^i{}_j(t,\vec{r}) = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}_k(t,\vec{r})}{\mathrm{d}x^l} - \frac{\mathrm{d}\mathsf{v}_l(t,\vec{r})}{\mathrm{d}x^k}\right] \tag{II.4b}$$

With these new forms, which are valid in any coordinate system, the raising or lowering of indices does not affect the visual symmetric or antisymmetric aspect of the tensor.

Using the tensors **D** and **R** we just introduced, whose physical meaning will be discussed at length in § II.1.2–II.1.3, relation (II.2b) can be recast as

$$\vec{\mathsf{v}}\big(t,\vec{r}+\delta\vec{r}\big) = \vec{\mathsf{v}}\big(t,\vec{r}\big) + \mathbf{D}(t,\vec{r})\cdot\delta\vec{r} + \mathbf{R}(t,\vec{r})\cdot\delta\vec{r} + \mathcal{O}\big(|\delta\vec{r}|^2\big) \tag{II.5}$$

where as stated at the beginning every field is considered at the same time.

Under consideration of relation (II.5) with $\delta\vec{r} = \delta\vec{\ell}(t)$, Eq. (II.1) for the time evolution of the material line element becomes

$$\delta\vec{\ell}(t+\delta t) = \delta\vec{\ell}(t) + \big[\mathbf{D}(t,\vec{r})\cdot\delta\vec{\ell}(t) + \mathbf{R}(t,\vec{r})\cdot\delta\vec{\ell}(t)\big]\delta t + \mathcal{O}\big(\delta t^2\big).$$

Subtracting $\delta\vec{\ell}(t)$ from both sides, dividing by δt and taking the limit $\delta t \to 0$, one finds for the rate of change of the material vector:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{\ell}(t) = \mathbf{D}(t,\vec{r})\cdot\delta\vec{\ell}(t) + \mathbf{R}(t,\vec{r})\cdot\delta\vec{\ell}(t) \tag{II.6}$$

In the following two subsections, we shall investigate the physical content of each of the tensors $\mathbf{R}(t,\vec{r})$ and $\mathbf{D}(t,\vec{r})$.

II.1.2 Rotation rate tensor and vorticity vector

The tensor $\mathbf{R}(t,\vec{r})$ defined by Eq. (II.3b) is called, for reasons that will become clearer below, *rotation rate tensor*.(xxii)

By construction, this tensor is antisymmetric. Accordingly, one can naturally associate with it a dual (pseudo)-vector $\vec{\Omega}(t,\vec{r})$, such that for any vector $\vec{V}$

$$\mathbf{R}(t,\vec{r})\cdot\vec{V}=\vec{\Omega}(t,\vec{r})\times\vec{V}\quad\forall\vec{V}\in\mathbb{R}^3. \tag{II.7}$$

In Cartesian coordinates, the components of $\vec{\Omega}(t,\vec{r})$ are related to those of the rotation rate tensor by

$$\Omega^i(t,\vec{r})\equiv-\frac{1}{2}\sum_{j,k=1}^{3}\epsilon^{ijk}\,\mathbf{R}_{jk}(t,\vec{r}) \tag{II.8a}$$

with ϵ^{ijk} the totally antisymmetric Levi-Civita symbol. Using the antisymmetry of $\mathbf{R}(t,\vec{r})$, this equivalently reads

$$\Omega^1(t,\vec{r})\equiv-\mathbf{R}_{23}(t,\vec{r}),\quad\Omega^2(t,\vec{r})\equiv-\mathbf{R}_{31}(t,\vec{r}),\quad\Omega^3(t,\vec{r})\equiv-\mathbf{R}_{12}(t,\vec{r}). \tag{II.8b}$$

Comparing with Eq. (II.3c), one finds

$$\vec{\Omega}(t,\vec{r})=\frac{1}{2}\vec{\nabla}\times\vec{\mathsf{v}}(t,\vec{r}). \tag{II.9}$$

Proof of Eqs. (II.7), (II.9): introducing the Cartesian components $\{V^j\}$ of $\vec{V}$ and dropping for brevity the $(t,\vec{r})$-dependence of fields, the i-th component of $\mathbf{R}\cdot\vec{V}$ reads

$$\mathbf{R}_{ij}V^j=\frac{1}{2}(\partial_j\mathsf{v}_i-\partial_i\mathsf{v}_j)V^j,$$

where we used the summation convention over the repeated index j and the shorthand notation ∂_i for the partial derivative with respect to x^i. This may further be rewritten as

$$\mathbf{R}_{ij}V^j=-\frac{1}{2}(\delta_i^k\delta_j^l-\delta_j^k\delta_i^l)(\partial_k\mathsf{v}_l)V^j,$$

which now involves three sums. The term with the four Kronecker symbols is in fact the sum (over a fifth index m) of the product $\epsilon_{ijm}\epsilon^{mkl}$ of Levi-Civita symbols:

$$\mathbf{R}_{ij}V^j=-\frac{1}{2}\epsilon_{ijm}\epsilon^{mkl}(\partial_k\mathsf{v}_l)V^j.$$

On the right hand side of this identity, $\epsilon^{mkl}\partial_k\mathsf{v}_l$ is the m-th component of the curl $\vec{\nabla}\times\vec{\mathsf{v}}$, i.e. using definition (II.9):

$$\mathbf{R}_{ij}V^j=-\epsilon_{ijm}\Omega^mV^j=\epsilon_{imj}\Omega_mV^j,$$

which is precisely the i-th component of $\Omega\times\vec{\mathsf{v}}$. □

Let us now rewrite relation (II.6) with the help of the vector $\vec{\Omega}(t,\vec{r})$, assuming that $\mathbf{D}(t,\vec{r})$ vanishes so as to isolate the effect of the remaining term. Under this assumption, the rate of change of the material vector between two neighboring points reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{\ell}(t)=\mathbf{R}(t,\vec{r})\cdot\delta\vec{\ell}(t)=\vec{\Omega}(t,\vec{r})\times\delta\vec{\ell}(t). \tag{II.10}$$

The term on the right hand side is then exactly the rate of rotation of a vector $\delta\vec{\ell}(t)$ in the motion of a rigid body with instantaneous angular velocity $\vec{\Omega}(t,\vec{r})$. Accordingly, the pseudovector $\vec{\Omega}(t,\vec{r})$ is referred to as *local angular velocity*.(xxiii) This a posteriori justifies the denomination *rotation rate tensor* for the antisymmetric tensor $\mathbf{R}(t,\vec{r})$.

(xxii) *Wirbeltensor* (xxiii) *Wirbelvektor*

Remarks:

∗ Besides the local angular velocity $\vec{\Omega}(t,\vec{r})$, one also defines the *vorticity vector*[(xxiv)] as the curl of the velocity field

$$\vec{\omega}(t,\vec{r}) \equiv \vec{\nabla}\times\vec{\mathsf{v}}(t,\vec{r}) = 2\vec{\Omega}(t,\vec{r}). \tag{II.11}$$

In fluid mechanics, the vorticity is actually more often used than the local angular velocity.

∗ The local angular velocity $\vec{\Omega}(t,\vec{r})$ or equivalently the vorticity vector $\vec{\omega}(t,\vec{r})$ define, at fixed t, divergence-free (pseudo)vector fields, since obviously $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\mathsf{v}}) = 0$. The corresponding field lines are called *vorticity lines*[(xxv)] and are given by [cf. Eq. (I.15)]

$$\mathrm{d}\vec{x}\times\vec{\omega}(t,\vec{r}) = \vec{0} \tag{II.12a}$$

or equivalently, at a point where none of the components of the vorticity vector vanishes,

$$\frac{\mathrm{d}x_1}{\omega^1(t,\vec{r})} = \frac{\mathrm{d}x_2}{\omega^2(t,\vec{r})} = \frac{\mathrm{d}x_3}{\omega^3(t,\vec{r})}. \tag{II.12b}$$

II.1.3 Strain rate tensor

According to the previous subsection, the local rotational motion of a material vector is governed by the (local and instantaneous) rotation rate tensor $\mathbf{R}(t,\vec{r})$. In turn, the translational motion is simply the displacement—which must be described in an affine space, not a vector one—of one of the endpoints of $\delta\vec{\ell}$ by an amount given by the product of velocity and length of time interval. That is, both components of the motion of a rigid body are already accounted for without invoking the symmetric tensor $\mathbf{D}(t,\vec{r})$.

In other words, the tensor $\mathbf{D}(t,\vec{r})$ characterizes the local deviation between the velocity fields in a deformable body, in particular a fluid, and in a rigid body rotating with angular velocity $\vec{\Omega}(t,\vec{r})$. Accordingly, it is called *strain rate tensor* or *deformation rate tensor*.[(xxvi)]

As we shall now see, the diagonal and off-diagonal components of $\mathbf{D}(t,\vec{r})$ actually describe the rates of change of different kinds of deformation. For simplicity, we assume throughout this subsection that $\vec{\Omega}(t,\vec{r}) = \vec{0}$.

II.1.3 a Diagonal components

We first assume that all off-diagonal terms in the strain rate tensor vanish: $\mathbf{D}_{ij}(t,\vec{r}) = 0$ for $i\neq j$, so as to isolate the meaning of the diagonal components.

Going back to Eq. (II.1), let us simply project it along one of the axes of the coordinate system, say along direction i. Denoting $\delta\ell^i$ the i-th component of $\delta\vec{\ell}$, one thus finds

$$\delta\ell^i(t+\delta t) = \delta\ell^i(t) + \big[\mathsf{v}^i\big(t,\vec{r}+\delta\vec{\ell}(t)\big) - \mathsf{v}^i\big(t,\vec{r}\big)\big]\delta t + \mathcal{O}\big(\delta t^2\big).$$

Taylor-expanding the term between square brackets to first order then yields

$$\delta\ell^i(t+\delta t) - \delta\ell^i(t) \simeq \sum_{j=1}^{3}\frac{\partial\mathsf{v}^i(t,\vec{r})}{\partial x^j}\,\delta\ell^j(t)\,\delta t,$$

up to terms of higher order in $|\delta\vec{\ell}(t)|$ or δt. Since we have assumed that both $\vec{\Omega}(t,\vec{r})$—or equivalently the components $\mathbf{R}_{ij}(t,\vec{r})$ of the rotation rate tensor—and the off-diagonal $\mathbf{D}_{ij}(t,\vec{r})$ with $i\neq j$ vanish, one checks that the partial derivative $\partial\mathsf{v}^i(t,\vec{r})/\partial x^j$ vanishes for $i\neq j$. That is, the only non-zero term in the sum is that with $j=i$, so that the equation simplifies to

$$\delta\ell^i(t+\delta t) - \delta\ell^i(t) \simeq \frac{\partial\mathsf{v}^i(t,\vec{r})}{\partial x^i}\,\delta\ell^i(t)\,\delta t = \mathbf{D}^i{}_i(t,\vec{r})\,\delta\ell^i(t)\,\delta t.$$

(xxiv) *Wirbligkeit* (xxv) *Wirbellinien* (xxvi) *Verzerrungsgeschwindigkeitstensor, Deformationsgeschwindigkeitstensor*

Thus, the relative elongation in δt of the i-th component—remember that there is no local rotation, so that the change in $\delta\ell^i$ is entirely due to a variation of the length of the material vector—is given by

$$\frac{\delta\ell^i(t+\delta t)-\delta\ell^i(t)}{\delta\ell^i(t)} = \mathbf{D}^i{}_i(t,\vec{r})\,\delta t \tag{II.13}$$

or alternatively, taking the limit $\delta t \to 0$

$$\frac{1}{\delta\ell^i(t)}\frac{\mathrm{d}}{\mathrm{d}t}\delta\ell^i(t) = \mathbf{D}^i{}_i(t,\vec{r}). \tag{II.14}$$

This equation means that the diagonal component $\mathbf{D}^i{}_i(t,\vec{r})$ represents the local rate of linear elongation in direction i.

Volume expansion rate

Instead of considering a one-dimensional material vector, one can study the evolution of a small "material rectangular parallelepiped" of continuous medium, situated at time t at position $\vec{r}$ with instantaneous edge lengths $\delta L^1(t)$, $\delta L^2(t)$, $\delta L^3(t)$—where for simplicity the coordinate axes are taken along the parallelepiped edges—, so that its volume at time t is simply $\delta\mathcal{V}(t) = \delta L^1(t)\,\delta L^2(t)\,\delta L^3(t)$.

Taking into account Eq. (II.13) for the relative elongation of each edge length, one finds that the relative change in volume between t and $t+\delta t$ is

$$\frac{\delta\mathcal{V}(t+\delta t)-\delta\mathcal{V}(t)}{\delta\mathcal{V}(t)} = \frac{\delta L^1(t+\delta t)-\delta L^1(t)}{\delta L^1(t)} + \frac{\delta L^2(t+\delta t)-\delta L^2(t)}{\delta L^2(t)} + \frac{\delta L^3(t+\delta t)-\delta L^3(t)}{\delta L^3(t)}$$
$$= \left[\mathbf{D}^1{}_1(t,\vec{r}) + \mathbf{D}^2{}_2(t,\vec{r}) + \mathbf{D}^3{}_3(t,\vec{r})\right]\delta t.$$

In the second line, one recognizes the *trace* of the tensor $\mathbf{D}(t,\vec{r})$, which going back to the definition of the latter is equal to the divergence of the velocity fluid:

$$\mathbf{D}^1{}_1(t,\vec{r}) + \mathbf{D}^2{}_2(t,\vec{r}) + \mathbf{D}^3{}_3(t,\vec{r}) = \frac{\partial\mathsf{v}^1(t,\vec{r})}{\partial x^1} + \frac{\partial\mathsf{v}^2(t,\vec{r})}{\partial x^2} + \frac{\partial\mathsf{v}^3(t,\vec{r})}{\partial x^3} = \vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}).$$

That is, this divergence represents the local and instantaneous *volume expansion rate* of the continuous medium. Accordingly, the flow of a fluid is referred to as *incompressible* in some region when the velocity field in that region is divergence-free:

$$\text{incompressible flow} \quad\Leftrightarrow\quad \vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}) = 0. \tag{II.15}$$

We shall comment on this definition in § II.2.3.

II.1.3 b Off-diagonal components

Let us now assume that $\mathbf{D}_{12}(t,\vec{r})$, and thereby automatically $\mathbf{D}_{21}(t,\vec{r})$, is the only non-vanishing component of the strain rate tensor. To see the influence of that component, we need to consider the time evolution of a different object than a material vector, since anything that can affect the latter—translation, rotation, dilatation—has already been described above.

Accordingly, we now look at the change between successive instants t and $t+\delta t$ of an elementary "material rectangle", as pictured in Fig. II.2. We denote by $\vec{\mathsf{v}}$ resp. $\vec{\mathsf{v}}+\delta\vec{\mathsf{v}}$ the velocity at time t at the lower left resp. upper right corner of the rectangle. Taylor expansions give for the Cartesian components of the shift $\delta\vec{\mathsf{v}}$

$$\delta\mathsf{v}_1 = \frac{\partial\mathsf{v}_1(t,\vec{r})}{\partial x^2}\,\delta\ell^2, \qquad \delta\mathsf{v}_2 = \frac{\partial\mathsf{v}_2(t,\vec{r})}{\partial x^1}\,\mathrm{d}\ell^1.$$

Figure II.2 shows that what is a right angle at time t becomes an angle $\pi/2-\delta\alpha$ at $t+\mathrm{d}t$, where $\delta\alpha = \delta\alpha_1 - \delta\alpha_2$. In the limit of small δt, both $\delta\alpha_1$ and $\delta\alpha_2$ will be small and thus approximately

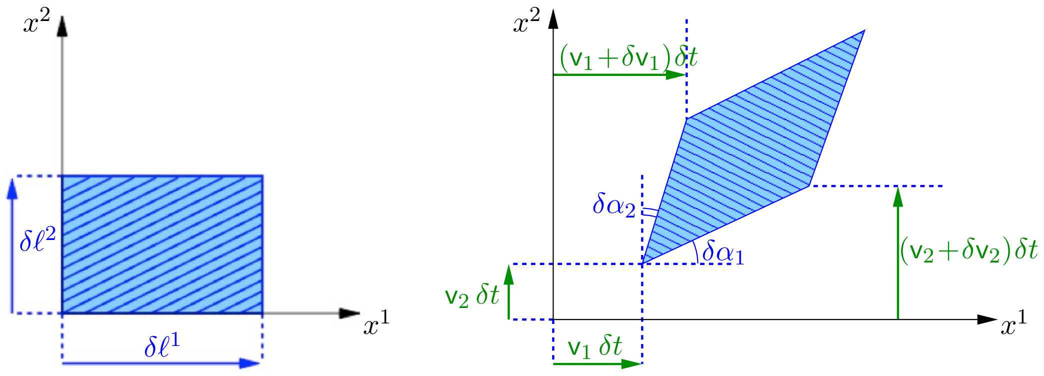

Figure II.2 – Evolution of a material rectangle caught in the motion of a continuous medium between times t (left) and $t+\delta t$ (right).

equal to their respective tangents. Using the fact that the parallelogram still has the same area—since the diagonal components of $\mathbf{D}$ vanish—the projection of any side of the deformed rectangle at time $t+\delta t$ on its original direction at time t keeps approximately the same length, up to corrections of order δt. One thus finds for the oriented angles

$$\delta\alpha_1 \simeq \frac{\delta\mathrm{v}_2\,\delta t}{\delta\ell^1} \quad\text{and}\quad \delta\alpha_2 \simeq -\frac{\delta\mathrm{v}_1\,\delta t}{\delta\ell^2}.$$

With the Taylor expansions given above, this leads to

$$\delta\alpha_1 \simeq \frac{\partial\mathrm{v}_2(t,\vec{r})}{\partial x^1}\,\delta t, \qquad \delta\alpha_2 \simeq -\frac{\partial\mathrm{v}_1(t,\vec{r})}{\partial x^2}\,\delta t.$$

Gathering all pieces, one finds

$$\frac{\delta\alpha}{\delta t} \simeq \frac{\partial\mathrm{v}_2(t,\vec{r})}{\partial x^1} + \frac{\partial\mathrm{v}_1(t,\vec{r})}{\partial x^2} = 2\,\mathbf{D}_{12}(t,\vec{r}). \tag{II.16}$$

In the limit $\delta t \to 0$, one sees that the off-diagonal component $\mathbf{D}_{12}(t,\vec{r})$ represents half the local velocity of the "angular deformation"—the *shear*—around direction x^3.

Remark: To separate the two physical effects present in the strain rate tensor, the latter is often written as the sum of a diagonal *rate-of-expansion tensor* proportional to the identity $\mathbb{1}$—which is in fact the $\binom{1}{1}$-form of the metric tensor $\mathbf{g}$ of Cartesian coordinates—and a traceless *rate-of-shear tensor* $\mathbf{S}$:

$$\mathbf{D}(t,\vec{r}) = \frac{1}{3}\big[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]\mathbb{1} + \mathbf{S}(t,\vec{r}) \tag{II.17a}$$

with

$$\mathbf{S}(t,\vec{r}) \equiv \frac{1}{2}\bigg(\vec{\nabla}\vec{\mathbf{v}}(t,\vec{r}) + \big[\vec{\nabla}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}} - \frac{2}{3}\big[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]\mathbb{1}\bigg). \tag{II.17b}$$

Component-wise, and generalizing to curvilinear coordinates, this reads

$$\mathbf{D}_{ij}(t,\vec{r}) = \frac{1}{3}\big[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]g_{ij}(t,\vec{r}) + \mathbf{S}_{ij}(t,\vec{r}) \tag{II.17c}$$

with [cf. Eq. (II.4a)]

$$\mathbf{S}_{ij}(t,\vec{r}) \equiv \frac{1}{2}\bigg[g^k{}_i(t,\vec{r})g^l{}_j(t,\vec{r})\bigg(\frac{\mathrm{dv}_k(t,\vec{r})}{\mathrm{d}x^l} + \frac{\mathrm{dv}_l(t,\vec{r})}{\mathrm{d}x^k}\bigg) - \frac{2}{3}\big[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]g_{ij}(t,\vec{r})\bigg]. \tag{II.17d}$$

Summary

Gathering the findings of this Section, the most general motion of a material volume element inside a continuous medium, in particular in a fluid, can be decomposed in four elements:

- a translation;
- a rotation, with a local angular velocity $\vec{\Omega}(t,\vec{r})$ given by Eq. (II.9)—i.e. related to the antisymmetric part $\mathbf{R}(t,\vec{r})$ of the velocity gradient—and equal to twice the (local) vorticity vector $\vec{\omega}(t,\vec{r})$;
- a local dilatation or contraction, in which the geometric form of the material volume element remains unchanged, whose rate is given by the divergence of the velocity field $\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})$, i.e. encoded in the diagonal elements of the strain rate tensor $\mathbf{D}(t,\vec{r})$;
- a change of shape ("deformation") of the material volume element at constant volume, controlled by the rate-of-shear tensor $\mathbf{S}(t,\vec{r})$ [Eqs. (II.17b), (II.17d)], obtained by taking the traceless symmetric part of the velocity gradient.

Remark: In the case of a uniform motion, all spatial derivatives are by definition zero, so that the vorticity $\vec{\omega}(t,\vec{r})$, the expansion rate $\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})$ and the rate-of-shear tensor $\mathbf{S}(t,\vec{r})$ actually vanish everywhere in the flow. Accordingly, the motion of a material element in that case is simply a pure translation, without deformation or rotation.

II.2 Classification of fluid flows

The motion, or *flow*[(xxvii)], of a fluid can be characterized according to several criteria, either purely geometrical (§ II.2.1), kinematic (§ II.2.2), or of a more physical nature (§ II.2.3), that takes into account the physical behavior of the flowing fluid in its evolution.

II.2.1 Geometrical criteria

In the general case, the quantities characterizing the properties of a fluid flow will depend on time as well as on three spatial coordinates.

For some more or less idealized models of actual flows, it may turn out that only two spatial coordinates play a role, in which case one talks of a *two-dimensional flow*. An example is the flow of air around the wing of an airplane, which in first approximation is "infinitely" long compared to its transverse profile: the (important!) effects at the ends of the wing, which introduce the dependence on the spatial dimension along the wing, may be left aside in a first approach, then considered in a second, more detailed step.

In some cases, e.g. for fluid flows in pipes, one may even assume that the properties only depend on a single spatial coordinate, so that the flow is *one-dimensional*. In that approximation, the physical local quantities are actually often replaced by their average value over the cross section of the pipe.

On a different level, one also distinguishes between *internal* und *external* fluid flows, according to whether the fluid is enclosed inside solid walls—e.g. in a pipe—or flowing around a body—e.g. around an airplane wing.

II.2.2 Kinematic criteria

The notions of *uniform*—that is independent of position—and *steady*—independent of time—motions were already introduced at the end of § I.3.3. Accordingly, there are *non-uniform* and *unsteady* fluids flows.

If the vorticity vector $\vec{\omega}(t,\vec{r})$ vanishes at every point $\vec{r}$ of a flowing fluid, then the corresponding motion is referred to as an *irrotational flow*[(xxviii)] or, for reasons that will be clarified in Sec. **??**, *potential flow*. The opposite case is that of a *vortical* or *rotational flow*.[(xxix)]

[(xxvii)] *Strömung* [(xxviii)] *wirbelfreie Strömung* [(xxix)] *Wirbelströmung*

According to whether the flow velocity v is smaller or larger than the (local) speed of sound c_s in the fluid, one talks of *subsonic* or *supersonic* motion(xxx), corresponding respectively to a dimensionless *Mach number*(g)

$$\mathrm{Ma} \equiv \frac{\mathsf{v}}{c_s} \tag{II.18}$$

smaller or larger than 1. Note that the Mach number can a priori be defined, and take different values $\mathrm{Ma}(t, \vec{r})$, at every point in a flow.

When the fluid flows in layers that do not mix with each other, so that the streamlines remain parallel, the flow is referred to as *laminar*. In the opposite case the flow is *turbulent*.

II.2.3 Physical criteria

All fluids are compressible, more or less according to the substance and its thermodynamic state. Nevertheless, this compressibility is sometimes irrelevant for a given motion, in which case it may be fruitful to consider that the fluid flow is *incompressible*, which, as seen in § II.1.3 a, technically means that its volume expansion rate vanishes, $\vec{\nabla} \cdot \vec{\mathsf{v}} = 0$. In the opposite case ($\vec{\nabla} \cdot \vec{\mathsf{v}} \neq 0$), the flow is said to be *compressible*. It is however important to realize that the statement is more a kinematic one, than really reflecting the thermodynamic compressibility of the fluid.

> In practice, flows are compressible in regions where the fluid velocity is "large", namely where the Mach number (II.18) is not much smaller than 1, i.e. roughly speaking $\mathrm{Ma} \gtrsim 0.2$.

In an analogous manner, one speaks of viscous resp. non-viscous flows to express the fact that the fluid under consideration is modeled as viscous resp. inviscid—which leads to different equations of motion—, irrespective of the fact that every real fluid has a non-zero viscosity.

Other thermodynamic criteria are also used to characterize possible fluid motions: isothermal flows—i.e. in which the temperature is uniform and remains constant—, isentropic flows—i.e. without production of entropy—, and so on.

Bibliography for Chapter II

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- Faber [1] Chapter 2.4;
- Feynman [10, 11] Chapter 39–1;
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- Sommerfeld [7, 8] Chapter I.

(xxx) *Unterschall-* bzw. *Überschallströmung*

(g) E. Mach, 1838–1916

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