I.2 Lagrangian description

The *Lagrangian*[(h)] perspective, which generalizes the approach usually adopted in the description of the motion of a (few) point particle(s), focuses on the trajectories of the material points, where the latter are labeled by their position in the reference configuration. Accordingly, physical quantities are expressed as functions of time t and initial position vectors $\vec{R}$, and any continuity condition has to be formulated with respect to these variables.

I.2.1 Lagrangian coordinates

Consider a material point M in a continuous medium. Given a reference frame $\mathcal{R}$, which allows the definition of its position vector at any time t, one can follow its *trajectory* $\vec{r}(t)$. With a choice of coordinate system, that trajectory is equivalently characterized by the functions $\{x^i(t)\}$ for $i = 1, 2, 3$.

Let $\vec{R}$ resp. $\{X^i\}$ denote the position resp. coordinates of the material point M at t_0. The trajectory obviously depends on this "initial" position, and $\vec{r}$ can thus be viewed as a function of t and $\vec{R}$, where the latter refers to the reference configuration κ_0:

$$\vec{r} = \vec{r}(t, \vec{R}) \tag{I.5a}$$

with the consistency condition

$$\vec{r}(t{=}t_0, \vec{R}) = \vec{R}. \tag{I.5b}$$

In the *Lagrangian description*, also referred to as *material description* or *particle description*, this point of view is generalized, and the various physical quantities $\mathcal{G}$ characterizing a continuous medium are viewed at *any* time as mathematical functions of the variables t and $\vec{R}$:

$$\mathcal{G} = \mathcal{G}(t, \vec{R}), \tag{I.6}$$

where the mapping $\mathcal{G}$—which as often in physics will be denoted with the same notation as the physical quantity represented by its value—is defined for every t on the initial volume $\mathcal{V}_0$ occupied by the reference configuration κ_0.

[(h)] J.-L. Lagrange, 1736–1813

Together with the time t, the position vector $\vec{R}$—or equivalently its coordinates X^1, X^2, X^3 in a given system—are called *Lagrangian coordinates*.

I.2.2 Continuity assumptions

An important example of physical quantity, function of t and $\vec{R}$, is simply the position vector (in the reference frame $\mathcal{R}$) of a material point at time t, i.e. $\vec{r}$ or equivalently its coordinates $\{x^i\}$, as given by relation (I.5a), which thus relates the configurations κ_0 and κ_t.

More precisely, $\vec{r}(t,\vec{R})$ maps for every t the initial volume $\mathcal{V}_0$ *onto* $\mathcal{V}_t$. To implement mathematically the physical picture of continuity, it will be assumed that the mapping $\vec{r}(t,\cdot): \mathcal{V}_0 \to \mathcal{V}_t$ is also one-to-one for every t—i.e. all in all bijective—, and that the function $\vec{r}$ and its inverse

$$\vec{R} = \vec{R}(t,\vec{r}) \tag{I.7}$$

are *continuous* with respect to both time and space variables. This requirement in particular ensures that neighboring points remain close to each other as time elapses. It also preserves the connectedness of volumes, (closed) surfaces or curves along the evolution: one may then define *material domains*, i.e. connected sets of material points which are transported together in the evolution of the continuous medium.

For the sake of simplicity, it will be assumed that the mapping $\vec{r}$ and its inverse, and more generally every mathematical function $\mathcal{G}$ representing a physical quantity, is at least twice continuously differentiable (i.e. of class $\mathscr{C}^2$). To be able to accommodate for important phenomena that are better modeled with discontinuities, like shock waves in fluids (Sec. ??) or ruptures in solids—for instance, in the Earth's crust—, the $\mathscr{C}^2$-character of functions under consideration may hold only piecewise.

I.2.3 Velocity and acceleration of a material point

As mentioned above, for a fixed reference position $\vec{R}$ the function $t \mapsto \vec{r}(t,\vec{R})$ is the trajectory of the material point which passes through $\vec{R}$ at the reference time t_0. As a consequence, the velocity at time t of this material point, measured in the reference frame $\mathcal{R}$, is simply

$$\vec{v}(t,\vec{R}) = \frac{\partial\vec{r}(t,\vec{R})}{\partial t}. \tag{I.8}$$

Since the variable $\vec{R}$ is independent of t, one could actually also write $\vec{v}(t,\vec{R}) = \mathrm{d}\vec{r}(t,\vec{R})/\mathrm{d}t$. In turn, the acceleration of the material point in $\mathcal{R}$ is given at time t by

$$\vec{a}(t,\vec{R}) = \frac{\partial\vec{v}(t,\vec{R})}{\partial t}. \tag{I.9}$$

Remark: The trajectory (or *pathline*(vii)) of a material point can be visualized, by tagging the point at its position $\vec{R}$ at time t_0, for instance with a fluorescent or radioactive marker, and then imaging the positions at later times $t > t_0$.

On the other hand, if one regularly—say for every instant $t_0 \le t' \le t$—injects some marker at a fixed geometrical point P, the resulting tagged curve at time t is the locus of the geometrical points occupied by medium particles which passed through P in the past. This locus is referred to as *streakline*.(viii) Denoting by $\vec{r}_P$ the position vector of point P, the streakline is the set of geometrical points with position vectors

$$\vec{r} = \vec{r}\big(t,\vec{R}(t',\vec{r}_P)\big) \quad \text{for } t_0 \le t' \le t. \tag{I.10}$$

(vii) *Bahnlinie* (viii) *Streichlinie*

I.3 Eulerian description

The Lagrangian approach introduced in the previous Section is actually not commonly used in fluid dynamics, at least not in its original form, except for specific problems.

> One reason is that physical quantities at a given time are expressed in terms of a reference configuration in the (far) past: a small uncertainty on this initial condition may actually yield after a finite duration a large uncertainty on the present state of the system, which is problematic. On the other hand, this line of argument explains why the Lagrangian point of view is adopted to investigate *chaos* in many-body systems!

The more usual description is the so-called *Eulerian*[(i)] perspective, in which the evolution between instants t and $t + \mathrm{d}t$ takes the system configuration at time t as a reference.

I.3.1 Eulerian coordinates. Velocity field

In contrast to the "material" Lagrangian point of view, which identifies the medium particles in a reference configuration and follows them in their respective motions, in the Eulerian description the emphasis is placed on the *geometrical* points. Thus, the *Eulerian coordinates* are time t and a spatial vector $\vec{r}$, where the latter does not label the position of a material point, but rather that of a geometrical point. Accordingly, the physical quantities in the Eulerian specification are described by *fields* on space-time.

Thus, the fundamental field that entirely characterizes the motion of a continuous medium in a given reference frame $\mathcal{R}$ is the *velocity field* $\vec{\mathsf{v}}_t(t,\vec{r})$. The latter is defined such that it gives the value of the Lagrangian velocity $\vec{v}$ [Eq. (I.8)] of a material point passing through $\vec{r}$ at time t:

$$\vec{v} = \vec{\mathsf{v}}_t(t,\vec{r}) \quad \forall t,\ \forall \vec{r} \in \mathcal{V}_t. \tag{I.11}$$

More generally, the value taken at given time and position by a physical quantity $\mathcal{G}$, whether attached to a material point or not, is expressed as a mathematical function $\mathcal{G}_t$ of the same Eulerian variables:

$$\mathcal{G} = \mathcal{G}_t(t,\vec{r}) \quad \forall t,\ \forall \vec{r} \in \mathcal{V}_t. \tag{I.12}$$

Note that the mappings $(t,\vec{R}) \mapsto \mathcal{G}(t,\vec{R})$ in the Lagrangian approach and $(t,\vec{r}) \mapsto \mathcal{G}_t(t,\vec{r})$ in the Eulerian description are in general different. For instance, the domains in $\mathbb{R}^3$ over which their spatial variables take their values differ: constant ($\mathcal{V}_0$) in the Lagrangian specification, time-dependent ($\mathcal{V}_t$) in the case of the Eulerian quantities. Accordingly the latter will be denoted with a subscript t in the next subsection.

I.3.2 Equivalence between the Eulerian and Lagrangian viewpoints

Despite the different choices of variables, the Lagrangian and Eulerian descriptions are fully equivalent. Accordingly, the prevalence in practice of the one over the other is more a technical issue than a conceptual one.

Thus, it is rather clear that the knowledge of the Lagrangian specification can be used to obtain the Eulerian formulation at once, using the mapping $\vec{r} \mapsto \vec{R}(t,\vec{r})$ between present and reference positions of a material point. For instance, the Eulerian velocity field can be expressed as

$$\vec{\mathsf{v}}_t(t,\vec{r}) = \vec{v}\big(t,\vec{R}(t,\vec{r})\big). \tag{I.13a}$$

This identity in particular shows that $\vec{\mathsf{v}}_t$ automatically inherits the smoothness properties of $\vec{v}$: if the mapping $(t,\vec{R}) \mapsto \vec{r}(t,\vec{R})$ and its inverse are piecewise $\mathscr{C}^2$ (cf. § I.2.2), then $\vec{\mathsf{v}}_t$ is (at least) piecewise $\mathscr{C}^1$ in both its variables.

[(i)] L. Euler, 1707–1783

For a generic physical quantity, the transition from the Lagrangian to the Eulerian point of view similarly reads

$$\mathcal{G}_t(t,\vec{r}) = \mathcal{G}\big(t,\vec{R}(t,\vec{r})\big). \tag{I.13b}$$

Reciprocally, given a (well-enough behaved) Eulerian velocity field $\vec{\mathsf{v}}_t$ on a continuous medium, one can uniquely obtain the Lagrangian description of the medium motion by solving the *initial value problem*

$$\begin{cases} \dfrac{\partial\vec{r}(t,\vec{R})}{\partial t} &= \vec{\mathsf{v}}_t\big(t,\vec{r}(t,\vec{R})\big) \\ \vec{r}(t_0,\vec{R}) &= \vec{R}, \end{cases} \tag{I.14a}$$

where the second line represents the initial condition. That is, one actually reconstructs the pathline of every material point of the continuous medium. Introducing differential notations, the above system can also be rewritten as

$$\mathrm{d}\vec{r} = \vec{\mathsf{v}}_t(t,\vec{r})\,\mathrm{d}t \quad \text{with} \quad \vec{r}(t_0,\vec{R}) = \vec{R}. \tag{I.14b}$$

Once the pathlines $\vec{r}(t,\vec{R})$ are known, one obtains the Lagrangian-description function $\mathcal{G}(t,\vec{R})$ for a given physical quantity $\mathcal{G}$ by writing

$$\mathcal{G}(t,\vec{R}) = \mathcal{G}_t\big(t,\vec{r}(t,\vec{R})\big). \tag{I.14c}$$

Since both Lagrangian and Eulerian descriptions are equivalent, we shall from now on drop the subscript t on the mathematical functions representing physical quantities in the Eulerian point of view.

I.3.3 Streamlines

At a given time t, the *streamlines*(ix) of the motion are defined as the field lines of $\vec{\mathsf{v}}$. That is, these are curves whose tangent is everywhere parallel to the instantaneous velocity field at the same geometrical point.

Let $\vec{x}(\lambda)$ denote a streamline, parameterized by λ. The definition can be formulated as

$$\frac{\mathrm{d}\vec{x}(\lambda)}{\mathrm{d}\lambda} = \alpha(\lambda)\vec{\mathsf{v}}\big(t,\vec{x}(\lambda)\big) \tag{I.15a}$$

with $\alpha(\lambda)$ a scalar function. Equivalently, denoting by $\mathrm{d}\vec{x}(\lambda)$ a differential line element tangent to the streamline, one has the condition

$$\mathrm{d}\vec{x}\times\vec{\mathsf{v}}\big(t,\vec{x}(\lambda)\big) = \vec{0}. \tag{I.15b}$$

Introducing a Cartesian system of coordinates, the equation for a streamline is conveniently rewritten as

$$\frac{\mathrm{d}x^1(\lambda)}{\mathsf{v}^1\big(t,\vec{x}(\lambda)\big)} = \frac{\mathrm{d}x^2(\lambda)}{\mathsf{v}^2\big(t,\vec{x}(\lambda)\big)} = \frac{\mathrm{d}x^3(\lambda)}{\mathsf{v}^3\big(t,\vec{x}(\lambda)\big)} \tag{I.15c}$$

at a point where none of the component v^i of the velocity field vanishes—if one of the v^i is zero, then so is the corresponding $\mathrm{d}x^i$, thanks to Eq. (I.15b).

Remark: Since the velocity field $\vec{\mathsf{v}}$ depends on the choice of reference frame, this is also the case of its streamlines at a given instant!

Consider now a closed geometrical curve in the volume $\mathcal{V}_t$ occupied by the continuous medium at time t. The streamlines tangent to this curve form in the generic case a tube-like surface, called *stream tube*.(x)

(ix) *Stromlinien* (x) *Stromröhre*

Let us introduce two further definitions related to properties of the velocity field:

- If $\vec{\mathsf{v}}(t,\vec{r})$ has at some t the same value at every geometrical point $\vec{r}$ of a (connected) domain $\mathcal{D} \subset \mathcal{V}_t$, then the velocity field is said to be *uniform* across $\mathcal{D}$.
  In that case, the streamlines are parallel to each other over $\mathcal{D}$.

- If $\vec{\mathsf{v}}(t,\vec{r})$ only depends on the position, not on time, then the velocity field and the corresponding motion of the continuous medium are said to be *steady* or equivalently *stationary*.
  In that case, the streamlines coincide with the pathlines and the streaklines.

  Indeed, one checks that Eq. (I.14b) for the pathlines, in which the velocity becomes time-independent, can then be recast (at a point where all v^i are non-zero) as
  $$\frac{\mathrm{d}x^1}{\mathsf{v}^1(t,\vec{r})} = \frac{\mathrm{d}x^2}{\mathsf{v}^2(t,\vec{r})} = \frac{\mathrm{d}x^3}{\mathsf{v}^3(t,\vec{r})},$$
  where the variable t plays no role: this is exactly the system (I.15c) defining the streamlines at time t. The equivalence between pathlines and streaklines is also trivial. □

I.3.4 Material derivative

Consider a material point M in a continuous medium, described in a reference frame $\mathcal{R}$. Let $\vec{r}$ resp. $\vec{r}+\mathrm{d}\vec{r}$ denote its position vectors at successive instants t resp. $t+\mathrm{d}t$. The velocity of M at time t resp. $t+\mathrm{d}t$ is by definition equal to the value of the velocity field at that time and at the respective position, namely $\vec{\mathsf{v}}(t,\vec{r})$ resp. $\vec{\mathsf{v}}(t+\mathrm{d}t,\vec{r}+\mathrm{d}\vec{r})$. For small enough $\mathrm{d}t$, the displacement $\mathrm{d}\vec{r}$ of the material point between t and $t+\mathrm{d}t$ is related to its velocity at time t by $\mathrm{d}\vec{r} = \vec{\mathsf{v}}(t,\vec{r})\,\mathrm{d}t$.

Let $\mathrm{d}\vec{\mathsf{v}} \equiv \vec{\mathsf{v}}(t+\mathrm{d}t,\vec{r}+\mathrm{d}\vec{r}) - \vec{\mathsf{v}}(t,\vec{r})$ denote the change in the material point velocity between t and $t+\mathrm{d}t$. Assuming that $\vec{\mathsf{v}}(t,\vec{r})$ is differentiable (cf. § I.3.2) and introducing for simplicity a system of Cartesian[(j)] coordinates, a Taylor[(k)] expansion to lowest order yields
$$\mathrm{d}\vec{\mathsf{v}} \simeq \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t}\,\mathrm{d}t + \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial x^1}\,\mathrm{d}x^1 + \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial x^2}\,\mathrm{d}x^2 + \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial x^3}\,\mathrm{d}x^3,$$
up to terms of higher order in $\mathrm{d}t$ or $\mathrm{d}\vec{r}$. Introducing the differential operator
$$\mathrm{d}\vec{r}\cdot\vec{\nabla} \equiv \mathrm{d}x^1\,\frac{\partial}{\partial x^1} + \mathrm{d}x^2\,\frac{\partial}{\partial x^2} + \mathrm{d}x^3\,\frac{\partial}{\partial x^3},$$
this can be recast in the more compact form
$$\mathrm{d}\vec{\mathsf{v}} \simeq \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t}\,\mathrm{d}t + \big(\mathrm{d}\vec{r}\cdot\vec{\nabla}\big)\vec{\mathsf{v}}(t,\vec{r}). \tag{I.16}$$

In the second term on the right-hand side, $\mathrm{d}\vec{r}$ can be replaced by $\vec{\mathsf{v}}(t,\vec{r})\,\mathrm{d}t$. On the other hand, the change in velocity of the material point between t and $t+\mathrm{d}t$ is simply the product of its acceleration $\vec{a}(t)$ at time t by the length $\mathrm{d}t$ of the time interval, at least to lowest order in $\mathrm{d}t$. Dividing both sides of Eq. (I.16) by $\mathrm{d}t$ and taking the limit $\mathrm{d}t \to 0$, in particular in the ratio $\mathrm{d}\vec{\mathsf{v}}/\mathrm{d}t$, yield
$$\vec{a}(t) = \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}). \tag{I.17}$$
That is, the acceleration of the material point consists of two contributions:

- the *local acceleration* $\dfrac{\partial\vec{\mathsf{v}}}{\partial t}$, which follows from the non-stationarity of the velocity field;

- the *convective acceleration* $\big(\vec{\mathsf{v}}\cdot\vec{\nabla}\big)\vec{\mathsf{v}}$, due to the non-uniformity of the motion.

More generally, one finds by repeating the same derivation as above that the time derivative of a physical quantity $\mathcal{G}$ attached to a material point or domain, expressed in terms of Eulerian fields,

[(j)] R. Descartes, 1596–1650 [(k)] B. Taylor, 1685–1731 is the sum of a local $(\partial\mathcal{G}/\partial t)$ and a convective $[(\vec{\mathsf{v}}\cdot\vec{\nabla})\mathcal{G}]$ part, irrespective of the tensorial nature of $\mathcal{G}$. Accordingly, one introduces the operator

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla} \tag{I.18}$$

called *material derivative*(xi) or (between others) *substantial derivative*,(xii) *derivative following the motion*, *hydrodynamic derivative*. Relation (I.17) can thus be recast as

$$\vec{a}(t) = \frac{\mathrm{D}\vec{\mathsf{v}}(t,\vec{r})}{\mathrm{D}t}. \tag{I.19}$$

Remarks:

∗ Equation (I.17) shows that even in the case of a steady motion, the acceleration of a material point may be non-vanishing, thanks to the convective part.

∗ The material derivative (I.18) is also often denoted (and referred to) as total derivative $\mathrm{d}/\mathrm{d}t$.

∗ One also finds in the literature the denomination *convective derivative*.(xiii) To the eyes and ears of the author of these lines, that name has the drawback that it does not naturally evoke the local part, but only. . . the convective one, which comes from the fact that matter is being transported, “conveyed”, with a non-vanishing velocity field $\vec{\mathsf{v}}(t,\vec{r})$.

∗ The two terms in Eq. (I.18) actually “merge” together when considering the motion of a material point in Galilean(l) space-time $\mathbb{R}\times\mathbb{R}^3$. As a matter of fact, one easily shows that $\mathrm{D}/\mathrm{D}t$ is the (Lie(m)) derivative along the world-line of the material point

The world-line element corresponding to the motion between t and $t+\mathrm{d}t$ goes from (t,x^1,x^2,x^3) to $(t+\mathrm{d}t, x^1+\mathsf{v}^1\,\mathrm{d}t, x^2+\mathsf{v}^2\,\mathrm{d}t, x^3+\mathsf{v}^3\,\mathrm{d}t)$. The tangent vector to this world-line thus has components $(1,\mathsf{v}^1,\mathsf{v}^2,\mathsf{v}^3)$, i.e. the derivative along the direction of this vector is $\partial_t + \mathsf{v}^1\partial_1 + \mathsf{v}^2\partial_2 + \mathsf{v}^3\partial_3$, with the usual shorthand notations $\partial_t \equiv \partial/\partial t$ and $\partial_i \equiv \partial/\partial x^i$. □

(xi) *Materielle Ableitung* (xii) *Substantielle Ableitung* (xiii) *Konvektive Ableitung*

(l) G. Galilei, 1564–1642 (m) S. Lie, 1842–1899