

CHAPTER I

Basic notions on continuous media

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A system of many microscopic degrees of freedom is often more conveniently described as a material body that fills some region of space continuously, rather than as a collection of discrete point particles (Sec. [I.1](#)). This theoretical approach, which is especially suited to represent systems whose internal deformations are relevant, is an instance of physical *modeling*, originally motivated by the agreement of its predictions with experimental observations. Like every model, that of a continuous medium is valid only in some range of physical conditions, in particular on macroscopic scales.

Mathematically, a classical continuous medium at a given instant is described as a volume—or more generally a manifold—in usual Euclidean space. The infinitesimal elements of this volume constitute the elementary “material points”, which are entirely characterized by their position.

To describe the time evolution of a physical system modeled as a continuous medium, two equivalent approaches are available. The first one consists in following the trajectories of the material points as time passes by (Sec. [I.2](#)). The physical picture of continuousness is then enforced by requesting that the mapping between the position of a given point at some reference initial time and its position at any later instant is continuous.

The second point of view, which will mostly be adopted in the remainder of these notes, focuses on the change in the various physical quantities at a fixed position as time elapses (Sec. [I.3](#)). The reference for the medium evolution between successive instants t and $t + dt$ is the “current” configuration of the material points, i.e. at time t , instead of their positions in the (far) past. In that description, the spatial variables are no longer dynamical, but only labels for the position at which some observable is considered. Accordingly, the dynamical quantities in the system are now time-dependent fields; the desired continuousness of the medium translates into continuity conditions on those fields.

Eventually, the mathematical object that models internal forces in a continuous medium, i.e. the influence from neighboring material points on each other, is shortly introduced (Sec. [I.4](#)). This allows the classification of deformable continuous media into two traditional large classes, and in particular the definition of fluids.

I.1 Continuous medium: a model for many-body systems

In this Section, we first spell out a few arguments which lead to the introduction of the model of a continuous medium (§ [I.1.1](#)). The basic ingredients of the mathematical implementation of the model are then presented and a few notions are defined (§ [I.1.2](#)). Eventually, the physical assumptions underlying the modeling are reexamined in greater detail, and some more or less obvious limitations of the continuous description are indicated (§ [I.1.3](#)).

I.1.1 Basic ideas and concepts

The actual structure of matter at the microscopic scale is discrete, and involves finite “elementary” entities: electrons, atoms, ions, molecules, . . . , which in the remainder of these notes will be collectively referred to as “atoms”. Any macroscopic sample of matter contains a large amount of these atoms. For instance, the number density in an ideal gas under normal conditions is about $2.7 \times 10^{25} \text{ m}^{-3}$, so that one cubic millimeter still contains 2.7×10^{16} atoms. Similarly, even though the number density in the interstellar medium might be as low as 10^2 m^{-3} , any volume relevant for astrophysics, i.e. with at least a kilometer-long linear size, involves a large number of atoms.

Additionally, these atoms are in constant chaotic motion, with individual velocities of order $10^2\text{--}10^3 \text{ m} \cdot \text{s}^{-1}$ for a system at thermal equilibrium at temperature $T \simeq 300 \text{ K}$. Given a mean free path⁽ⁱ⁾ of order 10^{-7} m in a gas under normal conditions, each atom undergoes $10^9\text{--}10^{10}$ collisions per second, i.e. its trajectory changes direction constantly.

As in Statistical Mechanics, it is in general unnecessary (and even useless) to know the details of the motion of each atom in a macroscopic system: as a matter of fact, there emerge global characteristics, which can be predicted to a high degree of accuracy thanks to the large number of degrees of freedom involved in their definition (mostly as averages), despite the chaoticity of the individual atomic behaviors. The macroscopic properties of systems at (global) thermodynamic equilibrium are thus entirely determined by a handful of collective variables, either extensive—like entropy, internal energy, volume, particle number, total momentum. . . —, or intensive—as e.g. the respective densities of the various extensive variables, or temperature, pressure, chemical potential, average velocity. . . —, where the latter take the same value throughout the system.

When thermodynamic equilibrium does not hold globally in a system, there is still the possibility that one may consider that it is valid locally, “at each point” in space. In that situation—whose underlying assumptions will be specified in greater detail in § [I.1.3](#)—the intensive thermodynamic variables characterizing the system macroscopically become *fields*, which can vary from point to point. More generally, experience shows that it is fruitful to describe a large amount of characteristics—not only thermodynamic, but also of mechanical nature, like forces and the displacements or deformations they induce—of macroscopic bodies as fields. A “continuous medium” is then intuitively a system described by such fields, which should satisfy some (mathematical) continuity property with respect to the spatial variables that parameterize the representation of the physical system as a geometrical quantity. This picture will be better specified in Secs. [I.2](#) and [I.3](#).

Assuming the relevance of the model of a medium whose properties are described by continuous fields is often referred to as *continuum hypothesis*⁽ⁱⁱ⁾.

The reader should keep in mind that the modeling of a given macroscopic system as a continuous medium does not invalidate the existence of its underlying discrete atomic structure. Specific

⁽ⁱ⁾ *mittlere freie Weglänge* ⁽ⁱⁱ⁾ *Kontinuumshypothese*

phenomena will still directly probe the latter, as e.g. X-ray scattering experiments for the gases or liquids of our everyday life. That is, the model has limitations to its validity, especially at small wavelengths or high frequencies, where “small” or “high” implies a comparison to some microscopic physical scale characteristic of the system under consideration. Turning the reasoning the other way around, the continuous-medium picture is often referred to as a long-wavelength, low-frequency approximation to a more microscopic description—from which it can actually be shown to emerge in the corresponding limits.

It is important to realize that the model itself is blind to its own limitations, i.e. there is no a priori criterion *within* the mathematical continuous-medium description that signals the breakdown of the relevance of the picture to actual physics. In practice, there might be hints that the equations of the continuous model are being applied in a regime where they should not, as for instance if they yield negative values for a quantity which should be positive, but such occurrences are not the general rule.

Remarks:

* The model of a continuous model is not only applicable—and applied—to obvious cases like gases, liquids or (deformable) solids, it may also be used to describe the behaviors of large crowds, fish schools, car traffic. . . provided the number of “elementary” constituents is large and the system is studied on a large enough scale.

* Even if the continuous description is valid on “long wavelengths”, it remains obvious that any physical system, viewed on a scale much larger than its spatial extent, is to first approximation best described as pointlike.

Consider for instance a molecular cloud of interstellar medium with a 10 parsec radius and about 10^{10} H₂ molecules per cubic meter. For a star forming at its core, it behaves as a continuous medium; seen from 1 kpc away, however, the inner degrees of the cloud are most likely already irrelevant and it is best described as a mere point.

1.1.2 General mathematical framework

Consider a non-relativistic classical macroscopic physical system Σ , described by Newtonian^(a) physics. The positions of its individual atoms, viewed as pointlike, at a given instant t —which is the same for all observers—are points in a three-dimensional Euclidean^(b) space \mathcal{E}_3 .

In the description as a continuous medium, the system Σ is represented by a geometrical manifold in \mathcal{E}_3 , which for the sake of simplicity will be referred to as a “volume” and denoted by \mathcal{V} . The basic constituents of \mathcal{V} are its infinitesimal elements $d^3\mathcal{V}$, called *material points*⁽ⁱⁱⁱ⁾ or *continuous medium particles*^(iv)—which explains a posteriori our designating the discrete constituents of matter as “atoms”—, or, in the specific case of the elementary subdivisions of a fluid, *fluid particles*^(v). As we shall state more explicitly in § 1.1.3, these infinitesimal elements are assumed to have the same physical properties as a finite macroscopic piece.

Associated with the physical picture attached to the notion of continuousness is the requirement that neighboring material points in the medium remain close to each other after some infinitesimal evolution of the system. We shall see below how this picture is implemented in the mathematical description.

Remark: The volume \mathcal{V} —with the topology inherited from \mathcal{E}_3 —need not be simply connected. For instance, one may want to describe the flow of a river around a bridge pier: the latter represents a physical region which water cannot penetrate, which is modeled as a hole in the volume \mathcal{V} occupied by fluid particles.

⁽ⁱⁱⁱ⁾ *Materielle Punkte* ^(iv) *Mediumteilchen* ^(v) *Fluidteilchen*

^(a) I. NEWTON, 1642–1727 ^(b) EUCLID (Εὐκλείδης), about 300 BC

To characterize the position of a given material point, as well as some of the observables relative to the physical system Σ , one still needs to specify the *reference frame* in which the system is studied, corresponding to the point of view of a given observer. This choice allows one to define vectors—like position vectors, velocities, or accelerations—and tensors relevant for kinematics.

In the reference frame, one can further choose a *coordinate system*. The basis vectors of the coordinate system will generically be designated as $\vec{e}_1, \vec{e}_2, \vec{e}_3$, while the components of a given vector will be denoted with upper (“contravariant”) indices, as e.g. $\vec{c} = c^i \vec{e}_i$, where the summation convention over repeated upper and lower indices was used.

Once the reference frame and coordinate system are determined, the macroscopic state of the physical system at time t is mapped onto a corresponding *configuration* κ_t of the medium, consisting of the continuous set of the position vectors $\vec{r} = x^i \vec{e}_i$ of its constituting material points. Since the volume occupied by the latter may also depend on time, it will also be labeled by t : \mathcal{V}_t .

To be able to formalize the necessary continuity conditions in the following Sections, one also introduces a reference time t_0 —conveniently taken as the origin of the time axis, $t_0 = 0$ —and the corresponding *reference configuration* κ_0 of the medium, which occupies a volume \mathcal{V}_0 . The generic position vector of a material point in this reference configuration will be denoted as $\vec{R} = X^i \vec{e}_i$.

Remark: In so-called “classical” continuous media, as have been introduced here, the material points are entirely characterized by their position vectors. In particular, they have no intrinsic angular momentum.

In contrast, the material points of so-called “micropolar fluids” (or “Cosserat^(c) fluids”) also have such an intrinsic angular momentum. Accordingly, the dynamical equations of motion (Chap. ??) also include one equation for angular-momentum conservation.

I.1.3 Local thermodynamic equilibrium

In a more bottom-up approach to the modeling of a system Σ of discrete constituents as a continuous medium, one should first divide Σ (in thought) into small cells of fixed—yet not necessarily universal—size fulfilling two conditions:

- (i) each individual cell can meaningfully be treated as a thermodynamic system, i.e. it must be large enough that the relative fluctuations of the usual extensive thermodynamic quantities computed for the content of the cell are negligible;
- (ii) the thermodynamic properties vary little over the cell scale, i.e. cells cannot be too large, so that (approximate) homogeneity is ensured.

The rationale behind these two requirements is illustrated by Fig. [I.1](#), which represents schematically how the value of a local macroscopic quantity, e.g. a density, depends on the resolution of the apparatus with which it is measured, i.e. equivalently on the length scale on which it is defined. If the apparatus probes too small a length scale, so that the discrete degrees of freedom become relevant, the measured value strongly fluctuates from one observation to the next one, as hinted at by the displayed envelope of possible results of measurements: this is the issue addressed by condition (i). Simultaneously, a small change in the measurement resolution, even with the apparatus still centered on the same point in the system, can lead to a large variation in the measured value of the observable, corresponding to the erratic behavior of the curve at small scales shown in Fig. [I.1](#). This fluctuating pattern decreases with increasing size of the observation scale, since this increase leads to a growth in the number of atoms inside the probed volume, and thus a drop in the size of relative fluctuations. At the other end of the curve, one reaches a regime where the low resolution of the observation leads to encompassing domains with enough atoms to be rid of fluctuations, yet with inhomogeneous

^(c)F. COSSERAT, 1852–1914 & E. COSSERAT, 1866–1931

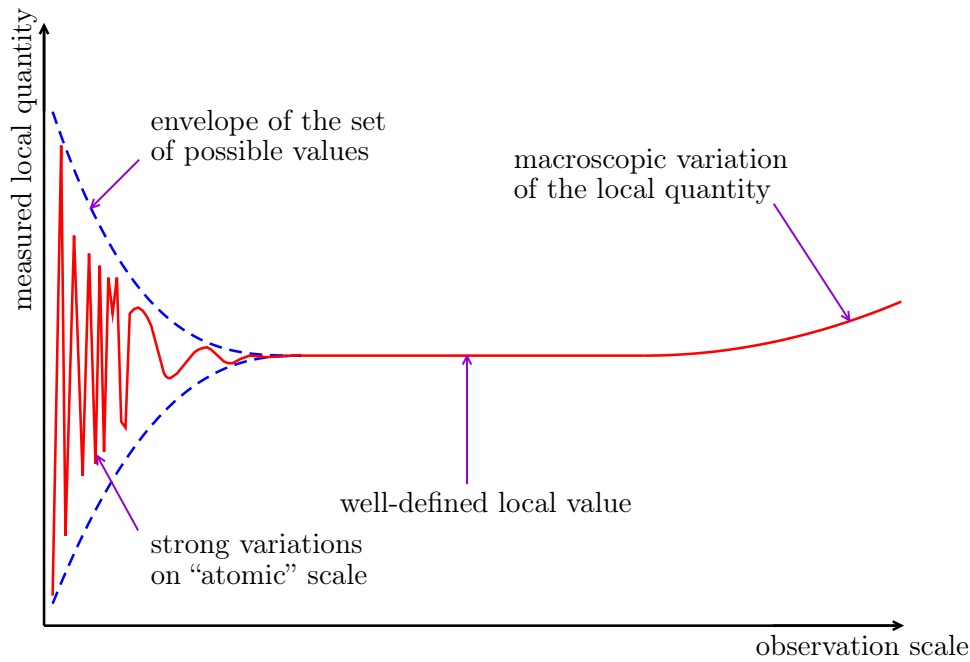


Figure I.1 – Typical variation of the measured value for a “local” macroscopic observable as a function of the size scale over which it is determined.

macroscopic properties, in a single probed region—in violation of condition (ii). As a result, the measured value of the density under consideration slowly evolves with the observation scale.

In between these two domains of strong statistical fluctuations and slow macroscopic variations lies a regime where the value measured for an observable barely depends on the scale over which it is determined. This represents the appropriate regime for meaningfully defining—and measuring—a local density, and more general local quantities.

It is important to note that this intermediate “mesoscopic” interval may not always exist. There are physical systems in which strong macroscopic variations are already present in a range of scales where microscopic fluctuations are still sizable. For such systems, one cannot find scale-independent local variables. That is, the proper definition of local quantities implicitly relies on the existence of a clear separation of scales in the physical system under consideration, which is what will be assumed in the remainder of these notes.

Remark: The smallest volume over which meaningful local quantities can be defined is sometimes called *representative volume element* (RVE), or *representative elementary volume*.

When conditions (i) and (ii) hold, one may in particular define *local thermodynamic variables*, corresponding to the values taken in each intermediate-size cell—labeled by its position \vec{r} —by the usual extensive parameters: internal energy, number of atoms... Since the separation between cells is immaterial, nothing prevents energy or matter from being transported from a cell to its neighbors, even if the global system is isolated. Accordingly, the local extensive variables in any given cell are actually time-dependent in the general case. In addition, it becomes important to add linear momentum—with respect to some reference frame—to the set of local extensive variables characterizing the content of a cell.

The size of each cell is physically irrelevant, as long as it satisfies the two key requirements; there is thus no meaningful local variable corresponding to volume. Similarly, the values of the extensive variables in a given cell, which are by definition proportional to the cell size, are as arbitrary as the latter. They are thus conveniently replaced by the respective local densities: internal energy density $e(t, \vec{r})$, number density $n(t, \vec{r})$, linear momentum density $\rho(t, \vec{r})\vec{v}(t, \vec{r})$, where ρ denotes the mass density, entropy density $s(t, \vec{r})$...

Remark: Rather than considering the densities of extensive quantities, some authors—in particular Landau^(d) & Lifshitz^(e) [4, 5]—prefer to work with *specific quantities*, i.e. their respective amounts per unit mass, instead of per unit volume. The relation between densities and specific quantities is trivial: denoting by χ_j resp. $\chi_{j,m}$ a generic local density resp. specific amount for the same physical quantity, one has the identity

$$\chi_j(t, \vec{r}) = \rho(t, \vec{r}) \chi_{j,m}(t, \vec{r}) \quad (\text{I.1})$$

in every cell—labeled by \vec{r} —and at every time t .

Once the local extensive variables have been meaningfully defined, one can develop the usual formalism of thermodynamics in each cell. In particular, one introduces the conjugate intensive variables, as e.g. local temperature $T(t, \vec{r})$ and pressure $\mathcal{P}(t, \vec{r})$. The underlying, important hypothesis is the assumption of a *local thermodynamic equilibrium*. According to the latter, the equation(s) of state of the system inside the small cell, expressed with local thermodynamic quantities, is the same as for a macroscopic system in the actual thermodynamic limit of infinitely large volume and particle number.

Consider for instance a non-relativistic classical ideal gas: its (mechanical) equation of state reads $\mathcal{P}\mathcal{V} = Nk_{\text{B}}T$, with N the number of atoms, which occupy a volume \mathcal{V} at uniform pressure \mathcal{P} and temperature T , while k_{B} is the Boltzmann^(f) constant. This is trivially recast as $\mathcal{P} = n k_{\text{B}}T$, with n the number density of atoms. The local thermodynamic equilibrium assumption then states that under non-uniform conditions of temperature and pressure, the equation of state in a local cell at position \vec{r} is given by

$$\mathcal{P}(t, \vec{r}) = n(t, \vec{r}) k_{\text{B}}T(t, \vec{r}) \quad (\text{I.2})$$

at every time t .

The last step towards the continuous-medium model is to promote \vec{r} , which till now was simply the discrete label attached to a given cell, to be a continuous variable taking its values in \mathbb{R}^3 —or rather, in the volume \mathcal{V}_t attached to the system at the corresponding instant t . Accordingly, taking into account the time-dependence of physical quantities, the local variables, in particular the thermodynamic parameters, become *fields* on $\mathbb{R} \times \mathbb{R}^3$.

The replacement of the fine-resolution description, in which atoms are the relevant degrees of freedom, by the lower-resolution model which assimilates small finite volumes of the former to structureless points is called *coarse graining*^(vi).

This is a quite generic procedure in theoretical physics, whereby the finer degrees of freedom of a more fundamental description are smoothed away—technically, this is often done by performing averages or integrals, so that these degrees of freedom are “integrated out”—and replaced by novel, effective variables in a theory with a more limited range of applicability, but which is more tractable for “long-range” phenomena.

Coming back to condition (ii), we already stated that it implicitly involves the existence of at least one large length scale L , over which the macroscopic physical properties of the system may vary. This scale can be a characteristic dimension of the system under consideration, as e.g. the diameter of the tube in which a liquid is flowing. In the case of periodic waves propagating in the continuous medium, L also corresponds to their wavelength. More generally, if \mathcal{G} denotes a macroscopic physical quantity, one may consider

$$L \cong \left[\frac{|\vec{\nabla}\mathcal{G}(t, \vec{r})|}{|\mathcal{G}(t, \vec{r})|} \right]^{-1}, \quad (\text{I.3})$$

where $\vec{\nabla}$ denotes the (spatial) gradient.

^(vi) *Vergrößerung*

^(d)L. D. LANDAU, 1908–1968 ^(e)E. M. LIFSHITZ, 1915–1985 ^(f)L. BOLTZMANN, 1844–1906

Condition (i) in particular implies that the typical size of the cells which are later coarse grained should be significantly larger than the *mean free path* ℓ_{mfp} of atoms, so that thermodynamic equilibrium holds in the local cells. Since on the other hand this same typical size should be significantly smaller than the scale L of macroscopic variations, one deduces the condition

$$\text{Kn} \equiv \frac{\ell_{\text{mfp}}}{L} \ll 1 \quad (\text{I.4})$$

on the dimensionless *Knudsen number* Kn .^(g)

In air under normal conditions $\mathcal{P} = 10^5$ Pa and $T = 300$ K, the mean free path is $\ell_{\text{mfp}} \approx 0.1 \mu\text{m}$. In the study of phenomena with variations on a characteristic scale $L \approx 10$ cm, one finds $\text{Kn} \approx 10^{-6}$, so that air can be meaningfully treated as a continuous medium.

The opposite regime $\text{Kn} > 1$ is that of a *rarefied medium*, as for instance of the so-called Knudsen gas, in which the collisions between atoms are negligible—and in particular insufficient to ensure thermal equilibrium as an ideal gas. The flow of such systems is not well described by hydrodynamics, but necessitates alternative descriptions like molecular dynamics, in which the degrees of freedom are explicitly atoms.

^(g)M. KNUDSEN, 1871–1949