I.3 Eulerian description

The Lagrangian approach introduced in the previous Section is actually not commonly used in fluid dynamics, at least not in its original form, except for specific problems.

> One reason is that physical quantities at a given time are expressed in terms of a reference configuration in the (far) past: a small uncertainty on this initial condition may actually yield

(vii) *Bahnlinie* (viii) *Streichlinie*

after a finite duration a large uncertainty on the present state of the system, which is problematic. On the other hand, this line of argument explains why the Lagrangian point of view is adopted to investigate *chaos* in many-body systems!

The more usual description is the so-called *Eulerian*[(d)] perspective, in which the evolution between instants t and $t + \mathrm{d}t$ takes the system configuration at time t as a reference.

I.3.1 Eulerian coordinates. Velocity field

In contrast to the "material" Lagrangian point of view, which identifies the medium particles in a reference configuration and follows them in their motion, in the Eulerian description the emphasis is placed on the *geometrical* points. Thus, the *Eulerian coordinates* are time t and a spatial vector $\vec{r}$, where the latter does not label the position of a material point, but rather that of a geometrical point. Accordingly, the physical quantities in the Eulerian specification are described by *fields* on space-time.

Thus, the fundamental field that entirely characterizes the motion of a continuous medium in a given reference frame $\mathcal{R}$ is the *velocity field* $\vec{\mathsf{v}}_t(t,\vec{r})$. The latter is defined such that it gives the value of the Lagrangian velocity $\vec{v}$ [Eq. (I.8)] of a material point passing through $\vec{r}$ at time t:

$$\vec{v} = \vec{\mathsf{v}}_t(t,\vec{r}) \quad \forall t,\ \forall \vec{r} \in \mathcal{V}_t. \tag{I.11}$$

More generally, the value taken at given time and position by a physical quantity $\mathcal{G}$, whether attached to a material point or not, is expressed as a mathematical function $\mathcal{G}_t$ of the same Eulerian variables:

$$\mathcal{G} = \mathcal{G}_t(t,\vec{r}) \quad \forall t,\ \forall \vec{r} \in \mathcal{V}_t. \tag{I.12}$$

Note that the mappings $(t,\vec{R}) \mapsto \mathcal{G}(t,\vec{R})$ in the Lagrangian approach and $(t,\vec{r}) \mapsto \mathcal{G}_t(t,\vec{r})$ in the Eulerian description are in general different. For instance, the domains in $\mathbb{R}^3$ over which their spatial variables take their values differ: constant ($\mathcal{V}_0$) in the Lagrangian specification, time-dependent ($\mathcal{V}_t$) in the case of the Eulerian quantities. Accordingly the latter will be denoted with a subscript t in the next subsection.

I.3.2 Equivalence between the Eulerian and Lagrangian viewpoints

Despite the different choices of variables, the Lagrangian and Eulerian descriptions are fully equivalent. Accordingly, the prevalence in practice of the one over the other is more a technical issue than a conceptual one.

Thus, it is rather clear that the knowledge of the Lagrangian specification can be used to obtain the Eulerian formulation at once, using the mapping $\vec{r} \mapsto \vec{R}(t,\vec{r})$ between present and reference positions of a material point. For instance, the Eulerian velocity field can be expressed as

$$\vec{\mathsf{v}}_t(t,\vec{r}) = \vec{v}\big(t,\vec{R}(t,\vec{r})\big). \tag{I.13a}$$

This identity in particular shows that $\vec{\mathsf{v}}_t$ automatically inherits the smoothness properties of $\vec{v}$: if the mapping $(t,\vec{R}) \mapsto \vec{r}(t,\vec{R})$ and its inverse are piecewise $\mathscr{C}^2$ (cf. § I.2.2), then $\vec{\mathsf{v}}_t$ is (at least) piecewise $\mathscr{C}^1$ in both its variables.

For a generic physical quantity, the transition from the Lagrangian to the Eulerian point of view similarly reads

$$\mathcal{G}_t(t,\vec{r}) = \mathcal{G}\big(t,\vec{R}(t,\vec{r})\big). \tag{I.13b}$$

Reciprocally, given a (well-enough behaved) Eulerian velocity field $\vec{\mathsf{v}}_t$ on a continuous medium, one can uniquely obtain the Lagrangian description of the medium motion by solving the *initial*

[(d)] L. Euler, 1707–1783

value problem

$$\begin{cases} \dfrac{\partial\vec{r}(t,\vec{R})}{\partial t} &= \vec{\mathsf{v}}_t\big(t,\vec{r}(t,\vec{R})\big) \\ \vec{r}(t_0,\vec{R}) &= \vec{R}, \end{cases} \tag{I.14a}$$

where the second line represents the initial condition. That is, one actually reconstructs the pathline of every material point of the continuous medium. Introducing differential notations, the above system can also be rewritten as

$$\mathrm{d}\vec{r} = \vec{\mathsf{v}}_t(t,\vec{r})\,\mathrm{d}t \quad \text{with} \quad \vec{r}(t_0,\vec{R}) = \vec{R}. \tag{I.14b}$$

Once the pathlines $\vec{r}(t,\vec{R})$ are known, one obtains the Lagrangian function $\mathcal{G}(t,\vec{R})$ for a given physical quantity $\mathcal{G}$ by writing

$$\mathcal{G}(t,\vec{R}) = \mathcal{G}_t\big(t,\vec{r}(t,\vec{R})\big). \tag{I.14c}$$

Since both Lagrangian and Eulerian descriptions are equivalent, we shall from now on drop the subscript t on the mathematical functions representing physical quantities in the Eulerian point of view.

I.3.3 Streamlines

At a given time t, the *streamlines*(ix) of the motion are defined as the field lines of $\vec{\mathsf{v}}$. That is, these are curves whose tangent is everywhere parallel to the instantaneous velocity field at the same geometrical point.

Let $\vec{x}(\lambda)$ denote a streamline, parameterized by λ. The definition can be formulated as

$$\frac{\mathrm{d}\vec{x}(\lambda)}{\mathrm{d}\lambda} = \alpha(\lambda)\vec{\mathsf{v}}\big(t,\vec{x}(\lambda)\big) \tag{I.15a}$$

with $\alpha(\lambda)$ a scalar function. Equivalently, denoting by $\mathrm{d}\vec{x}(\lambda)$ a differential line element tangent to the streamline, one has the condition

$$\mathrm{d}\vec{x} \times \vec{\mathsf{v}}\big(t,\vec{x}(\lambda)\big) = \vec{0}. \tag{I.15b}$$

Introducing a Cartesian system of coordinates, the equation for a streamline is conveniently rewritten as

$$\frac{\mathrm{d}x^1(\lambda)}{\mathsf{v}^1\big(t,\vec{x}(\lambda)\big)} = \frac{\mathrm{d}x^2(\lambda)}{\mathsf{v}^2\big(t,\vec{x}(\lambda)\big)} = \frac{\mathrm{d}x^3(\lambda)}{\mathsf{v}^3\big(t,\vec{x}(\lambda)\big)} \tag{I.15c}$$

at a point where none of the component v^i of the velocity field vanishes—if one of the v^i is zero, then so is the corresponding $\mathrm{d}x^i$, thanks to Eq. (I.15b).

Remark: Since the velocity field $\vec{\mathsf{v}}$ depends on the choice of reference frame, this is also the case of its streamlines at a given instant!

Consider now a closed geometrical curve in the volume $\mathcal{V}_t$ occupied by the continuous medium at time t. The streamlines tangent to this curve form in the generic case a tube-like surface, called *stream tube*.(x)

Let us introduce two further definitions related to properties of the velocity field:

- If $\vec{\mathsf{v}}(t,\vec{r})$ has at some t the same value at every geometrical point $\vec{r}$ of a (connected) domain $\mathcal{D} \subset \mathcal{V}_t$, then the velocity field is said to be *uniform* across $\mathcal{D}$.
  In that case, the streamlines are parallel to each other over $\mathcal{D}$.

(ix) *Stromlinien* (x) *Stromröhre*

- If $\vec{\mathsf{v}}(t,\vec{r})$ only depends on the position, not on time, then the velocity field and the corresponding motion of the continuous medium are said to be *steady* or equivalently *stationary*.
  In that case, the streamlines coincide with the pathlines and the streaklines.

  Indeed, one checks that Eq. (I.14b) for the pathlines, in which the velocity becomes time-independent, can then be recast (at a point where all v^i are non-zero) as
$$\frac{\mathrm{d}x^1}{\mathsf{v}^1(t,\vec{r})} = \frac{\mathrm{d}x^2}{\mathsf{v}^2(t,\vec{r})} = \frac{\mathrm{d}x^3}{\mathsf{v}^3(t,\vec{r})},$$
  where the variable t plays no role: this is exactly the system (I.15c) defining the streamlines at time t. The equivalence between pathlines and streaklines is also trivial. □

I.3.4 Material derivative

Consider a material point M in a continuous medium, described in a reference frame $\mathcal{R}$. Let $\vec{r}$ resp. $\vec{r}+\mathrm{d}\vec{r}$ denote its position vectors at successive instants t resp. $t+\mathrm{d}t$. The velocity of M at time t resp. $t+\mathrm{d}t$ is by definition equal to the value of the velocity field at that time and at the respective position, namely $\vec{\mathsf{v}}(t,\vec{r})$ resp. $\vec{\mathsf{v}}(t+\mathrm{d}t,\vec{r}+\mathrm{d}\vec{r})$. For small enough $\mathrm{d}t$, the displacement $\mathrm{d}\vec{r}$ of the material point between t and $t+\mathrm{d}t$ is related to its velocity at time t by $\mathrm{d}\vec{r} = \vec{\mathsf{v}}(t,\vec{r})\,\mathrm{d}t$.

Let $\mathrm{d}\vec{\mathsf{v}} \equiv \vec{\mathsf{v}}(t+\mathrm{d}t,\vec{r}+\mathrm{d}\vec{r}) - \vec{\mathsf{v}}(t,\vec{r})$ denote the change in the material point velocity between t and $t+\mathrm{d}t$. Assuming that $\vec{\mathsf{v}}(t,\vec{r})$ is differentiable (cf. § I.3.2) and introducing for simplicity a system of Cartesian coordinates, a Taylor expansion to lowest order yields
$$\mathrm{d}\vec{\mathsf{v}} \simeq \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t}\,\mathrm{d}t + \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial x^1}\,\mathrm{d}x^1 + \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial x^2}\,\mathrm{d}x^2 + \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial x^3}\,\mathrm{d}x^3,$$
up to terms of higher order in $\mathrm{d}t$ or $\mathrm{d}\vec{r}$. Introducing the differential operator
$$\mathrm{d}\vec{r}\cdot\vec{\nabla} \equiv \mathrm{d}x^1\,\frac{\partial}{\partial x^1} + \mathrm{d}x^2\,\frac{\partial}{\partial x^2} + \mathrm{d}x^3\,\frac{\partial}{\partial x^3},$$
this can be recast in the more compact form
$$\mathrm{d}\vec{\mathsf{v}} \simeq \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t}\,\mathrm{d}t + \big(\mathrm{d}\vec{r}\cdot\vec{\nabla}\big)\vec{\mathsf{v}}(t,\vec{r}). \tag{I.16}$$

In the second term on the right-hand side, $\mathrm{d}\vec{r}$ can be replaced by $\vec{\mathsf{v}}(t,\vec{r})\,\mathrm{d}t$. On the other hand, the change in velocity of the material point between t and $t+\mathrm{d}t$ is simply the product of its acceleration $\vec{a}(t)$ at time t by the length $\mathrm{d}t$ of the time interval, at least to lowest order in $\mathrm{d}t$. Dividing both sides of Eq. (I.16) by $\mathrm{d}t$ and taking the limit $\mathrm{d}t\to 0$, in particular in the ratio $\mathrm{d}\vec{\mathsf{v}}/\mathrm{d}t$, yield
$$\vec{a}(t) = \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}). \tag{I.17}$$
That is, the acceleration of the material point consists of two terms:

- the *local acceleration* $\dfrac{\partial\vec{\mathsf{v}}}{\partial t}$, which follows from the non-stationarity of the velocity field;
- the *convective acceleration* $\big(\vec{\mathsf{v}}\cdot\vec{\nabla}\big)\vec{\mathsf{v}}$, due to the non-uniformity of the motion.

More generally, one finds by repeating the same derivation as above that the time derivative of a physical quantity $\mathcal{G}$ attached to a material point or domain, expressed in terms of Eulerian fields, is the sum of a local $(\partial\mathcal{G}/\partial t)$ and a convective $[(\vec{\mathsf{v}}\cdot\vec{\nabla})\mathcal{G}]$ part, irrespective of the tensorial nature of $\mathcal{G}$. Accordingly, one introduces the operator
$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}} \tag{I.18}$$

called *material derivative*[(xi)] or (between others) *substantial derivative*,[(xii)] *derivative following the motion*, *hydrodynamic derivative*. Relation (I.17) can thus be recast as

$$\vec{a}(t) = \frac{\mathrm{D}\vec{\mathsf{v}}(t,\vec{r})}{\mathrm{D}t}. \tag{I.19}$$

Remarks:

∗ Equation (I.17) shows that even in the case of a steady motion, the acceleration of a material point may be non-vanishing, thanks to the convective part.

∗ The material derivative (I.18) is also often denoted (and referred to) as total derivative $\mathrm{d}/\mathrm{d}t$.

∗ One also finds in the literature the denomination *convective derivative*.[(xiii)] To the eyes and ears of the author of these lines, that name has the drawback that it does not naturally evoke the local part, but only... the convective one, which comes from the fact that matter is being transported, "conveyed", with a non-vanishing velocity field $\vec{\mathsf{v}}(t,\vec{r})$.

∗ The two terms in Eq. (I.18) actually "merge" together when considering the motion of a material point in Galilean space-time $\mathbb{R}\times\mathbb{R}^3$. As a matter of fact, one easily shows that $\mathrm{D}/\mathrm{D}t$ is the (Lie[(e)]) derivative along the world-line of the material point

The world-line element corresponding to the motion between t and $t+\mathrm{d}t$ goes from (t,x^1,x^2,x^3) to $(t+\mathrm{d}t, x^1+\mathsf{v}^1\,\mathrm{d}t, x^2+\mathsf{v}^2\,\mathrm{d}t, x^3+\mathsf{v}^3\,\mathrm{d}t)$. The tangent vector to this world-line thus has components $(1,\mathsf{v}^1,\mathsf{v}^2,\mathsf{v}^3)$, i.e. the derivative along the direction of this vector is $\partial_t + \mathsf{v}^1\partial_1 + \mathsf{v}^2\partial_2 + \mathsf{v}^3\partial_3$, with the usual shorthand notations $\partial_t \equiv \partial/\partial t$ and $\partial_i \equiv \partial/\partial x^i$. □

I.4 Mechanical stress

I.4.1 Forces in a continuous medium

Consider a closed material domain $\mathcal{V}$ inside the volume $\mathcal{V}_t$ occupied by a continuous medium, and let $\mathcal{S}$ denote the (geometric) surface enclosing $\mathcal{V}$. One distinguishes between two classes of forces acting on this domain:

- *Volume* or *body forces*,[(xiv)] which act at each point of the bulk volume of $\mathcal{V}$.
  Examples are weight, long-range electromagnetic forces or, in non-inertial reference frames, fictitious forces (Coriolis, centrifugal).
  For such forces, which tend to be proportional to the volume they act on, it will later be more convenient to introduce the corresponding volumic force density.

- *Surface* or *contact forces*,[(xv)] which act on the surface $\mathcal{S}$, like friction. These will be now discussed in further detail.

Consider an infinitesimally small geometrical surface element $\mathrm{d}^2\mathcal{S}$ at point P. Let $\mathrm{d}^2\vec{F}_s$ denote the surface force through $\mathrm{d}^2\mathcal{S}$. That is, $\mathrm{d}^2\vec{F}_s$ is the contact force, due to the medium exterior to $\mathcal{V}$, that a "test" material surface coinciding with $\mathrm{d}^2\mathcal{S}$ would experience. The vector

$$\vec{T}_{\mathrm{s}} \equiv \frac{\mathrm{d}^2\vec{F}_s}{\mathrm{d}^2\mathcal{S}}, \tag{I.20}$$

representing the surface density of contact forces, is called *(mechanical) stress vector*[(xvi)] on $\mathrm{d}^2\mathcal{S}$.

[(xi)] *Materielle Ableitung* [(xii)] *Substantielle Ableitung* [(xiii)] *Konvektive Ableitung* [(xiv)] *Volumenkräfte* [(xv)] *Oberflächenkräfte*

[(e)] S. Lie, 1842–1899

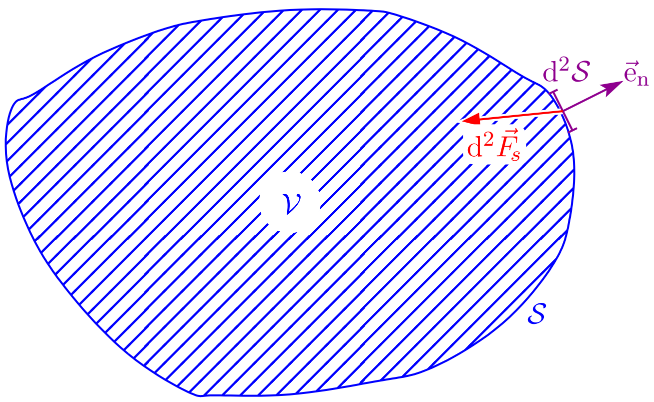

Figure I.2

The corresponding unit in the SI system is the Pascal, with 1 Pa $= 1\,\mathrm{N}\cdot\mathrm{m}^{-2}$.

Purely geometrically, the stress vector $\vec{T}_s$ on a given surface element $\mathrm{d}^2\mathcal{S}$ at a given point can be decomposed into two components, namely

- a vector orthogonal to the plane tangent at P to $\mathrm{d}^2\mathcal{S}$, the so-called *normal stress*[(xvii)]; when it is directed towards the interior resp. exterior of the medium domain being acted on, it is also referred to as *compression*[(xviii)] resp. *tension*[(xix)];
- a vector in the tangent plane at P, called *shear stress*[(xx)] and often denoted as $\vec{\tau}$.

Despite the short notation adopted in Eq. (I.20), the stress vector depends not only on the position of the geometrical point P where the infinitesimal surface element $\mathrm{d}^2\mathcal{S}$ lies, but also on the orientation of the surface. Let $\vec{e}_n$ denote the normal unit vector to the surface element, directed towards the exterior of the volume $\mathcal{V}$ (cf. Fig. I.2), and let $\vec{r}$ denote the position vector of P in a given reference frame. The relation between $\vec{e}_n$ and the stress vector $\vec{T}_s$ on $\mathrm{d}^2\mathcal{S}$ is then linear:

$$\vec{T}_s = \boldsymbol{\sigma}(\vec{r}) \cdot \vec{e}_n, \tag{I.21a}$$

with $\boldsymbol{\sigma}(\vec{r})$ a symmetric tensor of rank 2, the so-called (*Cauchy*[(f)]) *stress tensor*.[(xxi)]

In a given coordinate system, relation (I.21a) yields

$$T_s^i = \sum_{j=1}^{3} \boldsymbol{\sigma}^i{}_j\, e_n^j \tag{I.21b}$$

with T_s^i resp. e_n^j the coordinates of the vectors $\vec{T}_s$ resp. $\vec{e}_n$, and $\boldsymbol{\sigma}^i{}_j$ the $\binom{1}{1}$-components of the stress tensor.

While valid in the case of a three-dimensional position space, Eq. (I.21a) should actually be better formulated to become valid in arbitrary dimension. Thus, the unit-length "normal vector" to a surface element at point P is rather a *1-form* acting on the vectors of the tangent space to the surface at P. As such, it should be represented as the transposed of a vector $[(\vec{e}_n)^{\mathsf{T}}]$, which multiplies the stress tensor from the left:

$$\boxed{\vec{T}_s = (\vec{e}_n)^{\mathsf{T}} \cdot \boldsymbol{\sigma}(\vec{r}).} \tag{I.21c}$$

(xvi) *Mechanischer Spannungsvektor* (xvii) *Normalspannung* (xviii) *Druckspannung* (xix) *Zugsspannung* (xx) *Scher-, Tangential-* oder *Schubspannung* (xxi) (*Cauchy'scher*) *Spannungstensor*

(f) A.L. Cauchy, 1789–1857

This shows that the Cauchy stress tensor is a $\binom{2}{0}$-tensor (a "bivector"), which maps 1-forms onto vectors. In terms of coordinates, this gives, using Einstein's summation convention

$$T_s^j = \mathrm{e}_{\mathrm{n},i}\, \boldsymbol{\sigma}^{ij}\,, \tag{I.21d}$$

which thanks to the symmetry of $\boldsymbol{\sigma}$ is equivalent to the relation given above.

Remark: The symmetry property of the Cauchy stress tensor is intimately linked to the assumption that the material points constituting the continuous medium have no intrinsic angular momentum.

I.4.2 Fluids

With the help of the notion of mechanical stress, we may now introduce the definition of a *fluid*, which is the class of continuous media whose motion is described by hydrodynamics:

A fluid is a continuous medium that deforms itself as long as it is submitted to shear stresses. (I.22)

Turning this definition around, one sees that in a fluid *at rest*—or, to be more accurate, studied in a reference frame with respect to which it is at rest—the mechanical stresses are necessarily normal. That is, the stress tensor is in each point diagonal.

More precisely, for a locally isotropic fluid—which means that the material points are isotropic, which is the case throughout these notes—the stress $\binom{2}{0}$-tensor is everywhere proportional to the inverse metric tensor:

$$\boxed{\boldsymbol{\sigma}(t,\vec{r}) = -\mathcal{P}(t,\vec{r})\, \mathbf{g}^{-1}(t,\vec{r})} \tag{I.23}$$

with $\mathcal{P}(t,\vec{r})$ the *hydrostatic pressure* at position $\vec{r}$ at time t.

Going back to relation (I.21b), the stress vector will be parallel to the "unit normal vector" in any coordinate system if the square matrix of the $\binom{1}{1}$-components $\boldsymbol{\sigma}^i{}_j$ is proportional to the identity matrix, i.e. $\boldsymbol{\sigma}^i{}_j \propto \delta^i{}_j$, where we have introduced the Kronecker symbol. To obtain the $\binom{2}{0}$-components $\boldsymbol{\sigma}^{ik}$, one has to multiply $\boldsymbol{\sigma}^i{}_j$ by the component g^{jk} of the inverse metric tensor, summing over k, which precisely gives Eq. (I.23).

Remarks:

∗ Definition (I.22), as well as the two remarks hereafter, rely on an intuitive picture of "deformations" in a continuous medium. To support this picture with some mathematical background, we shall introduce in Sec. II.A an appropriate strain tensor, which quantifies these deformations, at least as long as they remain small.

∗ A deformable solid will also deform itself when submitted to shear stress! However, for a given fixed amount of tangential stress, the solid will after some time reach a new, deformed equilibrium position—otherwise, it is not a solid, but a fluid.

∗ The previous remark is actually a simplification, valid on the typical time scale of human beings. Thus, materials which in our everyday experience are solids—as for instance those forming the mantle of the Earth—will behave on a longer time scale as fluids—in the previous example, on geological time scales. Whether a given substance behaves as a fluid or a deformable solid is sometimes characterized by the dimensionless *Deborah number* [9], which compares the typical time scale for the response of the substance to a mechanical stress and the observation time.

∗ Even nicer, the fluid vs. deformable solid behavior may actually depend on the intensity of the applied shear stress: ketchup!