V.1.3 Plane Poiseuille flow

Let us now consider the flow of a Newtonian fluid between two motionless plane plates with a finite length along the x direction—yet still infinitely extended along the z direction—, as illustrated in Fig. V.2. The pressure is assumed to be different at both ends of the plates in the x direction, leading to the presence of a pressure gradient along x.

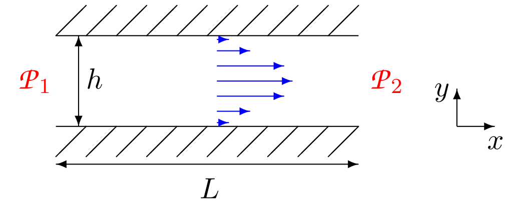

Figure V.2 – Flow between two motionless plates for $\mathcal{P}_1 > \mathcal{P}_2$, i.e. $\delta\mathcal{P} > 0$.

Assuming for the flow velocity $\vec{\mathsf{v}}(\vec{r})$ the same form $\mathsf{v}(y)\,\vec{e}_x$, independent of x, as in the case of the plane Couette flow, the equations of motion governing $\mathsf{v}(y)$ and pressure $\mathcal{P}(\vec{r})$ are the same as in the previous § V.1.2, namely Eqs. (V.3)–(V.4). The boundary conditions are however different. Thus, $\mathcal{P}_1 \neq \mathcal{P}_2$ results in a finite constant pressure gradient along x, $\alpha = \partial\mathcal{P}(\vec{r})/\partial x = -\delta\mathcal{P}/L \neq 0$, with $\delta\mathcal{P} \equiv \mathcal{P}_1 - \mathcal{P}_2$ the pressure drop. Equation (V.4) then leads to

$$\mathsf{v}(y) = -\frac{1}{2\eta}\frac{\delta\mathcal{P}}{L}y^2 + \gamma y + \delta,$$

with γ and δ two new constants.

The "no-slip" boundary conditions for the velocity at the two plates read

$$\mathsf{v}(y\!=\!0) = 0, \qquad \mathsf{v}(y\!=\!h) = 0,$$

which leads to $\delta = 0$ and $\gamma = \dfrac{1}{2\eta}\dfrac{\delta\mathcal{P}}{L}h$. The flow velocity thus has the parabolic profile

$$\mathsf{v}(y) = \frac{1}{2\eta}\frac{\delta\mathcal{P}}{L}\big[y(h-y)\big] \quad \text{for } 0 \leq y \leq h, \tag{V.5}$$

directed along the direction of the pressure gradient.

Remark: The flow velocity (V.5) becomes clearly problematic in the limit $\eta \to 0$! Tracing the problem back to its source, the equations of motion (V.3) cannot hold with a finite pressure gradient along the x direction and a vanishing viscosity. One quickly checks that the only possibility in the case of a perfect fluid is to drop one of the assumptions, either incompressibility or laminarity.

V.1.4 Hagen–Poiseuille flow

The previous two examples involved plates with an infinite length in at least one direction, thus were idealized constructions. In contrast, an experimentally realizable fluid motion is that of the *Hagen–Poiseuille flow*,[(ad)] in which a Newtonian fluid flows under the influence of a pressure gradient in a cylindrical tube with finite length L and radius a (Fig. V.3). Again, the motion is assumed to be steady, incompressible and laminar.

Using cylindrical coordinates, the ansatz $\vec{\mathsf{v}}(\vec{r}) = \mathsf{v}(r)\,\vec{e}_z$ with $r = \sqrt{x^2+y^2}$ satisfies the continuity equation $\vec{\nabla}\cdot\vec{\mathsf{v}}(\vec{r}) = 0$ and gives for the incompressible Navier–Stokes equation

$$\vec{\nabla}\mathcal{P}(\vec{r}) = \eta\triangle\vec{\mathsf{v}}(\vec{r}) \quad\Leftrightarrow\quad \begin{cases} \dfrac{\partial\mathcal{P}(\vec{r})}{\partial x} = \dfrac{\partial\mathcal{P}(\vec{r})}{\partial y} = 0 \\[2mm] \dfrac{\partial\mathcal{P}(\vec{r})}{\partial z} = \eta\left[\dfrac{\partial^2\mathsf{v}(r)}{\partial x^2} + \dfrac{\partial^2\mathsf{v}(r))}{\partial y^2}\right] = \eta\left[\dfrac{\mathrm{d}^2\mathsf{v}(r)}{\mathrm{d}r^2} + \dfrac{1}{r}\dfrac{\mathrm{d}\mathsf{v}(r)}{\mathrm{d}r}\right]. \end{cases} \tag{V.6}$$

(ad) G. Hagen, 1797–1884

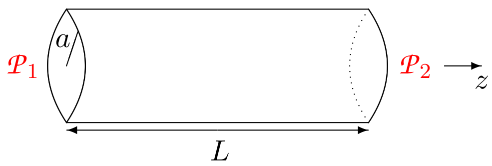

Figure V.3 – Setup of the Hagen–Poiseuille flow.

The right member of the equation in the second line is independent of z, implying that the pressure gradient along the z direction is constant. Using the boundary conditions yields

$$\frac{\partial\mathcal{P}(\vec{r})}{\partial z} = -\frac{\delta\mathcal{P}}{L},$$

with $\delta\mathcal{P} \equiv \mathcal{P}_1 - \mathcal{P}_2$. The z component of the Navier–Stokes equation (V.6) thus becomes

$$\frac{\mathrm{d}^2\mathsf{v}(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\mathsf{v}}{\mathrm{d}r} = -\frac{\delta\mathcal{P}}{\eta L}. \tag{V.7}$$

As always, this linear differential equation is solved in two successive steps, starting with the associated homogeneous equation. To find the general solution of the latter, one may introduce $\chi(r) \equiv \mathrm{d}\mathsf{v}(r)/\mathrm{d}r$, which obeys the simpler equation

$$\frac{\mathrm{d}\chi(r)}{\mathrm{d}r} + \frac{\chi(r)}{r} = 0.$$

The generic solution is $\ln\chi(r) = -\ln r + \text{const.}$, i.e. $\chi(r) = A/r$ with A a constant. This then leads to $\mathsf{v}(r) = A\ln r + B$ with B an additional constant.

A particular solution of the inhomogeneous equation (V.7) is $\mathsf{v}(r) = Cr^2$ with $C = -\delta\mathcal{P}/4\eta L$. The general solution of Eq. (V.7) is then given by

$$\mathsf{v}(r) = A\ln r + B - \frac{\delta\mathcal{P}}{4\eta L}r^2,$$

where the two integration constants still need to be determined.

To have a regular flow velocity at $r = 0$, the constant A should vanish. In turn, the boundary condition at the tube wall, $\mathsf{v}(r\!=\!a) = 0$, determines the value of the constant $B = (\delta\mathcal{P}/4\eta L)a^2$. All in all, the velocity profile thus reads

$$\mathsf{v}(r) = \frac{\delta\mathcal{P}}{4\eta L}\left(a^2 - r^2\right) \quad \text{for } r \leq a. \tag{V.8}$$

This is again parabolic, with $\vec{\mathsf{v}}$ pointing in the same direction as the pressure drop.

The mass flow rate across the tube cross section follows from a straightforward integration:

$$Q = \int_0^a \rho\mathsf{v}(r)\,2\pi r\,\mathrm{d}r = 2\pi\rho\frac{\delta\mathcal{P}}{4\eta L}\int_0^a \left(a^2 r - r^3\right)\mathrm{d}r = 2\pi\rho\frac{\delta\mathcal{P}}{4\eta L}\frac{a^4}{4} = \frac{\pi\rho a^4}{8\eta}\frac{\delta\mathcal{P}}{L}. \tag{V.9}$$

This result, known as *Hagen–Poiseuille law* (or equation), shows that the mass flow rate is proportional to the pressure drop per unit length.

Remarks:

∗ The Hagen–Poiseuille law only holds under the assumption that the flow velocity vanishes at the tube walls. The experimental confirmation of the law—which was actually deduced from experiment by Hagen (1839) and Poiseuille (1840)—is thus a proof of the validity of the no-slip assumption for the boundary condition.

∗ The mass flow rate across the tube cross section may be used to define the average flow velocity such that $Q = \pi a^2\rho\langle\mathsf{v}\rangle$ with

$$\langle\mathsf{v}\rangle \equiv \frac{1}{\pi a^2}\int_0^a \mathsf{v}(r)\,2\pi r\,\mathrm{d}r = \frac{1}{2}\mathsf{v}(r\!=\!0).$$

The Hagen–Poiseuille law then expresses a proportionality between the pressure drop per unit length and $\langle \mathsf{v} \rangle$ in a laminar flow.

> Viewing $\delta\mathcal{P}/L$ as the "generalized force" driving the motion, the corresponding "response" $\langle \mathsf{v} \rangle$ of the fluid is thus linear.

The relation is quite different in the case of a *turbulent* flow with the same geometry: for instance, measurements by Reynolds [21] gave $\delta\mathcal{P}/L \propto \langle \mathsf{v} \rangle^{1.722}$.

V.2 Dynamic similarity

The incompressible motion of a Newtonian fluid is governed by the kinetic condition $\vec{\nabla} \cdot \vec{\mathsf{v}}(t, \vec{r}) = 0$, the continuity equation (III.9), and the incompressible Navier–Stokes equation (III.32). In order to determine the relative influence of the various terms of the latter, it is often convenient to consider dimensionless forms of the equation, which leads to the introduction of a variety of dimensionless numbers.

For instance, the influence of the fluid mass density ρ and shear viscosity η, which are uniform throughout the fluid, on a flow in the absence of volume forces is entirely encoded in the Reynolds number (§ V.2.1). Allowing for volume forces, either due to gravity or to inertial forces, their relative importance is controlled by similar dimensionless parameters (§ V.2.2).

Let L_c resp. v_c be a characteristic length resp. velocity for a given flow. Since the Navier–Stokes equation itself does not involve any parameter with the dimension of a length or a velocity, both scales are controlled by "geometry", i.e. by the boundary conditions for the specific problem under consideration. Thus, L_c may be the size (diameter, side length) of a tube in which the fluid flows or of an obstacle around which the fluid moves. In turn, v_c may be the uniform velocity far from such an obstacle.

With the help of L_c and v_c, one may rescale the physical quantities in the problem, so as to obtain dimensionless quantities, which will hereafter be denoted with *:

$$\vec{r}^* \equiv \frac{\vec{r}}{L_c}, \quad \vec{\mathsf{v}}^* \equiv \frac{\vec{\mathsf{v}}}{\mathsf{v}_c}, \quad t^* \equiv \frac{t}{L_c/\mathsf{v}_c}, \quad \mathcal{P}^* \equiv \frac{\mathcal{P} - \mathcal{P}_0}{\rho \mathsf{v}_c^2}, \tag{V.10}$$

where $\mathcal{P}_0$ is some characteristic value of the (unscaled) pressure.

V.2.1 Reynolds number

Consider first the incompressible Navier–Stokes equation in the absence of external volume forces. Rewriting it in terms of the dimensionless variables and fields (V.10) yields

$$\frac{\partial \vec{\mathsf{v}}^*(t^*, \vec{r}^*)}{\partial t^*} + \big[\vec{\mathsf{v}}^*(t^*, \vec{r}^*) \cdot \vec{\nabla}^*\big]\vec{\mathsf{v}}^*(t^*, \vec{r}^*) = -\vec{\nabla}^*\mathcal{P}^*(t^*, \vec{r}^*) + \frac{\eta}{\rho \mathsf{v}_c L_c} \triangle^* \vec{\mathsf{v}}^*(t^*, \vec{r}^*), \tag{V.11}$$

with $\vec{\nabla}^*$ resp. $\triangle^*$ the gradient resp. Laplacian with respect to the reduced position variable $\vec{r}^*$. Besides the reduced variables and fields, this equation involves a single dimensionless parameter, the inverse of the *Reynolds number*

$$\boxed{\mathrm{Re} \equiv \frac{\rho \mathsf{v}_c L_c}{\eta} = \frac{\mathsf{v}_c L_c}{\nu}.} \tag{V.12}$$

This number measures the relative importance of inertia and viscous friction forces on a fluid element or a body immersed in the moving fluid: at large resp. small Re, viscous effects are negligible resp. predominant.

Remark: As stated above Eq. (V.10), both L_c and v_c are controlled by the geometry and boundary conditions. The Reynolds number—and every similar dimensionless we shall introduce hereafter—is thus a characteristic of a given flow, not of the fluid.

Law of similitude[li]

The solutions for the dynamical fields $\vec{\mathsf{v}}^*$, $\mathcal{P}^*$ at fixed boundary conditions and geometry—specified in terms of dimensionless ratios of geometrical lengths—are functions of the independent variables t^*, $\vec{r}^*$, and of the Reynolds number:

$$\vec{\mathsf{v}}^*(t^*,\vec{r}^*) = \vec{\mathsf{f}}_1^*(t^*,\vec{r}^*,\mathrm{Re}), \qquad \mathcal{P}^*(t^*,\vec{r}^*) = \mathsf{f}_2^*(t^*,\vec{r}^*,\mathrm{Re}), \tag{V.13}$$

with $\vec{\mathsf{f}}_1^*$ resp. f_2^* a vector resp. scalar function. The "physical" flow velocity and pressure fields are then given by

$$\vec{\mathsf{v}}(t,\vec{r}) = \mathsf{v}_c\vec{\mathsf{f}}_1^*\bigg(\frac{\mathsf{v}_c t}{L_c},\frac{\vec{r}}{L_c},\mathrm{Re}\bigg), \qquad \mathcal{P}(t,\vec{r}) = \mathcal{P}_0 + \rho\mathsf{v}_c^2\mathsf{f}_2^*\bigg(\frac{\mathsf{v}_c t}{L_c},\frac{\vec{r}}{L_c},\mathrm{Re}\bigg).$$

These equations underlie the use of fluid dynamical simulations with experimental models at a reduced scale, yet possessing the same (rescaled) geometry. Let L_c, v_c resp. L_{M}, v_{M} be the characteristic lengths and velocities of the real-size flow resp. of the reduced-scale experimental flow; for simplicity, we assume that the same fluid is used in both cases. If $\mathsf{v}_{\mathrm{M}}/\mathsf{v}_c = L_c/L_{\mathrm{M}}$, the Reynolds number for the experimental model is the same as for the real-size fluid motion: both flows then admit the same solutions $\vec{\mathsf{v}}^*$ and $\mathcal{P}^*$, and are said to be *dynamically similar*.

Remark: The functional relationships between the "dependent variables" $\vec{\mathsf{v}}^*$, $\mathcal{P}^*$ and the "independent variables" t^*, $\vec{r}^*$ and a dimensionless parameter (Re) represent a simple example of the more general (Vaschy[ae]–)Buckingham[af] π-theorem [22] in *dimensional analysis*, see Appendix D.

V.2.2 Other dimensionless numbers

If the fluid motion is likely to be influenced by gravity, the corresponding volume force density $\vec{f}_V = -\rho\vec{g}$ must be taken into account in the right member of the incompressible Navier–Stokes equation (III.32). Accordingly, if the latter is written in dimensionless form as in the previous paragraph, there is an additional term on the right hand side of Eq. (V.11), proportional to $1/\mathrm{Fr}^2$, with

$$\mathrm{Fr} \equiv \frac{\mathsf{v}_c}{\sqrt{gL_c}} \tag{V.14}$$

the *Froude number*.[ag] This dimensionless parameter measures the relative size of inertial and gravitational effects in the flow, the latter being important when Fr is small.

In the presence of gravity, the dimensionless dynamical fields $\vec{\mathsf{v}}^*$, $\mathcal{P}^*$ become functions of the reduced variables t^*, $\vec{r}^*$ controlled by both parameters Re and Fr.

The Navier–Stokes equation (III.31) holds in an inertial frame. In a non-inertial reference frame, there come additional terms, which may be expressed as fictive force densities on the right hand side, and come in addition to the "physical" volume force density $\vec{f}_V$. In the case of a reference frame in uniform rotation (with respect to an inertial frame) with angular velocity $\vec{\Omega}_0$, there are thus two extra contributions corresponding to centrifugal and Coriolis forces, namely $\vec{f}_{\mathrm{cent.}} = -\rho\vec{\nabla}\big[-\frac{1}{2}\big(\vec{\Omega}_0\times\vec{r}\big)^2\big]$ and $\vec{f}_{\mathrm{Cor.}} = -2\rho\vec{\Omega}_0\times\vec{\mathsf{v}}$, respectively.

The relative importance of the latter in a given flow can be estimated with dimensionless numbers. Thus, denoting $\Omega_0 \equiv |\vec{\Omega}_0|$, the *Ekman number*[ah]

$$\mathrm{Ek} \equiv \frac{\eta}{\rho\Omega_0 L_c^2} = \frac{\nu}{\Omega_0 L_c^2} \tag{V.15}$$

measures the relative size of (shear) viscous and Coriolis forces, with the latter predominating over the former when $\mathrm{Ek} \ll 1$.

[li] *Ähnlichkeitsgesetz*

[ae] A. Vaschy, 1857–1899 [af] E. Buckingham, 1867–1940 [ag] W. Froude, 1810–1879 [ah] V. Ekman, 1874–1954

One may also wish to compare the influences of the convective and Coriolis terms in the Navier–Stokes equation. This is done with the help of the *Rossby number*[(ai)]

$$\mathrm{Ro} \equiv \frac{\mathsf{v}_c}{\Omega_0 L_c} \tag{V.16}$$

which is small when the effect of the Coriolis force is the dominant one.

Remark: Quite obviously, the Reynolds (V.12), Ekman (V.15), and Rossby (V.16) numbers obey the simple identity

$$\mathrm{Ro} = \mathrm{Re} \cdot \mathrm{Ek}.$$

[(ai)] C.-G. Rossby, 1898–1957