4.3 Time evolution of neutron concentration

$$\frac{dX_n}{dt} = -\underbrace{(\lambda_{n\nu} + \lambda_{ne})}_{=:\,\lambda_{n\to p}} X_n + \underbrace{(\lambda_{pe} + \lambda_{p\bar{\nu}})}_{=:\,\lambda_{p\to n}} X_p$$

$$= -\lambda_{n\to p} \left(X_n - e^{-\Delta m/T} X_p \right)$$

$$X_p + X_n = 1$$

$$X_n - e^{-\Delta m/T} X_p = X_n - e^{-\Delta m/T} (1 - X_n)$$

$$= X_n (1 + e^{-\Delta m/T}) - e^{-\Delta m/T}$$

$$= (1 + e^{-\Delta m/T}) \left(X_n - \frac{1}{e^{\Delta m/T} + 1} \right)$$

in thermal equilibrium $X_n = X_n^{eq}$ with

$$X_n^{eq} \simeq \frac{e^{-m_n/T}}{e^{-m_p/T} + e^{-m_n/T}} = \frac{1}{1 + e^{\Delta m/T}}$$

$$\frac{dX_n}{dt} = -\lambda_{n\to p}\left(1 + e^{-\Delta m/T}\right)\left(X_n - X_n^{eq}\right)$$

homogeneous eq is solved by

$$X_n(t) = C \exp\left(-\int_0^t dt'\, \lambda_{n\to p}(t')\left[1 + e^{-\Delta m/T(t')}\right]\right)$$

variation constant C:

$$\dot{C} \exp\left(-\int_0^t dt'\, \lambda_{n\to p}(t')\left[1 + e^{-\Delta m/T}\right]\right) = \lambda_{n\to p}\left(1 + e^{-\Delta m/T}\right) X_n^{eq}$$

with $T' \equiv T(t')$, ...

$$C(t) = \int_0^t dt' \exp\left(\int_0^{t'} dt''\, \lambda''_{n\to p}\left[1 + e^{-\Delta m/T''}\right]\right) \lambda'_{n\to p}\left(1 + e^{-\Delta m/T'}\right) X_n^{eq\,\prime}$$

general solution (with $C = \text{const}$):

$$X_n(t) = \exp\left(-\int_0^t dt'\, \lambda_{n\to p}(t')\left[1 + e^{-\Delta m/T'}\right]\right)$$

$$\left\{C + \int_0^t dt' \underbrace{\exp\left(\int_0^{t'} dt''\, \lambda''_{n\to p}\left[1 + e^{-\Delta m/T''}\right]\right) \lambda'_{n\to p}\left(1 + e^{-\Delta m/T'}\right)}_{= \frac{d}{dt'} \exp\left(\int_0^{t'} dt''\, \lambda''_{n\to p}\left[1 + e^{-\Delta m/T''}\right]\right)} X_n^{eq\,\prime}\right\}$$

now integrate by parts

$$\{\dots\} = C + \exp\left(\int_0^t dt'' \, \lambda''_{n\to p} \left[1+e^{-\Delta m/T''}\right]\right) X_n^{eq}(t) - X_n^{eq}(0)$$

$$- \int_0^t dt' \exp\left(\int_0^{t'} dt'' \, \lambda''_{n\to p} \left[1+e^{-\Delta m/T''}\right]\right) \dot{X}_n^{eq\,\prime}$$

initial cond: $\quad X_n(0) = X_n^{eq}(0) \quad \Rightarrow \quad C = X_n^{eq}(0)$

$$(*) \quad \boxed{X_n(t) = X_n^{eq}(t) - \int_0^t dt' \exp\left(-\int_{t'}^t dt'' \, \lambda_{n\to p}(t'') \left[1+e^{-\frac{\Delta m}{T(t'')}}\right]\right) \dot{X}_n^{eq}(t')}$$

$\dot{X}_n^{eq}$ is due to the expansion and drives X_n out of equilibrium.

$\dot{X}_n^{eq} < 0 \quad \Rightarrow \quad$ the second term is positive.

For large t: $\quad X_n^{eq} \to 0$, while the integral has a finite limit $\quad X_n^* = \lim_{t\to\infty} X_n(t)$

X_n freezes out.

One can obtain an expansion in powers of derivatives of equilibrium quantities by additional partial integrations:

$$\exp\left(-\int_{t'}^t dt'' \, \lambda_{n\to p}(t'') \left[1+e^{-\frac{\Delta m}{T(t'')}}\right]\right)$$

$$= \frac{1}{\lambda_{n\to p}(t')\left[1+e^{-\Delta m/T(t')}\right]} \frac{d}{dt'} \exp\left(-\int_{t'}^t dt'' \, \lambda_{n\to p}(t'') \left[1+e^{-\frac{\Delta m}{T(t'')}}\right]\right)$$

$$(*) \quad X_n(t) = X_n^{eq}(t) \left\{ 1 - \frac{1}{\lambda_{n\to p}\left[1+e^{-\Delta m/T}\right]} \frac{\dot{X}_n^{eq}}{X_n^{eq}} + \dots \right\}$$

The second term is small when $\frac{\dot{X}_n^{eq}}{X_n^{eq}} \ll \lambda_{n\to p}$, that is, when the reaction rate is $\gg$ expansion rate.

The expansion breaks down at freeze-out. This happens when the second term in $\{\dots\}$ becomes of order 1, i.e., when the deviations from equilibrium become significant.

It turns out (see below) that this happens at $T < \Delta m$ and before e^+e^- annihilation, i.e. when T_ν still equals T.

In $(*)$ we can therefore neglect $e^{-\Delta m/T}$ and use $\lambda_{ne} = \lambda_{n\nu}$ so that $\lambda_{n\to p} \simeq 2\lambda_{n\nu}$

Thus freezeout happens when $-\frac{\dot{X}_n^{eq}}{X_n^{eq}} \simeq 2\lambda_{n\nu}$

$$X_n^{eq} = \frac{1}{1+e^{\Delta m/T}}$$

Exercise: time-temperature relation:

$$t/s \doteq 1.39 \left(\frac{\pi^2}{30} g_*\right)^{-1/2} (T/MeV)^{-2}$$

$$\dot{X}_n = \frac{dX_n}{dT}\frac{dT}{dt}$$

$$\frac{dt}{dT} \doteq 1.39 \left(\frac{\pi^2}{30} g_*\right)^{-1/2} (-2)\, T^{-3}\ MeV^2\, s$$

$$\frac{dX_n}{dT} = \frac{\frac{\Delta m}{T^2}\, e^{\Delta m/T}}{\left(1+e^{\Delta m/T}\right)^2}$$

$$-\frac{\dot{X}_n^{eq}}{X_n^{eq}} = \frac{1}{1+e^{\Delta m/T}} \; \frac{\Delta m}{T^2}\, e^{\Delta m/T} \; \frac{1}{2.78} \left(\frac{\pi^2}{30} g_*\right)^{1/2} T^3 \; \frac{1}{MeV^2 s}$$

$$= \underbrace{0.360 \left(\frac{\pi^2}{30} g_*\right)^{1/2}}_{\simeq 0.207 \sqrt{g_*}} \; \frac{\Delta m\, T}{\underbrace{e^{-\Delta m/T}+1}_{\simeq 1}} \; \frac{1}{MeV^2 s}$$

We had for $T_\nu = T$

$$\lambda_{n\nu} \simeq 1.61 \left(\frac{T}{\Delta m}\right)^3 \left(\frac{T}{\Delta m} + 0.24\right)^2 s^{-1}$$

$$-\frac{\dot{X}_n^{eq}}{X_n^{eq}} = 2\lambda_{n\nu} \iff$$

$$\boxed{\left(\frac{T_*}{\Delta m}\right)^2 \left(\frac{T_*}{\Delta m} + 0.24\right)^2 = 0.112 \left(\frac{\pi^2}{30} g_*\right)^{1/2} \left(\frac{\Delta m}{MeV}\right)^2 = 0.187 \left(\frac{\pi^2}{30} g_*\right)^{1/2}}$$

with $g_* = 10.8$: $\quad T_* = 0.84\ MeV$

$$X_n^{eq}(T_*) = 0.18$$

This determination of T_* is rough because we used the expansion (*) at a temperature where (*) already breaks down.

What one sees is that the neutron freeze-out concentration depends on g_*, i.e. on the number of relativistic species that are in thermal equilibrium. This gives strong restrictions for hypothetical light particles, requiring that they do not significantly change $X_n(T_*) =: X_n^*$

Increasing g_* increases T_* $\rightsquigarrow$ earlier freeze out, more neutrons survive.

Neutrons and protons fuse to ^4He.

More light particle species in equilibrium $\rightarrow$ larger ^4He abundance

The freeze-out concentration can be determined from the integral (*). For the SM (3 ν species):

$$X_n^* = 0.158$$

with one additional light neutrino flavor: $X_n^* = 0.163$

which is an 0.5% increase $\rightarrow$

$$\boxed{X_n^* = 0.158 + 0.005(N_\nu - 3)}$$

Neutron decay

so far neglected: $n \to p\, e^- \bar{\nu}$

lifetime $\tau_n \simeq 886\,s$ which is large compared to the freeze-out time $t_* \sim 1s$.

After freeze-out
$$X_n(t) = X_n^* e^{-t/\tau_n}$$

What we have discussed now is called chemical freeze-out of neutrons, which is the deviation of n_n from thermal equilibrium, or deviation from chemical equilibrium.

Neutrons still scatter frequently via nuclear, or strong interaction which are much stronger than the weak interactions mediated by W and Z bosons. The neutrons still remain in kinetic equilibrium, described by a Boltzmann distribution. Now the chemical potential μ_n is no longer equal to μ_B.