4.2 BE for neutron freeze-out [Mukhanov 3.5.1]

main reactions keeping n's in equilibrium:

$$n\,\nu \leftrightarrow p\,e^- \qquad\qquad n\,e^+ \leftrightarrow p\,\bar\nu \qquad\qquad (\nu = \nu_e)$$

Boltzmann eq: $\dot f_n - H |\vec p_n| \frac{\partial f_n}{\partial |\vec p|} = C$

The contribution of $n\nu \to pe^-$ to the loss term:

$$C_{loss} = -\frac{1}{2E_n}\int \prod_{j=\nu}^{e^-}\left[\frac{d^3p_j}{(2\pi)^3\, 2E_j}\right] f_n f_\nu |\mathcal{M}|^2 (2\pi)^4\, \delta(p_n + p_\nu - p_p - p_{e^-})$$

$\mathcal{M}$: invariant matrix element

$$f_\nu(E_\nu) = \frac{1}{e^{E_\nu/T_\nu}+1}$$

at $T \sim 1\,\mathrm{MeV}$ nucleons are non-relativistic

$E_n \simeq m_n \;, \quad E_p \simeq m_p$

phase space density of e^-: $f_{e^-} = f_F(E_e) = \frac{1}{e^{E_e/T}-1} \sim 1$

$\Rightarrow$ Pauli blocking gives an extra factor $[1 - f_F(E_e)]$

Integrate Boltzmann eq. over $\vec p_n$:

$$\int \frac{d^3p_n}{(2\pi)^3}\, \partial_t f_n = \dot n_n$$

$$\int \frac{d^3 p_n}{(2\pi)^3\, 2E_n} f_n = \frac{n_n}{2 m_n}$$

$$\frac{\vec{p}_n^{\,2}}{2m_n} \sim T \quad \Rightarrow \quad |\vec{p}_n| \sim \sqrt{m_n T} \gg p_{e^-}, p_\nu \quad \Rightarrow$$

momentum conservation gives $\vec{p}_n \simeq \vec{p}_p$

$$\int \frac{d^3 p_p}{(2\pi)^3\, 2E_p} (2\pi)^3 \delta(\vec{p}_n - \vec{p}_p) = \frac{1}{2 m_p}$$

energy conservation: $m_n + \underset{\sim}{\frac{\vec{p}_n^{\,2}}{2m_n}} + E_\nu = m_p + \underset{\sim}{\frac{\vec{p}_p^{\,2}}{2m_p}} + E_{e^-}$

$$\int \frac{d^3 p_\nu}{(2\pi)^3\, 2E_\nu} f_\nu \, (2\pi)\, \delta(E_\nu - \Delta m - E_e)$$

$$= \frac{4\pi}{(2\pi)^2\, 2} \int_0^\infty dE_\nu\, E_\nu^2 \frac{1}{E_\nu} f_\nu(E_\nu)\, \delta(E_\nu + \Delta m - E_e)$$

$$= \frac{1}{2\pi} E_\nu f_\nu(E_\nu)\, \Theta(E_e - \Delta m)$$

Heaviside step function $\Theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$

with $E_\nu = E_e - \Delta m$

In the p_e-integral substitute

$$p_e = \sqrt{E_e^2 - m_e^2}, \quad dp_e = \frac{E_e}{p_e} dE_e$$

$$d^3 p_e = d\Omega_e\, dE_e \frac{E_e}{p_e} p_e^2 = d\Omega_e\, dE_e\, E_e\, p_e$$

$$\int \frac{d^3 p_e}{(2\pi)^3 2E_e} \frac{1}{2\pi} E_\nu f_\nu(E_\nu)\, \Theta(E_e - \Delta m)$$

$$= \frac{4\pi}{16\pi^4} \frac{1}{2} \int_{\Delta m}^{\infty} dE_e\, E_e\, p_e\, \frac{1}{E_e}\, E_\nu f_\nu(E_\nu)$$

because $\Delta m > m_e$

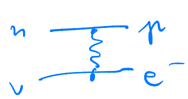
$$= \frac{1}{8\pi^3} \int_{\Delta m}^{\infty} dE_e\, p_e\, E_\nu f_\nu(E_\nu)$$

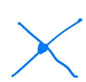
$n_p + n_n$ is conserved

$X_n := \frac{n_n}{n_p + n_n}$ w/o collision term $\dot{X}_n = 0 \Rightarrow$

$$\dot{X}_n = \frac{1}{n_p + n_n} \int \frac{d^3 p_n}{(2\pi)^3} C_{loss} = -\lambda_{n\nu} X_n$$

$$\lambda_{n\nu} = \underbrace{\frac{1}{4 m_n m_p} \frac{1}{8\pi^3}}_{\simeq \frac{1}{32\pi^3} \frac{1}{m_p^2}} \int_{\Delta m}^{\infty} dE_e\, p_e\, E_\nu f_\nu(E_\nu)\, |\mathcal{M}|^2 [1 - f_e(E_e)]$$

$\mathcal{M}$ = 

$\simeq$  4-fermion interaction

$|p_i| \ll m_W$

$$|\mathcal{M}|^2 = 16(1 + 3g_A^2)\, G_F^2\, p_n \cdot p_\nu\, p_p \cdot p_e$$

$G_F = \frac{\pi \alpha_W}{\sqrt{2}\, m_W^2} \simeq 1.17 \times 10^{-5}\ \text{GeV}^{-2}$ Fermi coupling

$g_A \simeq 1.26$

$$p_1 \cdot p_2 = E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2 \quad , \quad E_i = \sqrt{\vec{p}_i^{\,2} + m_i^2}$$

$$p_n \cdot p_\nu \simeq m_n E_\nu \quad , \quad p_p \cdot p_{e^-} \simeq m_p E_{e^-}$$

$$\lambda_{n\nu} = \frac{1}{32\pi^3} \frac{1}{m_p^2} \int_{\Delta m}^{\infty} dE_e \, p_e \, E_\nu f_\nu(E_\nu) [1 - f_e(E_e)]$$

$$16 (1 + 3 g_A^2) G_F^2 m_p^2 \qquad E_\nu E_e$$

$$= \frac{1}{2\pi^3} (1 + 3 g_A^2) G_F^2 \int_{\Delta m}^{\infty} dE_e \, E_e \, p_e \, E_\nu^2 f_\nu(E_\nu) [1 - f_e(E_e)]$$

with $p_e = \sqrt{E_e^2 - m_e^2}$, $E_\nu = E_e - \Delta m$

$$1 - f_e(E_e) = \frac{e^{E_e/T} + 1 - 1}{e^{E_e/T} + 1} = \frac{1}{1 + e^{-E_e/T}}$$

The integral depends on T, $m_e = 0.511 \text{ MeV}$

$m_n = 939.565 \text{ MeV}$ $\quad m_p = 938.272 \text{ MeV}$

$\Delta m = 1.293 \text{ MeV}$

m_e only enters quadratically.

$\left(\frac{m_e}{\Delta m}\right)^2 \simeq (0.395) \simeq 0.156$ is a small parameter

Try expansion in this parameter

$$q := \frac{E_e}{\Delta m}$$

$$\lambda_{n\nu} = \frac{1 + 3 g_A^2}{2\pi^3} G_F^2 (\Delta m)^5 \, I$$

$$I = \int_1^\infty dq \left[1 - \frac{m_e^2}{\Delta m^2}\frac{1}{q^2}\right]^{1/2} \frac{q^2\,(q-1)^2}{[1+\exp(\Delta m(q-1)/T_\nu)][1+e^{-q\Delta m/T}]}$$

neglect Pauli blocking , $q \to q+1$

$$I = \int_0^\infty dq \left[1 - \frac{m_e^2}{\Delta m^2}\frac{1}{(q+1)^2}\right]^{1/2} \frac{(q+1)^2\,q^2}{1+\exp(q\,\Delta m/T_\nu)}$$

$$= \int_0^\infty dq \frac{q^4 + 2q^3 + q^2\left(1 - \frac{1}{2}\frac{m_e^2}{\Delta m^2}\right)}{1+\exp(q\,\Delta m/T_\nu)} + \mathcal{O}\left(\left(\frac{m_e}{\Delta m}\right)^4\right)$$

$$\simeq \frac{45\,\zeta(5)}{2}\left(\frac{T_\nu}{\Delta m}\right)^5 + \frac{7\pi^4}{60}\left(\frac{T_\nu}{\Delta m}\right)^4 + \frac{3\zeta(3)}{2}\left(1 - \frac{m_e^2}{2\Delta m^2}\right)\left(\frac{T_\nu}{\Delta m}\right)^3$$

accuracy $\lesssim 2\%$ in the relevant temperature range

$\zeta(5) = 1.037$ $\qquad \zeta(3) = 1.202$

$$I \simeq \left(\frac{T_\nu}{\Delta m}\right)^3 \left\{ 23.33 \left(\frac{T_\nu}{\Delta m}\right)^2 + 11.36\,\frac{T_\nu}{\Delta m} + 1.803 \right\}$$

$$= 23.33 \left(\frac{T_\nu}{\Delta m}\right)^3 \left\{ \left(\frac{T_\nu}{\Delta m}\right)^2 + 0.4869\,\frac{T_\nu}{\Delta m} + 0.0773 \right\}$$

$$\simeq 23.33 \left(\frac{T_\nu}{\Delta m}\right)^3 \left(\frac{T_\nu}{\Delta m} + 0.24\right)^2$$

$$\lambda_{n\nu} = \frac{1+3g_A^2}{2\pi^3} G_F^2 (\Delta m)^5\, I = \frac{1+3(1.26)^2}{2\pi^3}\left(1.17\times10^{-5}\,\mathrm{GeV}^{-2}\right)^2 (1.29\,\mathrm{MeV})^5$$

$$\times\, 23.33 \left(\frac{T_\nu}{\Delta m}\right)^3 \left(\frac{T_\nu}{\Delta m} + 0.24\right)^2$$

$$= \underbrace{1.06\times10^{-9}\;10^{-12}\,\mathrm{MeV}^{-4}\,\mathrm{MeV}^5}_{=\,1.06\times10^{-21}\,\mathrm{MeV}/\hbar} \left(\frac{T_\nu}{\Delta m}\right)^3 \left(\frac{T_\nu}{\Delta m} + 0.24\right)^2$$

$\hbar = 6.58\times10^{-22}\,\mathrm{MeV}\times s$

$$\boxed{\lambda_{n\nu} \simeq 1.61 \left(\frac{T_\nu}{\Delta m}\right)^3 \left(\frac{T_\nu}{\Delta m} + 0.24\right)^2 s^{-1}}$$

without calculation:

rate for $n e^+ \to p \bar{\nu}$ $\quad \lambda_{ne} \simeq \lambda_{n\nu}$ for $T = T_\nu$, $T > m_e$

Contributions to the gain term:

$$p e^- \to n \nu \quad , \quad p \bar{\nu} \to n e^+ \quad \text{for } T_\nu = T$$

$$\lambda_{pe} \simeq e^{-\Delta m / T} \lambda_{n\nu} , \qquad \lambda_{p\bar{\nu}} = e^{-\Delta m / T} \lambda_{ne}$$