2.2 Equilibrium thermodynamics [Gorbunov 5.1]

statistical mechanics: system in thermal equilibrium

is described by the grand-canonical potential $\Omega(T,\mu)$

where μ is the set of chemical potentials for all

conserved charges, e.g. $\mu = (\mu_Q, \mu_B, \mu_L)$.

$$e^{-\beta\Omega} = \mathrm{tr}\left\{\exp\left(-\beta\left[H - \sum_i \mu_i Q_i\right]\right)\right\}$$

where $\beta \equiv \frac{1}{T}$

$$d\Omega = -S\,dT - P\,dV - \sum_\alpha Q_\alpha\, d\mu_\alpha \quad \text{or}$$

entropy $S = -\frac{\partial\Omega}{\partial T}$, $\quad P = -\frac{\partial\Omega}{\partial V}$, $\quad Q_\alpha = -\frac{\partial\Omega}{\partial\mu_\alpha}$

Duhem-Gibbs relation: $\Omega = -PV$

when interactions are weak, one can approximate the system as an ideal gas, i.e. a gas of non-interacting particles.

The phase space of particle species i is

$$f_i(\vec{p}) = \frac{1}{e^{\beta(E_i(\vec{p})-\mu_i)} \mp 1} \quad \text{for} \quad \begin{matrix}\text{bosons}\\ \text{fermions}\end{matrix}$$

$$E_i(\vec{p}) = \sqrt{\vec{p}^{\,2} + m_i^2}$$

μ_i chemical potential of particle species i:

$$\mu_i = \sum_\alpha \mu_\alpha Q_{\alpha i}$$

$Q_{\alpha i}$: charge Q_α carried by particle species i

example: if B and L are conserved

$$\mu_{e^-} = -\mu_Q + \mu_L \quad , \quad \mu_p = \mu_Q + \mu_B$$

$$\mu_u = \frac{2}{3}\mu_Q + \frac{1}{3}\mu_B \quad , \quad \mu_{\bar{d}} = \frac{1}{3}\mu_Q - \frac{1}{3}\mu_B$$

chemical potentials of particles and antiparticles have opposite signs.

N.B. In the literature chemical potentials are oftentimes introduced 'backwards'. First μ_i are defined, and then reactions are listed which give relations between the μ_i, like, e.g. $\mu_{e^-} = \mu_{\mu^-}$ or $\mu_\gamma = 0$

I think this is unnecessarily complicated.

Back to the phase-space densities: For $E_a \gg T$

$$f_i \simeq e^{-\beta(E_i - \mu_i)} \qquad \text{Boltzmann distribution}$$

non-relativistic (NR) limit

$$E_i \simeq m_i + \frac{\vec{p}^{\,2}}{2m_i}$$

$$f_i(\vec{p}) = \frac{1}{e^{\beta(\varepsilon_i(\vec{p}) - M_i)} \mp 1} \qquad \text{with}$$

$$\varepsilon_i(\vec{p}) = \frac{\vec{p}^{\,2}}{2m_i} \, ,$$

$M_i = \mu_i - m_i$ which is the chemical potential normally used in NR physics particle number density

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i(\vec{p})$$

g_i: number of spin (and color for quarks & gluons) states of particle species i

example: $g_{e^-} = 2$, $g_\gamma = 2$, $g_\pi = 1$

When $\mu_i \neq 0$ then $n_i \neq n_{\bar{i}}$. We will see that for $T \gtrsim 1\,\text{GeV}$, the ratio $\frac{n_q - n_{\bar{q}}}{n_q + n_{\bar{q}}}$ is very small of order 10^{-10}.

contribution to energy-momentum tensor from species i

$$T_i^{\mu\nu} = g_i \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu p^\nu}{E_i} \qquad \text{(see sect. 1.4)}$$

$$T^{\mu\nu} = \sum_i T_i^{\mu\nu}$$

energy density in particle species i

$$\rho_i = T_i^{00} = g_i \int \frac{d^3p}{(2\pi)^3} f_i(\vec{p})\, E_i(\vec{p})$$

pressure $\quad T_i^{mn} = \delta^{mn} P_i \Rightarrow P_i = \frac{1}{3} T_i^{mm}$

$$P_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i(\vec{p}) \frac{\vec{p}^{\,2}}{3 E_i(\vec{p})}$$

ultra-relativistic limit $T \gg m_i$ and small $\mu_i \ll T$

$$\rho_i = g_i \int \frac{d^3p}{(2\pi)^3} \frac{|\vec{p}\,|}{e^{\beta|\vec{p}\,|} \mp 1} = g_i \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^3}{e^{\beta p} \mp 1}$$

$$= \begin{cases} g_i \frac{\pi^2}{30} T^4 & \text{bosons} \\ \frac{7}{8} g_i \frac{\pi^2}{30} T^4 & \text{fermions} \end{cases}$$

For several ultra-relativistic species with small μ_i and the same temperature T:

$$\rho = g_* \frac{\pi^2}{30} T^4$$

with

$$g_* := \sum_{\substack{\text{bosons} \\ \text{with } m_i, \mu_i \ll T}} g_i + \frac{7}{8} \sum_{\substack{\text{fermions} \\ \text{with} \\ m_i, \mu_i \ll T}} g_i$$

called effective number of degrees of freedom.

number densities:

$$n_i = \frac{g_i}{2\pi^2} \int_0^\infty dp \frac{p^2}{e^{\beta p} \mp 1} = \begin{cases} g_i \frac{\zeta(3)}{\pi^2} T^3 & \text{bosons} \\ \frac{3}{4} g_i \frac{\zeta(3)}{\pi^2} T^3 & \text{fermions} \end{cases}$$

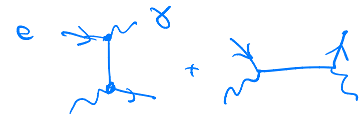
zeta function ζ, $\zeta(3) \simeq 1.20$

average energy per particle $\frac{\rho_i}{n_i} \simeq \begin{cases} 2.70\, T & \text{bosons} \\ 3.15\, T & \text{fermions} \end{cases}$

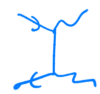
application: maximal temperature at which relativistic particles are in thermal equilibrium with respect to electromagnetic interactions.

Consider electrons at $T \gg 1\,\mathrm{MeV} \gtrsim m_e$

relevant processes: Compton scattering



e^+e^- annihilation



amplitudes $\propto \alpha = \frac{e^2}{4\pi}$

scattering probability $\propto \alpha^2$

interaction rate = (mean free time between scatterings)$^{-1}$

$$\Gamma \sim \alpha^2 T$$

by dimensional analysis.

Thermal equilibrium requires

(*) $$\Gamma \gg H$$

$$H^2 = \frac{8\pi}{3} G \rho = \frac{8\pi}{3 M_{pe}^2} g_* \frac{\pi^2}{30} T^4 = \frac{8\pi^3}{90} g_* \frac{T^4}{M_{pe}^2}$$

$$H = 1.66 \sqrt{g_*} \frac{T^2}{M_{pe}}$$

(*) $\Leftrightarrow$ $$T \ll \frac{1}{1.66\sqrt{g_*}} \alpha^2 M_{pe}$$

Standard Model of particle physics: $g_* \sim 100$

$\alpha \simeq \frac{1}{137}$

equilibrium for $T \ll 4 \cdot 10^{13}\,\mathrm{GeV}$

non-relativistic particles: Boltzmann distribution

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} e^{\beta \mu_i} \exp\left(-\frac{\vec{p}^2}{2 m_i T}\right)$$

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{(\mu_i - m_i)/T}$$

exponentially small for $m_i - \mu_i \gg T$

energy density, pressure:

$$\rho_i = n_i m_i + \frac{3}{2} n_i T$$

$$P_i = n_i T \ll \rho_i$$