1.5 Solutions to the Friedmann equations [Mukhanov]

a-dependence of ρ

matter (dust): $\rho a^3 = \text{const} \Rightarrow \rho \propto \frac{1}{a^3}$

radiation: $\rho \propto \frac{1}{a^4}$ (see exercises)

cosmological const: $\rho = \text{const}$

in general, one all kinds of contributions:

$$\rho = \rho_m + \rho_{rad} + \rho_\Lambda = \left(\frac{a_0}{a}\right)^3 \rho_{m0} + \left(\frac{a_0}{a}\right)^4 \rho_{rad\,0} + \rho_\Lambda$$

For small a, ρ_{rad} dominates, large a: ρ_Λ dominates
In between there could be matter domination.

notation for eq. of state $P = w\rho$

$w=0$ dust, $w = \frac{1}{3}$ radiation, $w = -1$ cosmological constant

An additional contribution to H comes from the curvature $\frac{k}{a^2}$ which would dominate at late times when Λ would vanish.

N.B. Recently the Dark Energy Survey Instrument (DESI) collaboration reported evidence for non-constant dark energy with $w = w_0 + w_a(1-a)$ ($a = a_0 = 1$ today)
with $w_a > -1$, $w_a < 0$

To solve the Friedmann eqs. use the so-called

conformal time η

defined through $\boxed{dt = a\, d\eta}$

The FRW metric then becomes

$$\boxed{d\tau^2 = a^2 \left(d\eta^2 - \left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right] \right)}$$

Why it is called "conformal":

for $k=0$: $g_{\mu\nu} = a^2 \eta_{\mu\nu}$ ← Minkowski metric

conformal transformation: $g_{\mu\nu} \to \lambda^2 g_{\mu\nu}$

N.B. this is not a coordinate transformation, it maps one manifold to another manifold.

conformal transformation are angle-preserving

$\frac{a \cdot b}{\sqrt{a^2 b^2}}$ with $a \cdot b = g_{\mu\nu} a^\mu b^\nu$ is invariant.

So the FRW with $k=0$ is conformally equivalent to Minkowski space.

Friedmann eqs. in terms of η

Hubble rate: $H = a^{-1}\frac{da}{dt} = a^{-1}\underbrace{\frac{d\eta}{dt}}_{=\frac{1}{a}}\frac{da}{d\eta} = a^{-2}a'$

The Friedmann eq. becomes

$$a^{-4}a'^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

(*) $$\boxed{a'^2 + k a^2 = \frac{8\pi G}{3}\rho a^4} \quad \Big| \frac{d}{d\eta}$$

$$2a'a'' + 2kaa' = \frac{8\pi G}{3}\left(\rho' a^4 + 4a^3 a'\rho\right)$$

$$\rho' = \frac{d\rho}{dt}\frac{dt}{d\eta} = \dot\rho a$$

we had $\dot\rho = -3H(\rho + P)$ $\Rightarrow$

$$\rho' = -3\frac{a'}{a^2}(\rho + P)a = -3\frac{a'}{a}(\rho + P)$$

$$\rho' a^4 + 4a^3 a'\rho = -3a'a^3(\rho + P) + 4a^3 a'\rho$$

$$= a'a^3(-3\rho - 3P + 4\rho) = a'a^3(\rho - 3P) \quad \Big| \frac{1}{2a'}$$

$$\boxed{a'' + ka = \frac{4\pi G}{3}a^3(\rho - 3P)}$$

examples:

(i) only radiation $\rho = 3P$

$$a'' + ka = 0$$

Solution for the initial condition $a(0) = 0$

$$a = a_m \begin{cases} \eta & 0 \\ \sin\eta & k = 1 \\ \sinh\eta & -1 \end{cases}$$

a_m can be fixed by using (*) :

$k=0$: $a'^2 = a_m^2 \Rightarrow$

$a_m^2 \; \frac{8\pi G}{3} \rho a^4 = \text{const}$. For $k = \pm 1$ one gets the same

t is obtained by integrating $\frac{dt}{d\eta} = a$ with $t(0) = 0$

$$t = a_m \begin{cases} \frac{1}{2}\eta^2 & \\ 1-\cos\eta & k = \\ \cosh\eta - 1 & \end{cases} \begin{matrix} 0 \\ 1 \\ -1 \end{matrix}$$

So for $k=0$ we have $a \propto \eta \Rightarrow a \propto \sqrt{t}$

and $H = \frac{1}{2t}$

For $k = 0, -1$: $0 < \eta < \infty$

$k = 1$: $0 < \eta < \pi$

(ii) flat universe with only matter

$$a'' + k a = \frac{4\pi}{3} G a^3 \rho = \text{const}$$

$k=0$: $a = \frac{2\pi}{3} G a^3 \rho \, \eta^2 + C\eta + D$

By shifting η by a constant we can eliminate D.

C can be fixed by using (*)

$$a' = \frac{4\pi}{3} G a^3 \rho \eta + C$$

$$a'^2 = \left(\frac{4\pi}{3} G a^3 \rho\right)^2 \eta^2 + 2 \frac{4\pi}{3} G a^3 \rho \eta C + C^2$$

$$\frac{8\pi G}{3} \rho a^4 = 2 \frac{4\pi}{3} G \rho a^3 a = 2 \frac{4\pi}{3} G \rho a^3 \left(\frac{2\pi}{3} G a^3 \rho \eta^2 + C\eta\right)$$

$$\Rightarrow \quad C = 0$$

$$a = a_m \eta^2 \quad \text{with} \quad a_m := \frac{2\pi}{3} G a^3 \rho$$

$$dt = a\, d\eta \quad \Rightarrow \quad t = a_m \int_0^\eta d\eta' \eta'^2 = \frac{1}{3} a_m \eta^3$$

$$\eta \propto t^{1/3}, \quad a \propto t^{2/3}$$

(iii) flat Universe with matter plus radiation:

$$\rho = \frac{1}{2} \rho_{eq} \left[\left(\frac{a_{eq}}{a}\right)^3 + \left(\frac{a_{eq}}{a}\right)^4 \right]$$

$$P = \frac{1}{3} \frac{1}{2} \rho_{eq} \left(\frac{a_{eq}}{a}\right)^4$$

$$a'' = \frac{4\pi G}{3} a^3 (\rho - 3P) = \frac{4\pi G}{3} a^3 \frac{\rho_{eq}}{2} \left(\frac{a_{eq}}{a}\right)^3 = \frac{2\pi G}{3} \rho_{eq} a_{eq}^3 = \text{const}$$

$$a = \frac{\pi G}{3} \rho_{eq} a_{eq}^3 \eta^2 + C\eta$$

C can be fixed by using $(*)$

$$a' = \frac{2\pi G}{3} \rho_{eq} a_{eq}^3 \eta + C$$

$$a'^2 = \left(\frac{2\pi G}{3} \rho_{eq} a_{eq}^3\right)^2 \eta^2 + 2C \frac{2\pi G}{3} \rho_{eq} a_{eq}^3 \eta + C^2$$

$$\frac{8\pi G}{3}\rho a^4 = \frac{8\pi G}{3}\frac{1}{2}\rho_{eq}\left[\left(\frac{a_{eq}}{a}\right)^3 + \left(\frac{a_{eq}}{a}\right)^4\right]a^4$$

$$= \frac{4\pi G}{3}\rho_{eq}\left[a_{eq}^3 a + a_{eq}^4\right]$$

$$= \frac{4\pi G}{3}\rho_{eq}\left[a_{eq}^3\left(\frac{\pi G}{3}\rho_{eq} a_{eq}^3 \eta^2 + C\eta\right) + a_{eq}^4\right] \Rightarrow$$

$$C^2 = \frac{4\pi G}{3}\rho_{eq}\, a_{eq}^4 \,, \qquad C = 2\sqrt{\frac{\pi G}{3}\rho_{eq}}$$

$$a = \frac{\pi G}{3}\rho_{eq} a_{eq}^3 \eta^2 + C\eta$$

$$= a_{eq}\left[\frac{\pi G}{3}\rho_{eq} a_{eq}^2 \eta^2 + 2\sqrt{\frac{\pi G}{3}\rho_{eq}}\, a_{eq}\eta\right]$$

$$= a_{eq}\left[\left(\frac{\eta}{\eta_*}\right)^2 + 2\frac{\eta}{\eta_*}\right]$$

with $\eta_* = \left[\sqrt{\frac{\pi G}{3}\rho_{eq}}\, a_{eq}\right]^{-1}$

$a = a_{eq}$ for $\eta = x\eta_*$, $\qquad x^2 + 2x = 1,\ x = -1 + \sqrt{2}$

$$\eta_{eq} = (\sqrt{2} - 1)\eta_*$$

For $\eta \ll \eta_{eq}$: $a \propto \eta$ radiation domination

$\eta \gg \eta_{eq}$ $a \propto \eta^2$ matter domination