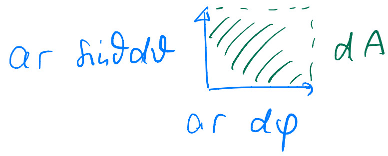
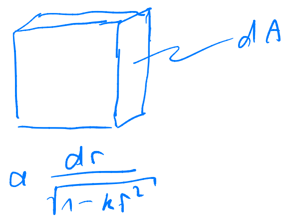
Surface integral on sphere at one value of r:



$$dA = a^2 r^2 \sin\vartheta \, d\vartheta \, d\varphi$$

area of the sphere: $a^2 r^2 \, 4\pi$

volume integral:



$$dV = a \frac{dr}{\sqrt{1-kr^2}} \, dA$$

volume of the ball inside a sphere with $r = r_s$:

$$4\pi a^3 \int_0^{r_s} dr \, \frac{r^2}{\sqrt{1-kr^2}}$$

For $k=0$ is equals $\frac{4\pi}{3}(ar_s)^3$, for $k=\pm 1$ it is bigger/smaller.

1.4 Einstein equations [Mukhanov 1.3.2]

Newton: gravitational potential ϕ, determined by

$$\Delta \phi = 4\pi G \rho$$

ϕ is related to the metric tensor through $g_{00} \simeq 1 + 2\phi$

General relativity: [conventions of Landau, Lifshitz II]

metric tensor $g_{\mu\nu} = g_{\nu\mu}$ 10 components

$$\boxed{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G\, T^{\mu\nu}}$$ Einstein eq.

$g^{\mu\nu}$: inverse metric $g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\rho}$

$R^{\mu\nu}$: Ricci tensor

$R = g_{\mu\nu} R^{\mu\nu}$ Ricci scalar

$T^{\mu\nu}$: energy-momentum tensor

$$R_{\mu\nu} = \frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial \Gamma^{\rho}_{\mu\rho}}{\partial x^{\nu}} + \Gamma^{\rho}_{\mu\nu} \Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\sigma}_{\mu\rho} \Gamma^{\rho}_{\nu\sigma}$$

with

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right)$$

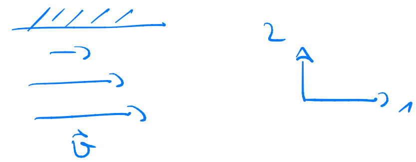
Christoffel symbols

E. eqs. are non-linear second order eqs. for $g_{\mu\nu}$.

Energy-momentum tensor

T^{00} : energy density

T^{0n} : density of energy current in n-direction
and density of the n-component of momentum

T^{mn} : density of current of n-component in m-direction example: fluid 

$T^{12} \neq 0$

One <u>point particle</u> with mass m_a, trajectory $\vec{x}(t)$:

$$p^\nu = m \frac{dx^\nu}{d\tau}$$

in local inertial frame:

$$p^0 = E = \frac{m}{\sqrt{1-\vec{v}^2}} \qquad \vec{v}_a = \dot{\vec{x}}$$

$$\vec{p} = \frac{m}{\sqrt{1-\vec{v}^2}} \frac{d\vec{x}_a}{dt}$$

phase space density $f_a(x, \vec{p})$

$$T^{0\nu}(x) = \int \frac{d^3p}{(2\pi)^3} \, p^\nu \, f(x, \vec{p})$$

$$T^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3} \frac{dx^\mu}{dt} p^\nu f$$

since $\quad p^\mu = E \dfrac{dx^\mu}{dt}$:

$$T^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu p^\nu}{E} f$$

<u>ideal fluid</u>

without gravity:

in local rest frame $T^{00} = \rho$, $T^{0n} = 0$

$$T^{mn} = P\,\delta^{mn} \qquad P: \text{pressure}$$

equation of state $\quad P = P(\rho)$

Lorentz boost to arbitrary inertial frame $\rightarrow$

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu - \eta^{\mu\nu} P$$

u^μ : 4-velocity of the fluid

with gravity:

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu - g^{\mu\nu} P$$

in FRW universe: $(u^\mu) = (1, \vec{0})$

$$T^{00} = \rho, \quad T^{0n} = 0, \quad T^{mn} = -P g^{mn}$$

$$T_{mn} = P \gamma_{mn}$$

example: (i) dust $P = 0$

(ii) radiation : massless (or ultrarelativistic $E \gg m$)

particles $p^\mu p_\mu = 0$ $\Rightarrow$ $T^\mu{}_\mu = 0$

local inertial rest frame $T^\mu{}_\mu = \rho + \delta_{mm} P$ $\Rightarrow$

$$P = \frac{1}{3}\rho$$

(iii) cosmological constant (dark energy)

$$8\pi G T^{\mu\nu} = g^{\mu\nu}\Lambda \quad , \quad \Lambda = \text{const}$$

$$P = -\rho$$

Friedmann equations

For dust we had $a^3 \rho = \text{const}$ $\Rightarrow$ $3\dot{a}a^2\rho + a^3\dot{\rho} = 0$ $\Leftrightarrow$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \quad \text{or}$$

(*) $\dot{\rho} + 3H\rho = 0$

In general, for an adiabatically expanding volume

$$dE = -P\,dV$$

$E = \rho V$

For a comoving volume: $V \propto a^3$

$d\rho\, a^3 + \rho\, 3a^2 da = -P\, 3a^2 da \implies$

$$\boxed{\dot{\rho} = -3H(\rho + P)}$$ generalizing (*).

In Newtonian cosmology we found

(*) $$H^2 - \frac{2E}{a^2} = \frac{8\pi}{3} G\rho$$

The 00-component of Einstein eq. gives

$$\boxed{H^2 + \frac{k}{a^2} = \frac{8\pi}{3} G\rho}$$

For $\Lambda = 0$ this gives (*) with $-2E = k$.

For (*) E was a constant of integration, in GR it a constant determined by the curvature of space.

Again one defines the critical density as

$$\rho_{crit} = \frac{3}{8\pi G} H^2$$

today: $\rho_{crit\,0} = 1.9 \times 10^{-29}\, h^2\, g\, cm^{-3}$

$= 1.1 \times 10^{-5}\, GeV\, cm^{-3}$

N.B. $m_{proton} = 0.938$ GeV

$$1 + \frac{k}{H^2 a^2} = \frac{8\pi G}{3} \frac{\rho}{H^2} = \frac{\rho}{\rho_{crit}}$$

Def. $\Omega : \frac{\rho}{\rho_{crit}}$

Then

$$\frac{k}{H^2 a^2} = \Omega - 1$$

$\Omega = 1$ $\quad k = 0$

$\Omega > 1$ $\quad k = 1$

$\Omega < 1$ $\quad k = -1$

Photon gas: $\rho = \frac{\pi^2}{15} T^4 = \frac{\pi^2}{15} \frac{1}{(\hbar c)^3} (k_B T)^4$

cosmic microwave background (CMB): $T_0 = 2.7$ K

$\rho_{0\,CMB} = 4.6 \times 10^{-34}\, g\, cm^{-3}$ $\Rightarrow$

$$\Omega_{0\,CMB} = \frac{\rho_{0\,CMB}}{\rho_{0\,crit}} = 2.9 \times 10^{-5}$$