1.3 Friedmann - Robertson - Walker (FRW) metric

= metric for homogeneous & isotropic universe

Fundamental observers : set of observers at different points in space, at rest with respect to the matter in their vicinity.

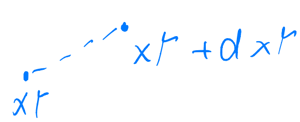
time = t = proper time of fundamental observers

The clocks of the fundamental observers are synchronized by setting it to a certain value when the (homogeneous) energy density has a certain value.

Choose spatial coordinates x^m so that fund. obs. have fixed x^m.
(comoving coordinates)

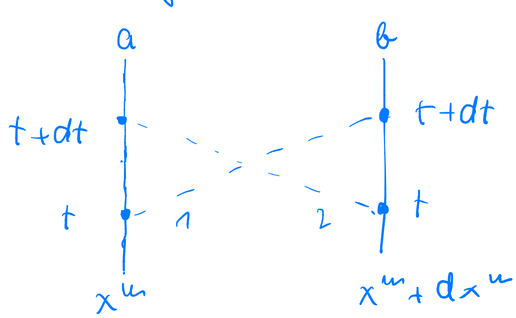
proper time of fund. obs. $d\tau^2 = g_{00}\,dt^2 = dt^2 \Rightarrow$

$$\boxed{g_{00} = 1}$$

light signal: 

$g_{\mu\nu}\,dx^\mu dx^\nu = 0$

two fundamental observers:



light signals emitted at time t both arrive at time $t+dt$

1: $0 = dt^2 + 2 g_{0n}\, dt\, dx^n + g_{mn}\, dx^m dx^n$ $\Rightarrow$

2: $0 = dt^2 + 2 g_{0n}\, dt\, (-dx^n) + g_{mn} (-dx^m)(-dx^n)$

$$\boxed{g_{0n} = 0}$$

So we have $d\tau^2 = dt^2 - d\vec{l}^2$ with $d\vec{l}^2 = \gamma_{mn}\, dx^m dx^n$

Symmetric Spaces [Mukhanov 1.3.1]

Now we want to describe a homogeneous and isotropic 3-dimensional space

To visualize such spaces start with homogeneous & anisotropic 2-d spaces.

(i) plane, coordinates x^1, x^2 $\quad dl^2 = (dx^1)^2 + (dx^2)^2$

(ii) 2-Sphere S^2, embedded in 3-d space with coordinates x^m

2-Sphere with radius a: $(x^1)^2 + (x^2)^2 + (x^3)^2 = a^2$

use the 3d Euclidean metric

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

choose x^1, x^2 as independent coordinates.

For each x^1, x^2 with $(x^1)^2 + (x^2)^2 < a^2$ $\exists$ 2 points on the sphere.

$$x^3 = \pm \sqrt{a^2 - (x^1)^2 - (x^2)^2}$$

$$dx_3 = -\frac{1}{x_3}\left(x^1 dx^1 + x^2 dx^2\right) = \pm \frac{x^n dx^m}{\sqrt{a^2 - x^n x^n}}$$

$$dl^2 = (dx^1)^2 + (dx^2)^2 + \frac{(x^1 dx^1 + x^2 dx^2)^2}{a^2 - (x^1)^2 - (x^2)^2}$$

polar coordinates: $\quad x^1 = r' \cos\varphi, \quad x^2 = r' \sin\varphi$

$$(dx^1)^2 + (dx^2)^2 = (dr')^2 + r'^2 (d\varphi)^2$$

$$r' := \sqrt{(x^1)^2 + (x^2)^2}$$

$$x^3 = \pm \sqrt{a^2 - r'^2}$$

$$dx^3 = -\frac{1}{2}\frac{1}{x^3}\, dr'^2 = \pm \frac{r' dr'}{\sqrt{a^2 - r'^2}}$$

$$dl^2 = (dx^1)^2 + (dx^2)^2 + \frac{r'^2 dr'^2}{a^2 - r'^2}$$

$$dl^2 = \left[1 + \frac{r'^2}{a^2 - r'^2}\right](dr')^2 + r'^2 (d\varphi)^2 \qquad \text{or}$$

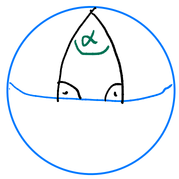
$$dl^2 = \frac{(dr')^2}{1 - (r'/a)^2} + r'^2 (d\varphi)^2$$

This describes a hom. & isot. space with constant positive curvature.

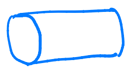
$a \to \infty$: flat space (plane)

How can one see that this space is curved (more precisely: that its inner curvature is nonzero)?

in flat space the sum of the interior angles of a triangle always equals π $(= 180^\circ)$



sum of interior angles $= \pi + \alpha > \pi$

N.B. tube  has zero internal curve

(iii) For $a^2 \to -a^2$ one obtains a hom. & isotr. space with negative curvature. It cannot be embedded in 3-dim. Euclidean space (it would correspond to a sphere with imaginary radius).

Define $r = \frac{r'}{\sqrt{|a^2|}}$ for all three cases. Then we can write for each of them

$$d\ell^2 = |a|^2 \left(\frac{dr^2}{1 - k r^2} + r^2 d\varphi^2 \right)$$

with $k = \begin{cases} 0 & \text{flat} \\ 1 & \text{pos. curvature} \\ -1 & \text{negative curvature} \end{cases}$

In curved space $|a|^{-1}$ is a measure for the curvature.
In flat space it has no meaning and could be eliminated by rescaling r.

generalization to <u>3dim-spaces</u>:

A 3-dim. sphere S^3 can be embedded in 4-dim Euclidean space (not to be confused with the 4-dim spacetime)

$$(x^1)^2 + \dots (x^4)^2 = a^2 \quad , \quad x^4 = \pm\sqrt{a^2 - r'^2}$$

$$dl^2 = dx^m dx^m + (dx^4)^2 = dx^m dx^m - \frac{(x^m dx^m)^2}{a^2 - x^n x^n}$$

$$= \left(\delta^{mn} - \frac{x^m x^n}{a^2 - x^l x^l}\right) dx^m dx^n$$

Spherical coordinates $\quad x^1 = r' \sin\vartheta \cos\varphi, \quad x^2 = r' \sin\vartheta \sin\varphi$

$$x^3 = r' \cos\vartheta$$

$$x^4 = \pm\sqrt{a^2 - r'^2} \quad , \quad dx^4 = \pm\frac{r' dr'}{\sqrt{a^2 - r'^2}}$$

$$dx^m dx^m = (dr')^2 + r'^2 d\Omega^2$$

with

$$d\Omega^2 := (d\vartheta)^2 + \sin^2\vartheta (d\varphi)^2$$

$$dl^2 = \frac{(dr')^2}{1 - (r'/a)^2} + r'^2 d\Omega^2$$

N. B. - S^3 has finite volume
- our coordinates cover only half of S^3.

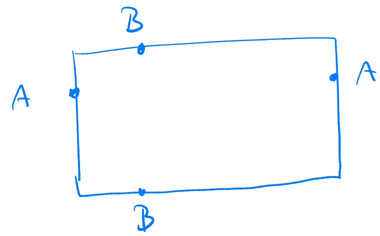
Again $a^2 \to -a^2$ gives a space with negative curvature.

The general case can be written as, using $r := \frac{r'}{\sqrt{|a^2|}}$

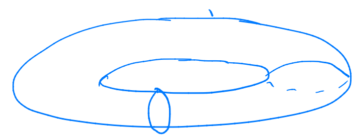
$$dl^2 = a^2\left(\frac{(dr)^2}{1-kr^2} + r^2 d\Omega^2\right)$$

N.B. Flat space can also be finite

2 dimensional example:



2 - torus T^2



Friedmann-Robertson-Walker metric:

$$\boxed{d\tau^2 = dt^2 - a^2\left\{\frac{dr^2}{1-kr^2} + r^2\left[(d\theta)^2 + \sin^2\theta\,(d\varphi)^2\right]\right\}}$$

The spatial coordinates are <u>comoving</u>: Any comoving observer has constant spatial coordinates r, θ, φ.