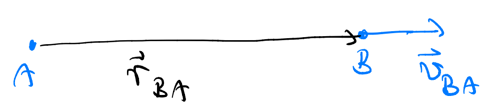
1 Expanding Universe

homogeneous and isotropic Universe

The CMB looks extremely isotropic. This indicates that at early times was nearly homogeneous and isotropic. Today there are large inhomogeneities on small scales. On the scale of $\sim$ 100 Mpc the Universe looks homogeneous and isotropic.

1.1. Newtonian cosmology [Mukhanov 1.2]

Let's assume a h&i Universe. All observers at different points in space moving with the matter around them (their galaxy or galaxy cluster) should see the same Hubble law

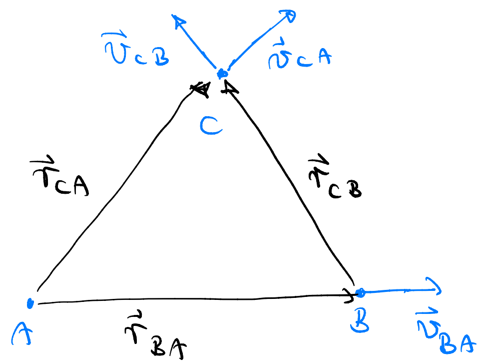


Velocity of observer B relative to A:

$$\vec{v}_{BA} = H \, \vec{r}_{BA}$$

where H will depend on time t. Is this compatible with a h&i Universe?

Consider how two observers A & B observe a third one C.



$$\vec{v}_{CB} = \vec{v}_{CA} - \vec{v}_{BA} = H\left(\vec{r}_{CA} - \vec{r}_{BA}\right) = H\,\vec{r}_{CB}$$

That is, observer B sees the same Hubble law as A.

<u>Continuity equation</u>

Consider non-relativistic (NR) matter $\vec{v}^2 \ll 1$ consisting of a single species with mass m.
(In cosmology NR matter is often called <u>dust</u>, or simply <u>matter</u>.)

energy density ρ

NR $\Rightarrow$ ρ is due to mass (= rest energy) of the particles,

$\rho = m\,n$, n = number density

pressure P (= momentum flux density) negligible

particle number and thus energy is conserved

Continuity equation $\dot{\rho} + \nabla \cdot \vec{j} = 0$

$\vec{j}$ = energy flux density

Assume that the particle's velocity is given by a velocity field $\vec{v}(t, \vec{x})$ like in a fluid.

Then $\vec{j}(t, \vec{x}) = \rho(t, \vec{x}) \, \vec{v}(t, \vec{x})$

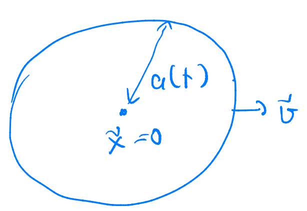
and the continuity equation becomes

$$\dot{\rho} + \nabla \cdot (\rho \vec{v}) = 0$$

Expansion

homogeneity, isotropy $\Rightarrow$ $\rho = \rho(t)$, $\vec{x}$-independent

Consider a sphere around $\vec{x} = 0$, of radius $a(t)$, moving with the particles.



$\vec{v} = \dot{a} \, \hat{x}$ $\quad \left(\hat{x} := \frac{\vec{x}}{|\vec{x}|} \right)$

mass inside the sphere $M = \frac{4\pi}{3} a^3 \rho$ time-independent

Assume that the mass outside the sphere does not exert a force on the particle. (This was true in Newtonian gravity if ρ was spherically symmetric and fell off outside the sphere. Here ρ does not fall off, and our assumptions can only be justified in GR)

Then the force on a particle at $|\vec{x}| = a$ is

$$\vec{F} = -G\frac{mM}{a^2}\hat{x}$$

acceleration: $\ddot{a} = -G\dfrac{M}{a^2}$

N.B. This is precisely the eq. of motion that follows from GR for a dust filled universe.

It is also the eq. of motion of a ball kicked upwards in the gravitational field of the earth (friction neglected).

Multiply by $\dot{a}$ and integrate

(*) $\frac{1}{2}\dot{a}^2 + V(a) = E$

with $V(a) = -\dfrac{GM}{a}$

For $E \geq 0$, the ball escapes to $|\vec{x}| = \infty$, for $E < 0$ it falls back to earth.

For our dust-filled Universe it means that for $E \geq 0$ it will expand forever growing to infinite size, and for $E < 0$ it will collaps.

Rewrite (*) with $H := \frac{\dot{a}}{a}$

$$H^2 - \frac{2E}{a^2} = \frac{2GM}{a^3}$$

$$M = \frac{4\pi}{3} a^3 \rho \quad \Rightarrow \quad \frac{2GM}{a^3} = \frac{8\pi G}{3} \rho$$

$$\boxed{H^2 - \frac{2E}{a^2} = \frac{8\pi G}{3} \rho}$$

At the boundary between expanding forever and collapse $E = 0$:

$$\rho = \rho_{crit}$$

with

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

<u>Critical density</u>

1.2 Metric tensor

metric of special relativity : $x^2 = t^2 - \vec{x}^2 = \eta_{\mu\nu} x^\mu x^\nu$

$(x^\mu) = \begin{pmatrix} t \\ \vec{x} \end{pmatrix}$, $(\eta_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$

proper time interval of a moving observer

(= time elapsed in the observer's rest frame) :

$$d\tau = \left(\eta_{\mu\nu}\, dx^\mu dx^\nu\right)^{1/2}$$

For a light signal: $d\tau^2 = 0$

In general relativity : coordinates x^μ

notation : $x = (x^\mu) := \begin{pmatrix} x^0 \\ \vdots \\ x^3 \end{pmatrix}$

metric tensor $g_{\mu\nu}(x)$, symmetric $g_{\mu\nu} = g_{\nu\mu}$

$g_{\mu\nu}$ is determined by distribution of energy & momentum

proper time of a moving observer

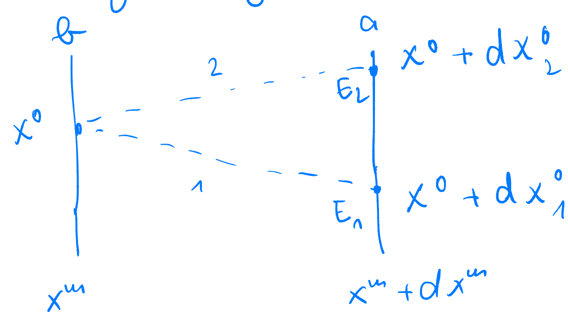
$$d\tau = \left(g_{\mu\nu}\, dx^\mu dx^\nu\right)^{1/2}$$

For a light signal: $d\tau = 0$

Spatial distances

dl = spatial distance between two observers a and b with fixed spatial coordinates x^i, $x^i + dx^i$

The distance can be measured by taking the time it takes a light signal from b to a and back.



$$dl = \tfrac{1}{2}\,(\text{time interval between } E_1 \text{ and } E_2) = \tfrac{1}{2}\sqrt{g_{00}}\,(dx^0_2 - dx^0_1)$$

$$1: \quad dx = \begin{pmatrix} -dx^0_1 \\ -dx^m \end{pmatrix}, \qquad 2: \quad d = \begin{pmatrix} dx^0_2 \\ dx^m \end{pmatrix}$$

for both $dx^0 = dx^0_1$, $dx^0 = dx^0_2$:

$$d\tau^2 = g_{00}(dx^0)^2 + 2\, g_{n0}\, dx^n dx^0 + g_{mn}\, dx^m dx^n = 0 \quad \Leftrightarrow$$

$$(dx^0)^2 + 2\,\frac{g_{n0}}{g_{00}}\, dx^n dx^0 + \frac{g_{mn}}{g_{00}}\, dx^m dx^n = 0$$

This eq. has two solutions which correspond to

$$dx^0_{1,2} = -\frac{g_{n0}}{g_{00}}\, dx^n \mp \left[\frac{g_{n0} g_{m0}}{g_0^2}\, dx^m dx^n - \frac{g_{mn}}{g_{00}}\, dx^m dx^n \right]^{1/2}$$

$$= -\frac{g_{0n}}{g_{00}}\, dx^n \mp \left[\left(\frac{g_{n0} g_{m0}}{g_{00}^2} - \frac{g_{mn}}{g_{00}} \right) dx^m dx^n \right]^{1/2}$$

$$dl = \sqrt{g_{00}}\,[\quad]^{1/2}$$

$$dl^2 = g_{00}\left(-\frac{g_{mn}}{g_{00}} + \frac{g_{m0}\,g_{n0}}{g_{00}^2}\right) dx^m\, dx^n$$

$$dl^2 = \gamma_{mn}\, dx^m\, dx^n$$ with

$$\boxed{\gamma_{mn} = \left(-g_{mn} + \frac{g_{m0}\,g_{n0}}{g_{00}}\right)}$$ spatial metric