Problem C11.1 The equilibrium hydrogen ionization fraction is found using the Saha equation

$$\frac{1 - X_e}{X_e^2} = \mathcal{N} T^{-3/2} e^{B_H/T}$$

where $\mathcal{N} = n_B \left(1 - Y_{\mathsf{He}}\right) \left(\frac{m_{\mathsf{H}}}{m_e m_p}\right)^{3/2}$. n_B is the baryon number density and $Y_{\mathsf{He}} \approx 0.245$ the helium mass fraction obtained from BBN. The above equation is a quadratic equation for X_e, which can be solved for all T. This equation can be greatly simplified in several regimes.

(a) When $T \ll B_H$, the equilibrium ionization fraction will be very small, since the photons are not energetic enough to ionize the hydrogen. Instead, find an expression for $X_e^{\mathsf{eq}}\left(T\right)$ in the $X_e \ll 1$ limit. How does X_e^{eq} scale with the temperature in this regime?

(b) When $T \gg B_H$, the equilibrium ionization fraction will be very close to 1. Write $X_e = 1 - \delta X_e$ for $\delta X_e \ll 1$ and $\delta X_e > 0$ to find how the departure of X_e with respect to 1 scales with the temperature in this regime.

Problem C11.2 Gravitational waves from a first order electroweak phase transition would have been emitted at $T = 130$ GeV. Estimate what their wavelengths are today. (*Hint : At* $T = 130$*GeV,* $g_\star = 96.25$)

Problem H11.1 Under the instantaneous freeze-out approximation, particles follow their equilibrium number density until their decoupling. Using the Saha equation, find what is the fraction of electron that are not bound in atoms when photons decouple, $T_{\mathsf{dec.}} = 0.26$ eV. (*Hint : Use the result of C11.1 (a).*)

Problem H11.2 Between the time of electron freeze-out ($T \simeq m_e/22$) and recombination ($T = 0.26$ eV), the Universe evolved but few new physical processes took place beyond Thomson scattering ($e^- + \gamma \leftrightharpoons e^- + \gamma$). Compute the number of scattering that a photon undergoes during that time. The quantity that counts the number of interaction over a period of time is the optical depth

$$\tau = \int \sigma_T n_e\left(t\right) c dt$$

where the integration is between the time of electron freeze-out and the time of recombination. In that expression, $\sigma_T = \frac{8\pi\alpha^2}{3m_e^2}$, $\alpha = \frac{1}{137}$ is the Thomson cross-section. For simplicity, assume that the Universe is flat and filled with matter.