Cosmology	**Sheet 8**	May 30, 2025

Problem C8.1 Starting from the collisionless Boltzmann equation in an homogeneous and isotropic universe, show that the temperature of non-relativistic matter scales as a^{-2}. Give an example which violates this scaling in the early universe.

Problem C8.2

(a) Integrate the Boltzmann equation in an homogeneous and isotropic universe over momentum to find an equation for the particle number density

$$n_i = \int \frac{d^3p}{(2\pi)^3} f_i$$

(b) Obtain the time-evolution of the so-called particle "yield"

$$Y_i \equiv \frac{n_i}{s}$$

where s is the total entropy of the universe

$$s = \frac{2\pi^2}{45} g_{\star,s} T^3$$

and $g_{\star,s}$ is the effective number of relativistic degrees of freedom associated with entropy

$$g_{\star,s} = \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i=\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3$$

Problem H8.1

(a) Show that the free Boltzman equation $\partial f/\partial t - Hp\partial f/\partial p = 0$ is solved by $f(t,p) = g(pa(t))$, where a is the scale factor and g is an arbitrary function.

(b) In which cases is the Fermi-Dirac or Bose-Einstein distribution function a solution of the free Boltzmann equation?
Hint : For these functions to be a solution, there must exist a function h such that $(E-\mu)/T = h(pa(t))$, where μ and T are functions of t, but not of p. Consider the ultra-relativistic and the non-relativistic limits with the first order including p. What happens when you include high-order terms?

Problem H8.2 Typically the temperature of the cosmic plasma cools as a^{-1} with the expansion. However, when particles annihilate, they deposit energy into the plasma, thereby slowing the cooling. Use the fact that the entropy density scales as a^{-3} to compute the ratio of $(aT)^3$ at $T = 10$ GeV to its present value today.