Problem sheet 5

2025/05/11

1 C5.1 : Metric in terms of the comoving distance

The FLRW metric, in the so-called "comoving coordinates" (r, θ, ϕ) is given by

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right) \tag{1}$$

Do a change of variable from the radial coordinates r to the comoving distance χ for a flat, spherical and hyperbolic universe. The comoving distance is defined as $d\chi^2 \equiv \frac{dr^2}{1-\kappa r^2}$.

2 C5.2 : Angular diameter distance in cosmology

In terms of the comoving coordinate χ, the angular diameter distance is given by

$$d_A = a(t)\, \chi \tag{2}$$

Obtain d_A as a function of redshift for a flat, dust-dominated universe. To do so, compute the comoving distance to an object which emitted light at a scale factor $a < a_0$. Deduce $\chi(z)$ and obtain $d_A(z)$. At which redshift $z_\star$ is the angular diameter distance maximized? Expand $d_A(z)$ for $z \ll 1$ at first order. Does this new notion of distance coincide with the luminosity distance and proper distance for $z \ll 1$?

3 H5.1 : The horizon problem

The Cosmic Microwave Background spectrum is, within one part in one hundred-thousand, a black-body radiation at 2.725K, independently of the direction of the observation. At the time of CMB emission ($z_{\mathrm{CMB}} \approx 1100$) could the entire universe be even causally connected? How many disconnected patches can we see in the CMB?

1. From the results of C4.1, obtain the current proper distance to the last scattering surface and infer the current proper distance between two antipodal points of the CMB. Is it bigger than the horizon distance?

2. From cosmological observations, we have that $\Omega_{\mathrm{rad}} h^2 = 4.18 \times 10^{-5}$, $\Omega_{\mathrm{m}} h^2 = 0.143$, where $\Omega_i = \rho_i(t_0)/\rho_c$ for ρ_c the critical density, $\rho_i(t_0)$ the energy density of the species i today. The subscript "rad" denotes radiation and "m" denotes matter. Using these figure, obtain the value of the redshift at matter-radiation equality. Was the CMB emitted in a radiation- or matter-dominated universe?

3. What was the comoving horizon distance at the time of the CMB? Deduce the proper horizon distance at that time. Take into account the stretching due to the expansion of the universe and obtain the current size D of that distance.

4. To obtain the angular size on the sky of causally-connected patches of the CMB, divide D by the current proper horizon distance. How many casually disconnected patches do we see in the CMB? What are their sizes in degrees?