Problem C3.1 Age of a radiation-dominated flat Universe The equation determining the evolution of the scale factor derived from General Relativity is

$$H^2 + \frac{k}{a^2} = \frac{8\pi}{3} G\rho$$

where $H = \dot{a}/a$ is the Hubble rate, k determines the curvature, and ρ is the total energy density of the Universe, which could contain many components, such as photons (radiation), non-relativistic protons (matter) or more exotic form of matter. For a flat universe ($k = 0$) filled with radiation ($a^4\rho = a_0^4\rho_0 =$ constant), express the age of the Universe in terms of today's Hubble rate H_0.

Problem C3.2

(a) Assume that $P = w\rho$ where w is a constant between -1 and 1/3. Using the equation for the time-evolution of the energy density, find how ρ depends on the scale factor.

(b) Using the Friedman equation, express the age of the universe for a flat Universe filled with this type of matter in terms of today's Hubble rate H_0 .

Problem H3.1 For a circle in a FRW universe (at an arbitrary fixed time) parametrized by $r = r_c$, $\theta = \pi/2$, and $0 \leq \varphi \leq 2\pi$, compute its circumference C and its radius R. In flat space we know that $C/R = 2\pi$. Expand your expression r_c for $r_c \ll 1$. Do you obtain the same result as in flat space?

Problem H3.2 From the equation in problem C3.1 together with

$$\dot{\rho} = -3H(\rho + P)$$

derive the following useful relation for the Hubble rate:

$$\dot{H} = -4\pi G(\rho + P) + \frac{k}{a^2}$$

Can you find an example for $T^{\mu\nu}$ which for $k = 0$ gives an accelerated expansion of the Universe?