Critical phenomena near four dimensions and in the large N limit

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Introduction

In these lectures, we first explain how perturbative calculations based on a local statistical field theory of the $\int \mathrm{d}^d x\, \phi^4(x)$ type, combined with renormalization group (RG) methods, allow proving universal properties of critical statistical systems and calculating universal quantities, at least in a vicinity of dimension 4.

Indeed, within the framework of the $\varepsilon = 4 - d$ expansion (d is the dimension of space), introduced by Wilson and Fisher, renormalization group equations (RGE) satisfied by correlation functions can be proved to all orders in a double expansion in the coefficient of $\int \mathrm{d}^d x\, \phi^4(x)$ and $\varepsilon = 4 - d$. The proof is based on the methods of perturbative statistical field theory, and a few assumptions that it is possible to clarify.

These RGE are an asymptotic form of more general functional RGE, in a situation in which the effective large distance hamiltonian remains sufficiently close to the gaussian fixed point.

They appear as a consequence of the renormalizability of certain local field theories. The renormalizability of a field theory expresses the relative insensitivity of correlation functions to the short distance structure and is, therefore, rather directly related to the universality of critical phenomena. It is also related to the nature of the large momentum divergences of Feynman diagrams calculated with a critical propagator. Dimension 4 finds a simple interpretation within this framework. Indeed, the ϕ^4 field theory is only renormalizable for $d \leq 4$. (Actually for $d < 4$ the theory is super-renormalizable.)

We then solve the RG equations. We show that when an IR stable fixed point (i.e., relevant for the large distance or low momentum behaviour) exists, the RGE of field theory allow proving universal properties of critical phenomena, determining the large distance behaviour of connected correlation functions or the singularities of thermodynamic functions at T_c.

Within the framework of the ε-expansion ($\varepsilon > 0$), a non-gaussian IR stable fixed point can indeed be found near dimension 4 and universal quantities can then be calculated in the form of an ε-expansion.

We restrict most of the discussion to theories with an Ising type symmetry and the field ϕ has only one component. Generalization to models with N-component fields and $O(N)$ symmetry is simple.

The ε-expansion is *a priori* valid only near dimension 4 while the physics applications correspond to dimension 3 or 2. It is, therefore, somewhat reassuring to verify that the results obtained in this way remain valid, at least in some limiting case, when ε is no longer infinitesimal. We thus discuss $O(N)$ symmetric field theories, at fixed dimension, in the large N limit and show that the same universal properties are recovered.

Large N techniques are also useful because they allow to discuss other non-perturbative questions, including issues relevant to four-dimensional physics like renormalons and triviality.

Finally, they allow verifying a surprising relation between the $(\phi^2)^2$ field theory and the non-linear σ-model valid to all orders in the $1/N$ expansion.

Effective field theory and quasi-gaussian approximation

We consider critical properties of statistical models with short range interactions, where a simple Ising $\mathcal{Z}_2$ or more generally $O(N)$ symmetry is spontaneously broken. Moreover, we work in the general perturbative framework of dimensional continuation, the dimension d of space being considered as a continuous parameter.

A general analysis then indicates that the large distance properties of statistical systems near the critical temperature can be inferred from the study of a general effective, local statistical field theory (SFT), that is, a field theory where the hamiltonian (or euclidean action) $\mathcal{H}(\phi)$ depends only on the field $\phi(x)$ and its derivatives (short range interactions).

The partition function is then given by a field integral of the form

$$\mathcal{Z} = \int [\mathrm{d}\phi(x)] \mathrm{e}^{-\mathcal{H}(\phi)} .$$

The simplest theory of critical phenomena is based on the quasi-gaussian or mean-field approximation, a theory in the spirit of the central limit theorem of probabilities.

Critical phenomena are then described by a quadratic hamiltonian (leading to a gaussian distribution), terms with higher powers of the field being treated as perturbations.

However, one verifies that such an approximation is inconsistent in space dimension $d \leq 4$.

To deal with the problem, a new tool has been invented, the renormalization group (RG).

The renormalization group idea

The RG describes a markovian flow of scale-dependent hamiltonians in which successive short distance (or in Fourier representation high momentum) degrees of freedom are successively integrated out:

$$\lambda \frac{\mathrm{d}\mathcal{H}_\lambda}{\mathrm{d}\lambda} = \mathcal{T}(\mathcal{H}_\lambda), \quad \mathcal{H}_{\lambda=1} = \mathcal{H}\,.$$

To this hamiltonian correspond (connected) scale-dependent correlation functions $W_\lambda^{(n)}$ related to the initial ones by

$$W_\lambda^{(n)}(x_1,\ldots,x_n) - Z^{-n/2}(\mathcal{H}_\lambda)W^{(n)}(\lambda x_1,\ldots,\lambda x_n) = R_\lambda^{(n)}(x_1,\ldots,x_n),$$

where $Z^{1/2}(\mathcal{H}_\lambda)$ is a field renormalization and the $R_\lambda^{(n)}$ decrease faster than any power for $\lambda \to \infty$.

Large distance behaviour of critical phenomena is then governed by RG fixed points, that is, hamiltonians solutions of

$$\mathcal{T}(\mathcal{H}_*) = 0\,.$$

The existence of a fixed point implies the universal behaviour

$$W^{(n)}(\lambda x_1, \ldots, \lambda x_n) \underset{\lambda \to \infty}{\propto} \lambda^{-n d_\phi} W_*^{(n)}(x_1, \ldots, x_n),$$

where d_ϕ is the field dimension (in inverse length or momentum unit). A linearization of the flow equation allows studying the fixed point stability:

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \Delta \mathcal{H}_\lambda = L^* \left(\Delta \mathcal{H}_\lambda \right),$$

where L^* is a linear operator, independent of λ, acting on the space of hamiltonians.

The local flow depends on the eigenvalues of L^* (which we assume real):

(i) Positive eigenvalues correspond to directions of instability and the corresponding eigenvectors or perturbations are called relevant.

(ii) Negative eigenvalues correspond to directions of stability and the corresponding eigenvectors are called irrelevant.

(ii) Vanishing eigenvalues requires going beyond the linear approximation and the corresponding eigenvectors are called marginal.

Gaussian fixed point

The linear RG transformations

$$\phi(x) \mapsto \lambda^{(2-d)/2}\phi(x/\lambda),$$

have the gaussian massless field theory,

$$\mathcal{H}_*(\phi) = \frac{1}{2}\int \mathrm{d}^d x \left[\sum_{\mu=1}^{d} \partial_\mu \phi(x) \partial_\mu \phi(x)\right],$$

as a RG fixed point and the gaussian field dimension is $d_\phi = (d-2)/2$.

One then verifies that the gaussian fixed point is unstable in dimension $d < 4$ and near dimension 4, the leading relevant perturbation has the form of a $\int \phi^4(x)\mathrm{d}^d x$ interaction.

Effective ϕ^4 field theory: Ising symmetry

It is then anticipated that near dimension 4 critical phenomena can be described by a hamiltonian sum of a quadratic part and a ϕ^4 interaction.

We thus consider an effective hamiltonian of the general form,

$$\mathcal{H}(\phi) = \mathcal{H}_{\mathrm{G}}(\phi) + \frac{1}{4!} g \int \mathrm{d}^d x \, \phi^4(x),$$

$$\mathcal{H}_{\mathrm{G}}(\phi) = \frac{1}{2} \int \mathrm{d}^d x \left[\sum_{\mu=1}^{d} \partial_\mu \phi(x) \left(1 + \sum_{k=1} u_{k+1} (-\nabla_x^2)^k \right) \partial_\mu \phi(x) + u_0 \phi^2(x) \right].$$

It is the sum of the gaussian fixed point hamiltonian, of two relevant ($u_0\phi^2$ and $g\phi^4$), even perturbations and of terms quadratic in the field with a sufficient number of derivatives, in order to regularize all large momentum divergences (called Landau–Ginzburg–Wilson hamiltonian).

These divergences are not present in the initial microscopic statistical models due to their short distance structure (lattice, atomic size,...).

To the critical temperature T_c, at which the correlation length ξ (the inverse of the physical mass in field theory language) diverges, corresponds a value

$$u_0 = u_{0c}(g) \;\Rightarrow\; \left\langle \tilde{\phi}(p)\tilde{\phi}(-p) \right\rangle^{-1} = \tilde{\Gamma}^{(2)}(p=0) = 0\,.$$

The parameter u_0 can then be decomposed into the sum

$$u_0 = u_{0c}(g) + \delta u_0\,, \quad \delta u_0 \propto T - T_c\,,$$

where δu_0 induces an infinitesimal relevant perturbation. Based on the general analysis, we assume that all parameters u_k and g are regular functions of the temperature for $|T - T_c| \ll 1$.

At leading order, such a hamiltonian reproduces all the results of the quasi-gaussian or mean field approximations. Beyond, it leads to a double, perturbative and dimensional, expansion of thermodynamic quantities. In particular, correlation functions have a finite expansion to all orders in $\varepsilon = 4 - d$ and g except at T_c at zero momentum.

Gaussian renormalization

In the hamiltonian, it is useful to rescale distances:

$$x \mapsto x/\Lambda \,,$$

where Λ is a parameter that has a momentum dimension.

Through this change, one takes as a reference scale the macroscopic scale relevant for large distance physics, rather than the initial microscopic scale (related, for example, to the lattice spacing of an initial statistical model). The latter is then characterized by the parameter $1/\Lambda$. Instead of studying the large distance limit, one then studies the limit $\Lambda \to \infty$.

One then performs the RG transformations corresponding to this change of scale adapted to the gaussian fixed point (the gaussian renormalization):

$$\phi(x) \mapsto \Lambda^{(2-d)/2} \phi(x/\Lambda) \,.$$

After the change of variables $x/\Lambda = x'$, the hamiltonian reads (omitting now primes)

$$\mathcal{H}(\phi) = \mathcal{H}_{\rm G}(\phi) + \frac{1}{4!} g\Lambda^{4-d} \int \mathrm{d}^d x\, \phi^4(x),$$

$$\mathcal{H}_{\rm G}(\phi) = \frac{1}{2}\int \mathrm{d}^d x \sum_\mu \left(\partial_\mu \phi(x)\right)^2 + \frac{1}{2}\Lambda^2 u_0 \int \mathrm{d}^d x\, \phi^2(x)$$
$$+ \frac{1}{2}\int \mathrm{d}^d x \sum_{k=1} u_{k+1}\Lambda^{-2k} \sum_\mu \partial_\mu\phi(x)(-\nabla^2)^k \partial_\mu\phi(x)\,.$$

In the context of the quantum field theory, the parameter Λ, which reflects the initial microscopic structure, is called cut-off.

After this transformation, the gaussian fixed point is stable and the quasi-gaussian model valid if the coefficients of all terms in the hamiltonian, but the two first in the effective action, go to zero for $\Lambda \to \infty$. For the quadratic terms with more than two derivatives, this condition is always satisfied.

Remarks

(i) This RG transformation corresponds to the first step in the RG process, where a first dilatation suppresses (at leading order) the terms that are irrelevant from the viewpoint of the fixed gaussian point, since, for $g \ll 1$, the RG flow is first dominated by the local flow near the gaussian fixed point.

(ii) After the introduction of the Λ parameter, all quantities have a dimension in Λ unit. Momenta have dimension 1, space coordinates dimension -1, the field ϕ has dimension $\frac{1}{2}(d-2)$, which is equal to its dimension from the viewpoint of the gaussian fixed point. More generally, all local monomials in the hamiltonian have as dimensions the eigenvalues of the corresponding eigenvectors of the RG flow in the linear approximation near the gaussian fixed point. With the introduction of the Λ parameter, the calculation of the dimension of the field and the analysis of the perturbations at the gaussian fixed point have been reduced to dimensional analysis.

Critical domain: gaussian fixed point

The critical domain is obtained by renormalizing the initial coefficients of the relevant terms in the hamiltonian to cancel the effect of RG transformations.

For example, at the gaussian fixed point the $\int \phi^2(x) \mathrm{d}^d x$ term is relevant and has dimension 2. By setting

$$\Lambda^2 u_0 = \Lambda^2 u_{0c}(g) + \tau \;\Leftrightarrow\; u_0 - u_{0c} \mapsto \tau/\Lambda^2 ,$$

we renormalize the deviation to the critical temperature to cancel the scaling factor generated by the RG in the linear approximation near the gaussian fixed point.

After this change, correlation functions have a finite limit in the critical domain when $\Lambda \to \infty$ at $\{p_i, \tau\}$ fixed if the gaussian fixed point is stable, that is, for $d > 4$.

For $d \leq 4$, instead, the $\int \phi^4(x) \mathrm{d}^d x$ perturbation is marginal or relevant and the correlation functions diverge when $\Lambda \to \infty$.

Correlation and vertex functions

The functional $\mathcal{W}(H) = \ln \mathcal{Z}(H)$, where $Z(H)$ is the partition function in an external, space-dependent (magnetic) field H, generates connected correlation functions:

$$\mathcal{W}(H) = \sum_{n=0} \frac{1}{n!} \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, W^{(n)}(x_1, \ldots, x_n) H(x_1) \ldots H(x_n).$$

In local field theories, connected correlation functions satisfy the cluster property: $W^{(n)}(x_1, \ldots, x_n)$ decays at large point separations.

Generating functional of vertex functions. The generating functional $\Gamma(M)$ of (1PI) vertex functions

$$\Gamma(M) = \sum_{n=0} \frac{1}{n!} \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, \Gamma^{(n)}(x_1, \ldots, x_n) M(x_1) \ldots M(x_n),$$

is the Legendre transform of $\mathcal{W}(H)$:

$$\mathcal{W}(H) + \Gamma(M) = \int \mathrm{d}^d x \, H(x) M(x), \quad M(x) = \frac{\delta \mathcal{W}(H)}{\delta H(x)}.$$

$\Gamma(M)$ is also the thermodynamic potential expressed in terms of the local magnetization $M(x) = \langle\phi(x)\rangle$. In terms of the Fourier components $\tilde{M}(p)$ of $M(x)$,

$$\Gamma(M) = \sum_n \frac{1}{n!} \int \mathrm{d}^d p_1 \ldots \mathrm{d}^d p_n \, \delta^{(d)} \left(\textstyle\sum_i p_i\right) \tilde{M}(p_1) \ldots \tilde{M}(p_n) \tilde{\Gamma}^{(n)}(p_i),$$

where $\tilde{\Gamma}^{(n)}$ is the Fourier transform of $\Gamma^{(n)}$.

Dimension of vertex functions

Later we shall need the (gaussian) dimensions of the Fourier components of vertex functions. The dimension $[\tilde{\phi}]$ of the Fourier components $\tilde{\phi}$ of the field can be inferred from the relation

$$\phi(x) = \int \mathrm{d}^d p \, \mathrm{e}^{ipx} \tilde{\phi}(p) \quad \Rightarrow \quad [\tilde{\phi}] = -(d+2)/2 \,.$$

Since $\tilde{M}$ and $\tilde{\phi}$ have the same dimension, one infers the dimension

$$[\tilde{\Gamma}^{(n)}] = -nd + n(d+2)/2 + d = d - n(d-2)/2 \,.$$

Renormalization theorem

In dimension 4 (the upper-critical dimension) in an expansion in powers of g, the coefficient g of the ϕ^4 term, and, more generally, in dimension $d < 4$, in a double expansion in powers of the coefficient g and $\varepsilon = 4-d$, correlation functions have divergences that take the form of powers of $\ln\Lambda$, the degree increasing with the order in g and in ε. These divergences are controlled by renormalization theorems.

First, we state the renormalization theorem for the critical theory $\tau = 0$.

To formulate the theorem, one introduces a momentum $\mu \ll \Lambda$, called renormalization scale, and a parameter g_{r} characterizing the effective interaction at scale μ, called renormalized interaction.

Then, one can find two functions $Z(\Lambda/\mu, g_{\rm r})$ and $Z_g(\Lambda/\mu, g_{\rm r})$, calculable order by order in an expansion in powers of $g_{\rm r}$ and ε, that satisfy

$$\Lambda^{4-d} g = \mu^{4-d} Z_g(\Lambda/\mu, g_{\rm r}) g_{\rm r} = \mu^{4-d} g_{\rm r} + O(g_{\rm r}^2), \quad Z(\Lambda/\mu, g_{\rm r}) = 1 + O(g_{\rm r}),$$

such that all vertex functions

$$\tilde{\Gamma}_{\rm r}^{(n)}(p_i; g_{\rm r}, \mu, \Lambda) = Z^{n/2}(g_{\rm r}, \Lambda/\mu) \tilde{\Gamma}^{(n)}(p_i; g, \Lambda),$$

called renormalized, have, order by order, finite limits $\tilde{\Gamma}_{\rm r}^{(n)}(p_i; g_{\rm r}, \mu)$ when $\Lambda \to \infty$ at $p_i, \mu, g_{\rm r}$ fixed.

Remark. The field renormalization $Z^{1/2}(\Lambda/\mu, g_{\rm r})$ defined here corresponds to the ratio between the true renormalization in the RG sense and the gaussian renormalization $(\Lambda/\mu)^{(d-2)/2}$, a factorization well adapted to the search of fixed points close to the gaussian fixed point, as one assumes in the framework of the ε-expansion.

Remarks.

(i) There is some amount of arbitrariness in the choice of the renormalization constants Z and Z_g since they can be multiplied by arbitrary finite functions of $g_{\rm r}$. The constants can be completely determined by renormalization conditions, for example,

$$\frac{\mathrm{d}}{\mathrm{d}p^2}\tilde{\Gamma}_{\rm r}^{(2)}(p=\mu,\mu,g_{\rm r})=1\,,\quad \tilde{\Gamma}_{\rm r}^{(4)}(p_i=\mu\theta_i,\mu,g_{\rm r})=\mu^{4-d}g_{\rm r}\,,$$

where the θ_i are four vectors such that $\sum_i\theta_i=0$ and $\theta_i\cdot\theta_j=\frac{4}{3}\delta_{ij}-\frac{1}{3}$.

Then, one proves, order by order in an expansion in powers of g and ε, that the functions $\tilde{\Gamma}_{\rm r}^{(n)}$ are unique, in the sense that they are independent of the special choice of the coefficients of the quadratic terms irrelevant with respect to the gaussian fixed point in $\mathcal{H}_{\rm G}(\phi)$.

This result can be generalized: one can prove, in the sense of formal series, that the contributions of all irrelevant perturbations, from the viewpoint of the gaussian fixed point, go to zero.

(ii) In the language of quantum field theory, the parameters or correlation functions of the initial theory are called bare parameters or correlation functions.

(iii) Physically, the momentum μ is the inverse of the macroscopic distance scale and the parameter g_{r} plays the role of $g(\Lambda/\mu)$, where Λ/μ is the dilatation parameter.

(iv) Beyond the perturbative expansion, the interpretation of renormalized correlation functions is subtle. In the formal renormalization theory, the parameters of the renormalized theory are fixed, and the initial parameters, which are the coefficients of all operators relevant or marginal with respect to the gaussian fixed point, are adjustable parameters and thus vary when the scale factor Λ/μ varies.

This corresponds to applying, in the vicinity of the gaussian fixed point and for $d = 4 - \varepsilon$, a generalization of the strategy that has been already used for the coefficient of ϕ^2 and which allows deriving universal properties in the critical domain.

In this sense, the existence of renormalized functions implies universal properties in the largest possible critical domain near the gaussian fixed point.

However, the existence of such an extended critical domain beyond the perturbative expansion has to be investigated. The study of the RGE provides some information about this question.

Finally, of course, the renormalization theorem has implications for the large distance behaviour of statistical models only if the property of renormalizability remains true at fixed dimension.

Super-renormalizable field theories.

At any fixed dimension $d < 4$, the perturbative expansion has no meaning for $\Lambda \to \infty$, because a factor Λ^{4-d} multiplies the relevant ϕ^4 interaction. Moreover, one verifies that, even at Λ fixed, in the case of the critical or massless theory, the perturbative expansion does not exist due to IR (zero momentum) divergences. In the framework of dimensional continuation, for any $\varepsilon = 4 - d > 0$, one can find an order $k = O(1/\varepsilon)$ in the expansion in powers of g such that some diagrams diverge.

By contrast, for $\tau > 0$ and $g_0 = g\Lambda^{4-d}/\tau^{2-d/2} \sim g\Lambda^{4-d}\xi^{4-d}$ fixed (ξ is the correlation length), correlation functions have a limit for $\Lambda \to \infty$. This condition implies decreasing g as a function of the dilatation parameter with a power appropriated to the gaussian fixed point in order to cancel the dilatation factor, as we have done systematically for the deviation to the critical theory due to the ϕ^2 operator. One then obtains a finite perturbative expansion, and the corresponding SFT is called super-renormalizable.

The interesting theory, from the viewpoint of statistical physics, then corresponds, in general, to the limit $g_0 \to \infty$. However, there exists exceptional situations where such a SFT has a physics application, for example, in the problem of cold and very diluted quantum gases, where the parameter g is naturally very small and simultaneously the coefficients of all interaction terms of higher degree in the field are even more negligible.

Renormalization and dimensional regularization

In the framework of dimensional regularization, the field renormalization as well as the relations between initial (bare) and renormalized parameters have a finite limit for $\Lambda \to \infty$. The corresponding renormalization constants become meromorphic functions of the dimension, singular at the dimensions corresponding to logarithmic divergences in Λ.

Renormalization group equations (RGE)

In the framework of the ε-expansion, that is, in a double series expansion in powers of the parameter g and of ε, the critical behaviour differs from the quasi-gaussian behaviour only by powers of logarithms. These logarithms are organized by the RGE.

RGE for the critical theory

From the existence of renormalized functions in the limit $\Lambda \to \infty$, follows a new equation obtained by differentiation of the equation with respect to Λ at μ, g_{r} fixed:

$$\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\mathrm{r}},\mu \ \text{fixed}} Z^{n/2}(g,\Lambda/\mu)\tilde{\Gamma}^{(n)}(p_i;g,\Lambda) = o(\Lambda^{-2+\upsilon}), \quad \upsilon > 0\,,$$

where the renormalization factor Z has been expressed in terms of g instead of g_{r}.

In agreement with the perturbative philosophy, one first neglects all contributions that, order by order, decay as powers of Λ. This defines asymptotic functions $\tilde{\Gamma}^{(n)}_{\text{as.}}(p_i; g, \Lambda)$ and $Z_{\text{as.}}(g, \Lambda/\mu)$ as sums of the perturbative contributions to the functions $\tilde{\Gamma}^{(n)}(p_i; g, \Lambda)$ and $Z(g, \Lambda/\mu)$, respectively, that do not go to zero when $\Lambda \to \infty$. Such vertex functions can also be obtained by adding to the hamiltonian all possible irrelevant terms and adjusting order by order their amplitudes as functions of g in order to eliminate systematically the contributions that go to zero. The asymptotic functions then satisfy the RGE with a r.h.s. that vanishes exactly. This modification of the effective action is the source of the method of improved action also used in numerical simulations.

Using chain rule, one derives from the renormalization equation,

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g, \Lambda/\mu) \frac{\partial}{\partial g} - \frac{n}{2} \eta(g, \Lambda/\mu)\right] \tilde{\Gamma}^{(n)}_{\mathrm{as.}}(p_i; g, \Lambda) = 0 \,,$$

where the functions β and η are defined by

$$\beta(g, \Lambda/\mu) = \Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\mathrm{r}}, \mu} g \,,$$

$$\eta(g, \Lambda/\mu) = -\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\mathrm{r}}, \mu} \ln Z_{\mathrm{as.}}(g, \Lambda/\mu).$$

Since the two functions are dimensionless, they can depend on g, Λ, μ only through the dimensionless combinations g and Λ/μ. Moreover, the functions β and η can be calculated directly, by using the RGE, in terms of the initial vertex functions, which do not depend on μ. Thus, the functions β and η cannot depend on Λ/μ.

The RGE then take the simpler form (Zinn-Justin 1973)

$$\left(\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}-\frac{n}{2}\eta(g)\right)\tilde{\Gamma}^{(n)}_{\text{as.}}(p_i;g,\Lambda)=0\,.$$

Formulated within the formalism with cut-off Λ, the fundamental idea of the RG becomes: it is possible to modify the parameter Λ and in a correlated way the normalization of the field ϕ and the coefficients of all interactions in a way that leaves all correlation functions invariant.

The RG equations are satisfied by the functions $\tilde{\Gamma}^{(n)}$ asymptotically in the limit $|p_i| \ll \Lambda$. It is possible to verify to all orders in the expansion in g and ε that the neglected terms are of the form $(\ln\Lambda)^L/\Lambda^2$, where the degree L increases with the order.

Notation. In what follows, the vertex functions are implicitly the functions $\tilde{\Gamma}^{(n)}_{\text{as.}}$ without corrections that satisfy the RGE exactly and we omit the subscript as..

Solution of RGE: the ε-expansion

The solution of the RGE, combined with one-loop perturbative calculations, allows proving the universality of the large distance asymptotic behaviour of correlation functions and to determine it to first order in ε.

General solution. The RGE can be solved, for example, by the method of characteristics. One introduces a dilatation parameter λ and one looks for two functions $g(\lambda)$ and $Z(\lambda)$ such that

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}\big(p_i; g(\lambda), \Lambda/\lambda\big)\right] = 0\,.$$

By differentiating explicitly with respect to λ, one finds that the equation is consistent with the RGE if

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda) = -\beta\big(g(\lambda)\big), \qquad g(1) = g\,;$$

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln Z(\lambda) = -\eta\big(g(\lambda)\big), \qquad Z(1) = 1\,.$$

Then,

$$\tilde{\Gamma}^{(n)}(p_i; g, \Lambda) = Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}\big(p_i; g(\lambda), \Lambda/\lambda\big).$$

Remark. This equation is analogous to the renormalization equation if one chooses $\lambda = \Lambda/\mu$ and if one identifies g_{r} with $g(\Lambda/\mu)$, that is, the effective interaction at scale μ, and $Z(\Lambda/\mu)$ with the renormalization of the field. One may then wonder why it was necessary introduce the partial differential equations.

The main reason is the following: it allows showing that the coefficients of the RGE do not depend on the ratio μ/Λ, in contrast to the renormalization constants. This implies properties of renormalization constants that are contained in the renormalization equation only implicitly. This also proves the markovian character of the flow equations, that is, that $\lambda = \Lambda/\mu$ does not appear explicitly in the RGE.

Other form. Multiplying Λ by a factor λ in the solution, one obtains

$$\tilde{\Gamma}^{(n)}(p_i; g, \Lambda\lambda) = Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}\big(p_i; g(\lambda), \Lambda\big).$$

Dimensional considerations imply

$$\tilde{\Gamma}^{(n)}(p_i; g, \Lambda\lambda) = \lambda^{d-(n/2)(d-2)}\tilde{\Gamma}^{(n)}(p_i/\lambda; g, \Lambda).$$

Thus, the solution can be rewritten as

$$\tilde{\Gamma}^{(n)}(p_i/\lambda; g, \Lambda) = \lambda^{-d}\left(\lambda^{(d-2)}Z^{-1}(\lambda)\right)^{n/2}\tilde{\Gamma}^{(n)}\big(p_i; g(\lambda), \Lambda\big).$$

These various equations realize asymptotically (because the terms subleading by powers of Λ have been neglected) the RG general ideas. The parameter $g(\lambda)$ characterizes the effective hamiltonian $\mathcal{H}_\lambda$ at scale λ. However, the function $Z^{1/2}(\lambda)$ defined here differs from the real field RG transformation by a factor $\lambda^{(d-2)/2}$.

Discussion. Integrating, one finds the finite form of the RGE,

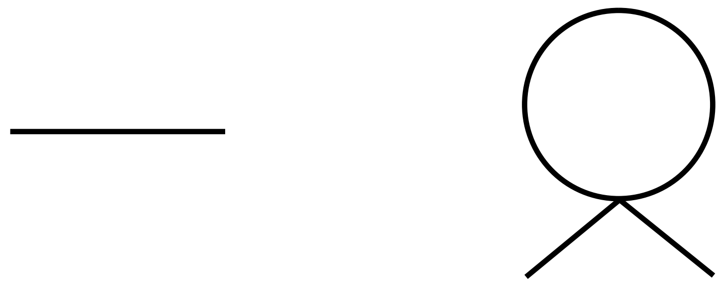
$$\int_g^{g(\lambda)} \frac{\mathrm{d}g'}{\beta(g')} = -\ln\lambda\,, \quad \int_1^{\lambda} \frac{\mathrm{d}s}{s}\eta\big(g(s)\big) = -\ln Z(\lambda).$$

In what follows, we assume explicitly that the RG functions, $\beta(g)$ and $\eta(g)$, are regular functions of g, for $g > 0$.

Since it is equivalent to increase Λ or λ, to study the limit $\Lambda \to \infty$, one must thus determine the behaviour of the amplitude $g(\lambda)$ of the effective interaction for $\lambda \to \infty$. The first equation shows that $g(\lambda)$ increases if the β-function is negative, or decreases in the opposite case.

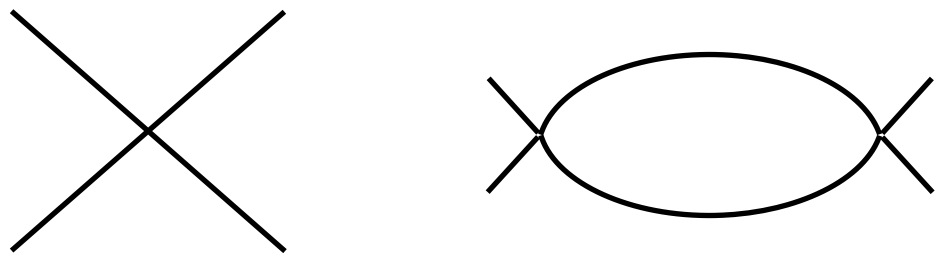
Fixed points correspond to zeros g^* of the β-function which, therefore, play an essential role in the study of the critical behaviour. Those for which the function β has a negative slope are repulsive fixed points in the IR: $g(\lambda)$ moves away from such zeros, except if initially $g(1) = g^*$. On the contrary, those for which the slope is positive, are attractive fixed points from the viewpoint of the large distance behaviour.

Explicit calculations: Feynman at one-loop order



(a) (b)

Fig. 1 The 2-point vertex function: contributions of order 1 and g.



(c) (d)

Fig. 2 The 4-point vertex function: contributions of order g and g^2.

Calculations to one-loop order: Fixed point and scaling properties

β- and η-functions. The RG functions β and η can be calculated perturbatively by expressing that the two- and four-point functions satisfy the RGE. The first correction to the two-point function is a constant (diagram (b)) and induces only a modification of the critical parameter u_{0c} in such a way that

$$\tilde{\Gamma}^{(2)}(p) = p^2 + O(g^2) \ \Rightarrow \ \eta(g) = O(g^2).$$

For the 4-point vertex function, one finds

$$\begin{aligned}\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4) &= \Lambda^{\varepsilon} g \\ &- \tfrac{1}{2} g^2 \Lambda^{2\varepsilon} \left[B_\Lambda(p_1 + p_2) + B_\Lambda(p_1 + p_3) + B_\Lambda(p_1 + p_4)\right] + O(g^3)\end{aligned}$$

with (diagram (d), $\tilde{\Delta}$ is the Fourier transform of the propagator)

$$B_\Lambda(p) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d q \, \tilde{\Delta}(q) \tilde{\Delta}(p - q).$$

At this order, we need B_Λ only for $d = 4$. One guesses that $\Lambda \to \infty$, $B_\Lambda(p) \sim K_B \ln \Lambda$, which is large momentum contribution. Thus,

$$B_\Lambda(p) \underset{\Lambda\to\infty}{\sim} \frac{1}{16\pi^4} \int_{|q|>1} \mathrm{d}^4 q \, \tilde{\Delta}^2(q).$$

We parametrize the propagator with cut-off as

$$\tilde{\Delta}(p) = \frac{D(p/\Lambda)}{p^2}, \quad D(0) = 1, \; D(\infty) = 0.$$

Then,

$$\begin{aligned} K_B = \Lambda \frac{\partial B_\Lambda(p)}{\partial \Lambda} &\underset{\Lambda\to\infty}{\sim} \frac{1}{16\pi^4} \int_{|q|>1} \frac{\mathrm{d}^4 q}{q^4} \Lambda \frac{\partial}{\partial \Lambda} D^2(q/\Lambda) \\ &= -\frac{1}{8\pi^2} \int_{q>1}^{\infty} \frac{\mathrm{d}q}{q} q \frac{\mathrm{d}}{\mathrm{d}q} D^2(q/\Lambda) \\ &= \frac{1}{8\pi^2} D^2(1/\Lambda) \sim \frac{1}{8\pi^2}. \end{aligned}$$

One infers

$$\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4) \underset{\Lambda\to\infty}{=} \Lambda^{\varepsilon} g - \frac{3g^2}{16\pi^2} \ln\Lambda + O(g^2 \times 1, g^2\varepsilon)$$

and, thus,

$$\varepsilon\Lambda^{\varepsilon} g - \frac{3g^2}{16\pi^2} + \beta(g)\Lambda^{\varepsilon} = O(g^3, g^2\varepsilon),$$

which implies

$$\beta(g,\varepsilon) = -\varepsilon g + \frac{3g^2}{16\pi^2} + O\left(g^3, g^2\varepsilon\right).$$

Below four dimensions, if g is initially small, the expression shows that $g(\lambda)$ first increases as a consequence of the instability of the gaussian fixed point. But, within the framework of the ε-expansion, $\beta(g)$ has another zero g^*:

$$\beta(g^*) = 0 \quad \text{for} \quad \mathrm{g}^* = \frac{16\pi^2}{3}\varepsilon + \mathrm{O}\left(\varepsilon^2\right).$$

The asymptotic limit of $g(\lambda)$ is g^*. The local IR stability is ensured by the sign of ω, the slope of $\beta(g)$ at the fixed point g^*:

$$\omega = \beta'(g^*) = \varepsilon + O\left(\varepsilon^2\right) > 0\,.$$

In four dimensions, this fixed point merges with the gaussian fixed point and the eigenvalue ω vanishes indicating the presence of a marginal operator.

From the flow of $Z(\lambda)$ for $\lambda \to \infty$, one then infers

$$\ln Z(\lambda) \underset{\lambda\to\infty}{\sim} -\eta \ln \lambda \quad \text{with } \eta = \eta(\mathrm{g}^*).$$

Correlation functions. In the framework of the ε-expansion, $\tilde{\Gamma}^{(n)}(g^*)$ is finite. Then, the RGE imply

$$\tilde{\Gamma}^{(n)}(p_i/\lambda; g, \Lambda) \underset{\lambda\to\infty}{\propto} \lambda^{-d+(n/2)(d-2+\eta)} \tilde{\Gamma}^{(n)}(p_i; g^*, \Lambda).$$

The equation shows that critical vertex or correlation functions have an asymptotic power law behaviour that does not depend on the initial value of the coefficient g of ϕ^4.

In particular, for $n = 2$, one obtains the behaviour of the inverse of the connected two-point function $\widetilde{W}^{(2)}(p)$. Inverting, one infers

$$\widetilde{W}^{(2)}(p) \equiv \left[\tilde{\Gamma}^{(2)}(p)\right]^{-1} \underset{|p|\to 0}{\propto} 1 \big/ p^{2-\eta}.$$

The spectral representation of the two-point function implies that $\eta > 0$.

The $\eta(g)$ function at two-loop and the exponent η

To determine the first correction to the dimension of the field ϕ, it is necessary to calculate the two-point function $\tilde{\Gamma}^{(2)}(p)$ up to order g^2 for $d = 4$ and, thus, the diagram (f) in figure 3:

$$(\mathrm{e}) = \frac{1}{(2\pi)^8} \int \mathrm{d}^4 q_1 \, \mathrm{d}^4 q_2 \, \tilde{\Delta}(q_1) \tilde{\Delta}(q_2) \tilde{\Delta}(p - q_1 - q_2).$$

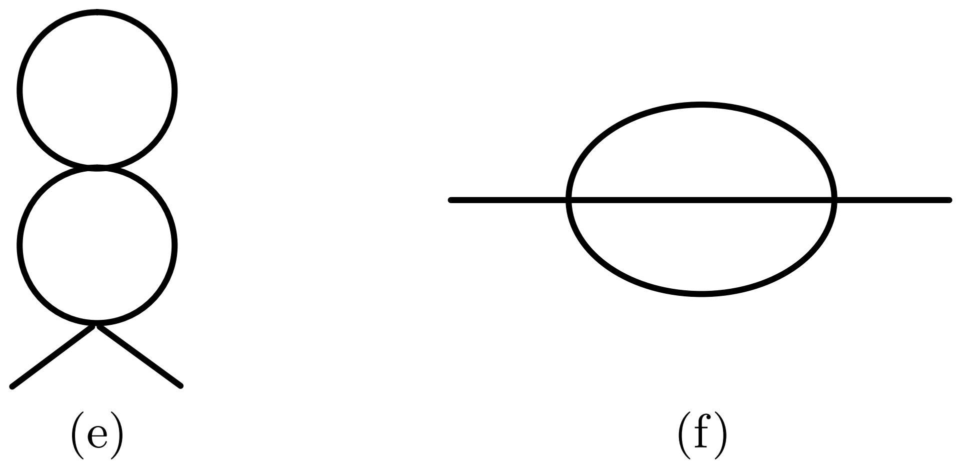


Fig. 3 The 2-point function: contributions of order g^2.

The Feynman diagram can be calculated more easily in position variables where it takes the form

$$(\mathrm{e}) = \int \mathrm{e}^{ipx} \Delta^3(x) \mathrm{d}^4 x \,.$$

In dimension 4, as a function of position variables, we parametrize the propagator in the form

$$\Delta(x) = \frac{\vartheta(\Lambda x)}{4\pi^2 x^2}\,,$$

where the cut-off ensures regularity at short distance:

$$\lim_{x\to\infty}\vartheta(x)=1\,,\quad \vartheta(x)\underset{x\to 0}{\propto}x^2.$$

The condition $\tilde{\Gamma}^{(2)}(0)=0$ determines u_{0c} at order g^2. The vertex function then takes the form

$$\tilde{\Gamma}^{(2)}(p)=p^2-\frac{g^2}{6}Kp^2\ln(\Lambda/p)+O(g^2\times 1,g^2\varepsilon).$$

The coefficient K is given by

$$K=\frac{\partial}{\partial p^2}\Lambda\frac{\partial}{\partial\Lambda}\int \mathrm{e}^{ipx}\Delta^3(x)\mathrm{d}^4x\bigg|_{p=0}\,.$$

We use the identity

$$\sum_{\mu=1}^{4}\left(\frac{\partial}{\partial p_\mu}\right)^2\Phi(p^2)=8\Phi'(p^2)+4p^2\Phi''(p^2).$$

Thus,

$$K = -\frac{1}{8(4\pi^2)^3} \int \frac{\mathrm{d}^4 x}{x^4} \Lambda \frac{\partial}{\partial \Lambda} \vartheta^3(\Lambda x) = -\frac{1}{(4\pi)^4} \int_0^\infty \mathrm{d}x \, \frac{\partial}{\partial x} \vartheta^3(\Lambda x).$$

The integrand is an explicit derivative. Only $x = \infty$ contributes and the result thus is independent of the function ϑ. One finds

$$K = -\frac{1}{(4\pi)^4}\,.$$

One infers

$$\tilde{\Gamma}^{(2)}(p) = p^2 + \frac{1}{24} \frac{g^2}{(8\pi^2)^2} p^2 \ln(\Lambda/p) + O(g^2 \times 1, g^2 \varepsilon).$$

The RGE then implies

$$\frac{g^2}{6(4\pi)^4} p^2 - \eta(g) p^2 = 0\,, \;\Rightarrow\; \eta(g) = \frac{1}{6(4\pi)^4} g^2 + O(g^3).$$

Substituting the value of g^*, one obtains the first correction to the gaussian value of the exponent η:

$$\eta = \frac{\varepsilon^2}{54} + O\left(\varepsilon^3\right).$$

The field ϕ, which at the gaussian fixed point has the canonical dimension $(d-2)/2$, has now the so-called anomalous dimension

$$d_\phi = \tfrac{1}{2}(d - 2 + \eta).$$

Universality

These results call for a few comments. In the framework of the ε-expansion, one shows that all correlation functions have, for $d < 4$, a large distance behaviour different from the one predicted by the quasi-gaussian approximation. Moreover, this critical behaviour does not depend on the initial value of the coefficient g of ϕ^4. At least for $\varepsilon \ll 1$, one can hope that the analysis of leading IR singularities remains valid and, as a consequence, the critical behaviour does not depend on any other parameter in the hamiltonian. The critical behaviour thus is universal, though less universal than in the quasi-gaussian approximation or the mean-field approximations: it depends on a limited number of qualitative characteristic properties of the statistical system.

The critical domain above T_c

We now study the critical domain $\tau > 0$. We thus modify the hamiltonian:

$$\mathcal{H}(\phi) \mapsto \mathcal{H}(\phi) + \frac{\tau}{2} \int \mathrm{d}^d x\, \phi^2(x),$$

where τ, the coefficient of ϕ^2, characterizes the deviation from the critical temperature: $\tau \propto T - T_c$.

The renormalization theorem generalizes to correlation functions of ϕ and ϕ^2, and leads to the appearance of a new renormalization factor $Z_2(\Lambda/\mu, g_{\mathrm{r}})$ associated with the parameter τ. With the same arguments as in the critical situation, one derives a more general RGE of the form

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \frac{n}{2} \eta(g) - \eta_2(g) \tau \frac{\partial}{\partial \tau}\right] \tilde{\Gamma}^{(n)}(p_i; \tau, g, \Lambda) = 0\,,$$

where a new function $\eta_2(g)$ appears. The additional term is proportional to τ since it must vanish for $\tau = 0$.

The dimensionless function η_2 may still have a regular dependence in the ratio τ/Λ^2, but we have neglected such a possible dependence for the same reason that we have already neglected all other contributions of the same order.

To determine $\eta_2(g)$, one can calculate the two-point function and apply the RGE. At order g, one finds (diagram (b))

$$\tilde{\Gamma}^{(2)}(p,\tau) = p^2 + \tau + \frac{g}{32\pi^4}\int^{\Lambda} \mathrm{d}^4 q \left(\frac{1}{q^2+\tau} - \frac{1}{q^2}\right) + \cdots \,.$$

Using

$$\frac{1}{32\pi^2}\int^{\Lambda} \mathrm{d}^4 q \left(\frac{1}{q^2+\tau} - \frac{1}{q^2}\right) \sim -\frac{\tau}{16\pi^2}\ln(\Lambda/\sqrt{\tau}),$$

and applying the RGE, one finds

$$-\frac{g\tau}{16\pi^2} - \eta_2(g)\tau = 0 \;\Rightarrow\; \eta_2(g) = -\frac{g}{16\pi^2} + O(g^2).$$

Solution of RGE

To study the critical behaviour above T_c, one again integrates the RGE by the method of characteristics. In addition to the functions $g(\lambda)$ and $Z(\lambda)$, one must now introduce a new function $\tau(\lambda)$. Solving the RGE, one obtains an asymptotic behaviour of the form

$$\tilde{\Gamma}^{(n)}(p_i; \tau, g, \Lambda = 1) \underset{\substack{\tau \ll 1 \\ |p_i| \ll 1}}{\sim} m^{d-n(d-2+\eta)/2} F_+^{(n)}(p_i/m),$$

$$m(\Lambda = 1) \propto \xi^{-1} \propto t^{\nu}$$

the exponent ν being related to the function $\eta_2(g)$ by $\nu = [\eta_2(g^*) + 2]^{-1}$.

Thus, one has proved a general scaling form in the critical domain. Moreover, one has obtained the singular behaviour at T_c of the parameter m, which is proportional to the physical mass or the inverse of the correlation length ξ. The divergence of the correlation length ξ at T_c is thus characterized by the exponent ν.

For $\tau > 0$, correlation functions are finite at zero momentum and thus behave like

$$\tilde{\Gamma}^{(n)}(0; \tau, g, \Lambda) \propto \tau^{\nu(d-n(d-2+\eta)/2)}.$$

In particular, for $n = 2$, one obtains the inverse of the magnetic susceptibility χ:

$$\chi^{-1} = \tilde{\Gamma}^{(2)}(p = 0; \tau, g, \Lambda) \propto \tau^{\nu(2-\eta)}.$$

The divergence of χ is characterized by an exponent that is usually denoted by γ. This establishes the relation between exponents

$$\gamma = \nu(2 - \eta).$$

ϕ and ϕ^2 vertex functions

We denote by $\Gamma^{(\ell,n)}$ the vertex (or 1PI) functions of ℓ $\boldsymbol{\phi}^2$ and n $\boldsymbol{\phi}$ fields in the Fourier representation. In the symmetric phase ($\tau \geq 0$) in zero field, the $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^2$ vertex functions then satisfy the RG equations:

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \frac{n}{2}\eta(g) - \left(\tau\frac{\partial}{\partial\tau} + \ell\right)\eta_2(g)\right]\tilde{\Gamma}^{(\ell,n)} = \delta_{n0}\delta_{\ell 2}\Lambda^{d-4}B(g),$$

where $B(g)$ is associated to the additive renormalization of the ϕ^2 2-point function. The equation can be solved by a slight extension of the method used so far.

For $\ell = 2, n = 0$ and $p = 0$, one infers the behaviour of the singular part of the specific heat,

$$\mathcal{C} \propto \tau^{-\alpha},$$

with the relation $\alpha = 2 - d\nu$.

$(\phi^2)^2$ field theory and RG near $d = 4$

For later purpose, let us indicate how the results generalize to $O(N)$ symmetric systems. In terms of the initial N-component scalar field $\boldsymbol{\phi}$, we write the action as (omitting higher order derivatives)

$$\mathcal{S}(\boldsymbol{\phi}) = \int \left\{ \frac{1}{2} \left[\partial_\mu \boldsymbol{\phi}(x)\right]^2 + \frac{1}{2} u_0 \boldsymbol{\phi}^2(x) + \frac{1}{4!} g \left[\boldsymbol{\phi}^2(x)\right]^2 \right\} \mathrm{d}^d x \,.$$

The RG β-function in dimension $d = 4 - \varepsilon$,

$$\beta(g, \varepsilon) = -\varepsilon g + \frac{N+8}{48\pi^2} g^2 + O(g^3),$$

has again for $d < 4$ a non-trivial IR attractive zero

$$g^* = \frac{48\pi^2 \varepsilon}{N+8} + O(\varepsilon^2), \;\Rightarrow\; \beta'(g^*) \equiv \omega = \varepsilon + O(\varepsilon^2) > 0 \;\; .$$

Practical calculations: ε expansion and fixed dimension scheme

The easiest quantities to calculate by field theory methods are critical exponents, although some results also exist for the scaling equation of state. The exponents are known up to order ε^5. However, it can be shown by semi-classical analysis (instantons), that the series are divergent and summation methods are required to extract useful numbers from the series.

Following Parisi'suggestion, one can also evaluate the β-function directly in dimension 3 but, then, one has no longer a "small" expansion parameter. However, it has been noticed by Nickel that Feynman diagrams in dimension 3 can be more easily evaluated than near dimension 4. At present, Nickel has managed to calculate all diagrams up to seven loops (in the terminology of Feynman diagrams) contributing to η, η_2, but the diagrams contributing to the β-function, which are more difficult, only up to six loops.

For example, to six loop order, for $N = 1$, Nickel has obtained

$$\beta(\tilde{g}) = -\tilde{g} + \tilde{g}^2 - \tfrac{308}{729}\tilde{g}^3 + 0.3510695978\tilde{g}^4 - 0.3765268283\tilde{g}^5 + 0.49554751\tilde{g}^6 - 0.749689\tilde{g}^7 + O\left(\tilde{g}^8\right),$$

where $\tilde{g} = 3g/(16\pi))$.

One must first determine numerically the zero of the β-function, which is a number of order 1. This clearly requires some summation of the series. Since the series again are divergent, summation is based on Borel transformation techniques (Le Guillou and ZJ, Guida and ZJ).

Critical exponents from the $O(N)$ symmetric $(\phi^2)^2_3$ field theory

N	0	1	2	3
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
α	0.235 ± 0.003	0.109 ± 0.004	-0.011 ± 0.004	-0.122 ± 0.010
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
$\omega\nu$	0.478 ± 0.010	0.504 ± 0.008	0.529 ± 0.009	0.553 ± 0.012

Critical exponents from $O(N)$ symmetric lattice models

N	0	1	2	3
γ	1.1575 ± 0.0006	1.2385 ± 0.0025	1.322 ± 0.005	1.400 ± 0.006
ν	0.5877 ± 0.0006	0.631 ± 0.002	0.674 ± 0.003	0.710 ± 0.006
α	0.237 ± 0.002	0.103 ± 0.005	-0.022 ± 0.009	-0.133 ± 0.018
β	0.3028 ± 0.0012	0.329 ± 0.009	0.350 ± 0.007	0.365 ± 0.012
$\omega\nu$	0.56 ± 0.03	0.53 ± 0.04	0.60 ± 0.08	0.54 ± 0.10

Scalar field theory: the large N formalism

One considers a general $O(N)$ symmetric euclidean action (or classical hamiltonian) for an N-component scalar field $\boldsymbol{\phi}$ of the form

$$\mathcal{S}(\boldsymbol{\phi}) = \int \left[\tfrac{1}{2}\left[\partial_\mu \boldsymbol{\phi}(x)\right]^2 + NU\left(\boldsymbol{\phi}^2(x)/N\right)\right] \mathrm{d}^d x \,,$$

where $U(\rho)$ is a polynomial and the explicit N dependence has been chosen to lead to a large N limit. The corresponding partition function is given by the field integral

$$\mathcal{Z} = \int [\mathrm{d}\boldsymbol{\phi}(x)] \exp\left[-\mathcal{S}(\boldsymbol{\phi})\right].$$

To render the perturbative expansion finite, a cut-off Λ consistent with the symmetry is implied, implemented by the addition of quadratic terms with higher derivatives to the action.

The solution of the model in the large N limit is inspired by the central limit theorem: it can be expected that, for N large, $O(N)$ invariant quantities like

$$\boldsymbol{\phi}^2(x) = \sum_{i=1}^{N} \phi_i^2(x)$$

self-average and therefore have small fluctuations (as for the central limit theorem this relies on the assumption that the components ϕ_i are somehow weakly correlated). Thus, for example,

$$\left\langle \boldsymbol{\phi}^2(x)\boldsymbol{\phi}^2(y) \right\rangle \underset{N\to\infty}{\sim} \left\langle \boldsymbol{\phi}^2(x) \right\rangle \left\langle \boldsymbol{\phi}^2(y) \right\rangle .$$

This observation suggests to take $\boldsymbol{\phi}^2(x)$ as a dynamical variable, rather than $\boldsymbol{\phi}(x)$ itself.

Implementation. To implement the idea, one introduces two auxiliary fields λ and ρ and impose the constraint $\rho(x) = \boldsymbol{\phi}^2(x)/N$ by an integral over λ. For each point of space x, one uses the identity

$$1 = N \int \mathrm{d}\rho\, \delta(\boldsymbol{\phi}^2 - N\rho) = \frac{N}{4i\pi} \int \mathrm{d}\rho \mathrm{d}\lambda \mathrm{e}^{\lambda(\boldsymbol{\phi}^2 - N\rho)/2},$$

where the λ-integration contour runs parallel to the imaginary axis.

The insertion of the identity into the field integral yields a new representation of the partition function:

$$\mathcal{Z} = \int [\mathrm{d}\boldsymbol{\phi}][\mathrm{d}\rho][\mathrm{d}\lambda] \exp\left[-\mathcal{S}(\boldsymbol{\phi}, \rho, \lambda)\right]$$

with

$$\mathcal{S}(\boldsymbol{\phi}, \rho, \lambda) = \int \left[\tfrac{1}{2}\left[\partial_\mu \boldsymbol{\phi}(x)\right]^2 + NU\big(\rho(x)\big) + \tfrac{1}{2}\lambda(x)\big(\boldsymbol{\phi}^2(x) - N\rho(x)\big)\right] \mathrm{d}^d x\,.$$

The field integral is then gaussian in $\boldsymbol{\phi}$, the integral can be performed and the dependence of the partition function on N becomes explicit.

Actually, it is convenient to separate in $\boldsymbol{\phi}(x)$ one component, setting

$$\boldsymbol{\phi}(x) = \{N^{1/2}\sigma(x), \boldsymbol{\pi}(x)\},$$

and to integrate over the $N-1$ components $\boldsymbol{\pi}$ only. This does not affect the large N limit but only the $1/N$ corrections.

To generate σ-correlation functions, one also adds a source $N^{1/2}H(x)$ (a space-dependent magnetic field in the ferromagnetic language) to the action. The partition function then becomes

$$\mathcal{Z}(H) = \int [\mathrm{d}\sigma][\mathrm{d}\rho][\mathrm{d}\lambda] \exp\left[-\mathcal{S}_N(\sigma,\rho,\lambda) + N\int \mathrm{d}^d x\, H(x)\sigma(x)\right]$$

with

$$\begin{aligned}\mathcal{S}_N(\sigma,\rho,\lambda) = & N\int \left[\tfrac{1}{2}\left(\partial_\mu\sigma\right)^2 + U(\rho) + \tfrac{1}{2}\lambda(x)(\sigma^2(x) - \rho(x))\right]\mathrm{d}^d x \\ & + \tfrac{1}{2}(N-1)\mathrm{tr}\,\ln\left[-\nabla^2 + \lambda(\bullet)\right].\end{aligned}$$

In the large N limit, $\mathcal{S}_N$ is of order N and the integral can thus be evaluated by the steepest descent method.

Remarks

(i) In this formalism it is natural to consider also ρ-field correlation functions. The ρ-field is proportional to the $\boldsymbol{\phi}^2$ composite field that, near the critical temperature, plays the role of the energy operator.

(ii) One can solve also more general $O(N)$ symmetric field theory with two derivatives. Indeed, this involves adding the two terms

$$Z(\boldsymbol{\phi}^2/N)(\partial_\mu \boldsymbol{\phi})^2, \quad V(\boldsymbol{\phi}^2/N)(\partial_\mu \boldsymbol{\phi}\cdot\boldsymbol{\phi})^2/N,$$

where Z and V are two arbitrary functions. These terms can be rewritten

$$Z(\rho)(\partial_\mu \boldsymbol{\phi})^2, \quad NV(\rho)(\partial_\mu \rho)^2,$$

in such a way that the $\boldsymbol{\phi}$ integral remains gaussian and can be performed. This reduces again the study of the large N limit to the steepest descent method.

Large N limit: saddle points and phase transitions

The function $U(\rho)$ being considered as N independent, in the large N limit, the field integral can be calculated by the steepest descent method.

Saddle points. We look for a uniform saddle points ($\sigma(x), \rho(x), \lambda(x)$ are space-independent)

$$\sigma(x) = \sigma\,, \quad \rho(x) = \rho \quad \text{and} \quad \lambda(x) = m^2$$

because the λ saddle point value must be positive.

The action density $\mathcal{E}$ in zero field H per degree of freedom ($\mathcal{S}_N/N/\text{vol.}$) then becomes

$$\mathcal{E} = U(\rho) + \tfrac{1}{2}m^2(\sigma^2 - \rho) + \frac{1}{2}\int^\Lambda \frac{\mathrm{d}^d k}{(2\pi)^d}\, \ln[(k^2 + m^2)/k^2],$$

where the symbol $\int^\Lambda$ means calculated with a regularized propagator inducing a cut-off at scale Λ.

Differentiating then $\mathcal{E}$ with respect to σ, ρ and m^2, respectively, one obtains the saddle point equations

$$m^2\sigma = 0\,, \quad \tfrac{1}{2}m^2 = U'(\rho), \quad \sigma^2 - \rho + \Omega_d(m) = 0$$

with

$$\Omega_d(m) = \frac{1}{(2\pi)^d}\int^{\Lambda} \frac{\mathrm{d}^d k}{k^2+m^2}$$

Below we need the first terms of the expansion of $\Omega_d(m)$ for $m^2 \to 0$. One finds for $m \to 0$ and $d > 2$ an expansion which we parametrize as

$$\Omega_d(m) = \Omega_d(0) - K(d)m^{d-2} + a(d)m^2\Lambda^{d-4} + O\left(m^4\Lambda^{d-6}, m^d\Lambda^{-2}\right).$$

The constant $K(d)$ is universal, that is, independent of the cut-off procedure:

$$K(d) = -\frac{1}{(4\pi)^{d/2}}\Gamma(1-d/2)\,.$$

The constant $a(d)$, in contrast, depends explicitly on the regularization but for $\varepsilon = 4 - d \to 0$ satisfies $a(d) \sim 1/(8\pi^2\varepsilon)$.

Integrating $\Omega_d(m)$ over m^2, one then obtains a finite expression for the regularized integral arising from the $\boldsymbol{\phi}$ integration:

$$\frac{1}{(2\pi)^d}\int^{\Lambda} \mathrm{d}^d k \; \ln[(k^2+m^2)/k^2] = \int_0^m 2s\mathrm{d}s\,\Omega_d(s).$$

One infers (except for $d=4$ where a logarithm is generated)

$$\int_0^m 2s\mathrm{d}s\,\Omega_d(s) = -2\frac{K(d)}{d}m^d + \Omega_d(0)m^2 + \frac{a(d)}{2}m^4\Lambda^{d-4} + O(m^6\Lambda^{d-6}, m^{d+2}\Lambda^{-2}).$$

Phase transitions

We recall the saddle point equations:

$$m^2\sigma = 0\,, \quad \tfrac{1}{2}m^2 = U'(\rho), \quad \sigma^2 - \rho + \Omega_d(m) = 0\,.$$

The first equation implies either $\sigma = 0$ or $m = 0$. We see from the $\operatorname{tr}\ln$ term that m, at this order, is also the mass of the $\boldsymbol{\pi}$ field.

If $\sigma = 0$ the $O(N)$ symmetry is unbroken and the N $\boldsymbol{\phi}$-field components have the same mass m.

Instead, when $\sigma \neq 0$, the $O(N)$ symmetry is spontaneously broken, $m = 0$ and the massless $\boldsymbol{\pi}$-field corresponds to the expected $N-1$ Goldstone modes.

The solution $m = 0$ can exist only for $d > 2$, because at $d = 2$ the integral is IR divergent, a result consistent with Mermin–Wagner–Coleman's theorem, which forbids for $d \leq 2$ a phase transition with order in a system with short range forces and a continuous symmetry.

Thus, below, we discuss only the dimensions $d > 2$; the dimension $d = 2$ can be examined more directly in the formalism of the non-linear σ model.

We now assume that the polynomial $U(\rho)$ has for $\rho \geq 0$ a unique minimum at a strictly positive value of ρ where $U''(\rho)$ does not vanish, otherwise the critical point would turn out to be a multicritical point.

(i) *Broken phase.* When $m = 0$, the saddle point equations reduce to

$$U'(\rho) = 0\,, \quad \sigma^2 - \rho + \Omega_d(0) = 0\,.$$

The first equation implies that ρ is given by the minimum of U and the second equation then determines the field expectation value. Clearly a solution can be found only if

$$\rho > \rho_c = \Omega_d(0),$$

and then

$$\sigma = \sqrt{\rho - \rho_c}.$$

(ii) *The symmetric phase.* When $\sigma = 0$, the saddle point equations can be written as

$$\rho - \rho_c = \Omega_d(m) - \Omega_d(0), \quad m^2 = 2U'(\rho)\,.$$

The first saddle point equation implies $\rho \leq \rho_c$. At the value $\rho = \rho_c$ a transition takes place between an ordered phase $\rho > \rho_c$ and a symmetric phase $\rho \leq \rho_c$. The condition

$$U'(\rho_c) = 0$$

determines critical potentials.

From the large N action, we infer that the σ-propagator then becomes

$$\Delta_\sigma \underset{|p|,m \ll \Lambda}{\sim} \frac{1}{p^2 + m^2}\,.$$

Therefore, m is at this order the physical mass or the inverse of the correlation length ξ of the field σ (and thus of all components of the $\boldsymbol{\phi}$-field).

The condition $m \ll \Lambda$, (i.e. $\xi \gg 1/\Lambda$) defines the critical domain. The first equation then implies that $\rho - \rho_c$ is small in the critical domain.

From the second equation follows that $U'(\rho)$ is small and thus ρ is close to the minimum of $U(\rho)$. We can then expand $U(\rho)$ around ρ_c:

$$U(\rho) = U'(\rho_c)(\rho - \rho_c) + \tfrac{1}{2} U''_c (\rho - \rho_c)^2 + O\big((\rho - \rho_c)^3\big),$$

and it is convenient to set

$$U'(\rho_c) = \tfrac{1}{2}\tau \,, \quad |\tau| \ll \Lambda^2 \,.$$

With this parametrization

$$m^2 = 2U''_c(\rho - \rho_c) + \tau + O\big((\rho - \rho_c)^2\big).$$

With our assumptions U''_c is strictly positive (the sign ensures that the extremum is a minimum). Then τ is positive in the symmetric phase, while $\tau < 0$ corresponds to the broken phase.

At this point we realize that, in the case of a generic critical point, $U(\rho)$ can be approximated by a quadratic polynomial. The problem then reduces to a discussion of the $(\boldsymbol{\phi}^2)^2$ field theory that indeed, in the framework of the $\varepsilon = (4-d)$-expansion, describes generic critical phenomena.

$(\phi^2)^2$ field theory in the large N limit

From now on, the discussion will be specific to the $(\phi^2)^2$ field theory. In terms of the initial N-component scalar field ϕ, we write the action as

$$\mathcal{S}(\phi) = \int \left\{ \frac{1}{2} \left[\partial_\mu \phi(x)\right]^2 + \frac{1}{2} r \phi^2(x) + \frac{1}{4!} \frac{u}{N} \left[\phi^2(x)\right]^2 \right\} \mathrm{d}^d x$$

with u N-independent. The action corresponds to the function

$$U(\rho) = \tfrac{1}{2} r\rho + \tfrac{1}{4!} u\rho^2 .$$

In the large N action, the integral over ρ is then gaussian. The integration results in simply replacing $\rho(x)$ by the solution of

$$\tfrac{1}{6} u\rho(x) + r = \lambda(x).$$

One then obtains the large N action $\mathcal{S}_N(\sigma, \lambda)$,

$$\mathcal{S}_N = \frac{N}{2} \int \mathrm{d}^d x \left[(\partial_\mu \sigma)^2 + \lambda \sigma^2 - \frac{3}{u} \lambda^2 + \frac{6r}{u} \lambda \right] + \frac{(N-1)}{2} \operatorname{tr} \ln \left[-\nabla^2 + \lambda(\bullet) \right].$$

Diagrammatic interpretation. In the $(\phi^2)^2$ field theory, the leading perturbative contributions in the large N limit come from chains of 'bubble' diagrams of the form displayed in the figure. These diagrams asymptotically form a geometric series, which the algebraic techniques described above allow to sum.

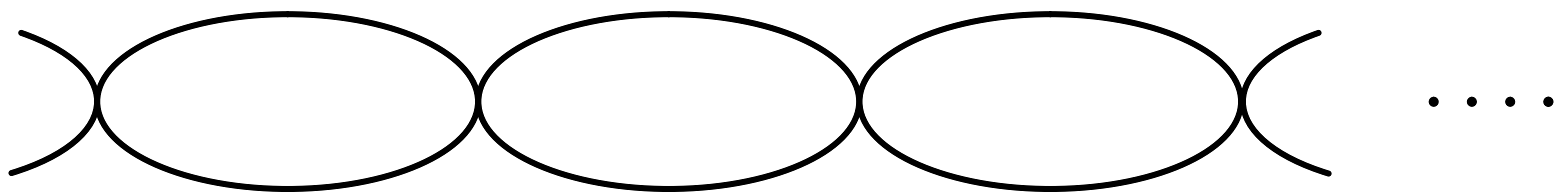


Fig. 4 The dominant diagrams in the large N limit.

The low temperature phase. We first assume that $\sigma = \langle\phi\rangle N^{-1/2}$ does not vanish, and thus the $O(N)$ symmetry is spontaneously broken. The constant ρ is given by the second saddle point equation, which reduces to $U'(\rho) = \frac{1}{2}r + \frac{u}{12}\rho = 0$.

The solution must satisfy $\rho > \rho_c$ and then $\sigma = \sqrt{\rho - \rho_c}$.

The critical potential U satisfies:

$$U'(\rho_c) = 0 \ \Rightarrow \ r = r_c = -u\rho_c/6 \, .$$

The expectation value of the field vanishes for $r = r_c$, which thus corresponds to the critical temperature T_c. Then, setting $r - r_c = \tau$,

$$U'(\rho) = \tfrac{1}{2}\tau + \tfrac{u}{12}(\rho - \rho_c) = 0 \ \Rightarrow \ \rho - \rho_c = -(6/u)\tau \, .$$

The symmetry is broken for $\tau < 0$, that is at low temperature, and then

$$\sigma^2 = -(6/u)\tau \propto (-\tau)^{2\beta} \quad \text{with} \quad \beta = \tfrac{1}{2} \, .$$

For $N \to \infty$, the exponent β remains quasi-gaussian or mean-field like in all dimensions.

The high temperature phase. For $\tau > 0$ (i.e., above T_c), σ vanishes. One then finds

$$m^2 = 2U'(\rho) = (u/6)(\rho - \rho_c) + \tau\,, \quad \rho - \rho_c = \Omega_d(m) - \Omega_d(0).$$

(i) For $d > 4$, the leading contribution to $\rho - \rho_c$ is proportional to m^2, and thus, at leading order, $m^2 = \xi^{-2} \sim \tau^{2\nu}$ with $\nu = \frac{1}{2}\,,$
which is the gaussian or mean-field result for the correlation exponent ν.

(ii) For $2 < d < 4$, the leading term now is of order m^{d-2}:

$$\rho - \rho_c \sim -K(d)m^{d-2}.$$

The leading m-dependent contribution for $m \to 0$ now comes from $\rho - \rho_c$. Keeping only the leading term, one obtains ($\varepsilon = 4 - d$)

$$m = \xi^{-1} \sim \tau^{1/(2-\varepsilon)}, \ \Rightarrow \ \nu = \frac{1}{2-\varepsilon} = \frac{1}{d-2}\,,$$

which shows that the exponent ν is no longer gaussian (or mean-field like).

(iii) For $d = 4$, the leading m-dependent contribution still comes from $\rho - \rho_c$:

$$m^2 \sim \frac{48\pi^2}{u} \frac{\tau}{\ln(\Lambda/m)}.$$

The correlation length has no longer a power law behaviour but, instead, the behaviour of the gaussian model modified by a logarithm. This is typical of a situation where the gaussian fixed point is stable, in the presence of a marginal operator.

(iv) Examining the third saddle point equation for $\sigma = 0$ and $d = 2$, one finds that the correlation length becomes large only for $r \to -\infty$. This peculiar situation can be better discussed in the framework of the non-linear σ-model.

Critical limit $\tau = 0$. At $\tau = 0$, m vanishes and the form of the σ-propagator implies the critical exponent η remains gaussian for all d:

$$\eta = 0 \ \Rightarrow \ d_\phi = \tfrac{1}{2}(d-2)\,.$$

One then verifies that for $d \le 4$, the exponents β, ν, η satisfy the scaling relation proved within the framework of the ε-expansion:

$$\beta = \nu d_\phi = \tfrac{1}{2}\nu(d-2+\eta).$$

Singular free energy and scaling equation of state. In a constant magnetic field $HN^{1/2}$ in the σ direction, the free energy density per degree of freedom $W(H)$ is given by (Ω is the volume)

$$\begin{aligned} W(H) &= \ln \mathcal{Z}/N\Omega = -\mathcal{E} \\ &= \frac{3}{2u}m^4 - \frac{3r}{u}m^2 - \frac{1}{2}m^2\sigma^2 + H\sigma - \int_0^m s\mathrm{d}s\,\Omega_d(s). \end{aligned}$$

The saddle point values m^2, σ are given by the second saddle point equation and the first modified saddle point:

$$m^2\sigma = H\,.$$

The magnetization M, expectation value of $\phi/\sqrt{N}$, is

$$M = \frac{\partial W}{\partial H} = \sigma\,,$$

because partial derivatives of W with respect to m^2 and σ vanish as a consequence of the saddle point equations. The thermodynamic potential density $\mathcal{G}(M)$, Legendre transform of $W(H)$, follows:

$$\mathcal{G}(M) = HM - W(H) = -\frac{3}{2u}m^4 + \frac{3r}{u}m^2 + \frac{1}{2}m^2M^2 + \int_0^m s\mathrm{d}s\,\Omega_d(s).$$

As a property of the Legendre transformation, the saddle point equation for m^2 is now obtained by expressing that the derivative of $\mathcal{G}$ with respect to m^2 vanishes.

Introducing r_c and $u^* = 6\Lambda^{-\varepsilon}/a(d)$, one obtains

$$\mathcal{G}(M) = \frac{3}{2}\left(\frac{1}{u^*} - \frac{1}{u}\right)m^4 + \frac{3(r - r_c)}{u}m^2 + \frac{1}{2}m^2M^2 - \frac{K(d)}{d}m^d .$$

For $d < 4$, the term proportional to m^4 is negligible for m small with respect to the singular term m^d. Thus, at leading order in the critical domain,

$$\mathcal{G}(M) = \frac{3}{u}\tau m^2 + \frac{1}{2}m^2M^2 - \frac{K(d)}{d}m^d .$$

One then expresses that the derivative with respect to m^2 vanishes,

$$(6/u)\tau + M^2 - K(d)m^{d-2} = 0\,, \Rightarrow\ m = \left[\left((6/u)\tau + M^2\right)/K(d)\right]^{1/(d-2)} .$$

It follows that the leading contribution to the thermodynamic potential, in the critical domain, is given by

$$\mathcal{G}(M) \propto \left[(6/u)\tau + M^2\right]^{d/(d-2)} .$$

Differentiating with respect to M, one obtains an equation of state with a scaling form:

$$H = \frac{\partial \mathcal{G}}{\partial M} \propto M^{(d+2)/(d-2)} \left(1 + (6/u)\tau/M^2\right)^{2/(d-2)} \equiv M^\delta f(\tau/M^{1/\beta}).$$

One finds a value of the exponent $\delta = (d+2)/(d-2)$ consistent with the general scaling relation $\delta = d/d_\phi - 1$ and $f(x) = (1+ax)^{2/(d-2)}$, and recovers $\beta = 1/2$.

Leading corrections to scaling. The m^4 term yields the leading corrections to scaling. It is subleading by a power of τ, $m^4/m^d = O(\tau^{(4-d)/(d-2)})$.

The exponent governing the leading corrections to scaling in the temperature variable is $\omega\nu$ and thus

$$\omega\nu = (4-d)/(d-2) \;\Rightarrow\; \omega = 4-d\,.$$

For the special value $u = u^*$, this correction vanishes, a remark relevant only if u^* and thus $a(d)$ are positive.

Specific heat exponent. Differentiating twice $\mathcal{G}(M)]$ with respect to τ, one obtains the scaling specific heat at fixed magnetization

$$\mathcal{C} \propto \left\{\tau \left[1 + (u/6)M^2/\tau\right]\right\}^{(4-d)/(d-2)} \ \Rightarrow \ \alpha = \frac{4-d}{d-2} \equiv 2 - d\nu \,.$$

The λ and ϕ^2 two-point functions. In the high temperature phase, differentiating twice the large N action with respect to $\lambda(x)$ and setting $\lambda(x) = m^2$, one obtains the λ-propagator

$$\Delta_\lambda(p) \propto \left[(6/u) + B_\Lambda(p, m)\right]^{-1} \,,$$

where $B_\Lambda(p, m)$ is the bubble diagram of Fig. 5:

$$B_\Lambda(p, m) = \frac{1}{(2\pi)^d} \int^\Lambda \frac{\mathrm{d}^d q}{(q^2 + m^2)\left[(p-q)^2 + m^2\right]} .$$

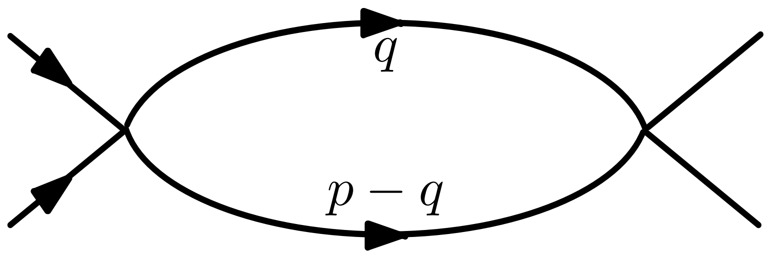


Fig. 5 The ‘bubble’ diagram $B_\Lambda(p, m)$.

In the critical theory ($m = 0$ at this order) for $2 \le d < 4$, the denominator is dominated for $p \to 0$ by the integral

$$B_\Lambda(p, 0) = \frac{1}{(2\pi)^d} \int^\Lambda \frac{\mathrm{d}^d q}{q^2 (p-q)^2} \underset{2<d<4}{=} b(d) p^{-\varepsilon} - a(d) \Lambda^{-\varepsilon} + O\left(\Lambda^{d-6} p^2\right),$$

where

$$b(d) = K(d) \frac{\Gamma^2(d/2)}{\Gamma(d-1)},$$

and thus,

$$\Delta_\lambda(p) \underset{p \to 0}{\propto} p^\varepsilon .$$

This behaviour implies that, in the large N limit the *dimension of the field λ is*

$$[\lambda] = [\phi^2] = d - 1/\nu = \tfrac{1}{2}(d + \varepsilon) = 2,$$

a result important for the $1/N$ perturbation theory.

Remarks.

(i) For $d = 4$, the integral has a logarithmic behaviour:

$$B_\Lambda(p,0) \underset{p \ll \Lambda}{\sim} \frac{1}{8\pi^2} \ln(\Lambda/p) + \text{ const.},$$

and still gives the leading contribution to the propagator $\Delta_\lambda \propto 1/\ln(\Lambda/p)$.

(ii) Therefore, that for $d \le 4$ the contributions generated by the term proportional to $\lambda^2(x)$ in the large N action are always negligible in the critical domain. Neglecting the λ^2 term in the large N action and reintroducing the initial field $\boldsymbol{\phi}$, one discovers that the $(\phi^2)^2)$ model is equivalent to the non-linear σ-model for what concerns large distance properties.

Leading corrections to scaling

In the free energy the leading corrections to scaling vanish for $u = u^*$ (provided u^* and thus $a(d)$ are positive). Similarly, in the correlation length or mass m the leading corrections again vanish for $u = u^*$:

$$1 - \frac{u}{u^*} + (u/6)K(d)m^{-\varepsilon} + O\left(m^{2-\varepsilon}\Lambda^{-2}\right) = \frac{\tau}{m^2}\,.$$

The same property holds for the leading correction to the λ-propagator in the critical theory which cancels for $u = u^*$:

$$\Delta_\lambda(p) = -\frac{2}{N}\left[\frac{6}{u} - \frac{6}{u^*} + b(d)p^{-\varepsilon}\right]^{-1}.$$

Actually, all correction terms suppressed by powers of order ε for $d \to 4$ vanish simultaneously as expected from the RG analysis of the ϕ^4 field theory if u^* is an IR fixed point. Moreover, one verifies that the leading correction is proportional to $(u-u^*)\tau^{\varepsilon/(2-\varepsilon)}$, which leads to $\omega\nu = \varepsilon/(2-\varepsilon)$.

Finally, one then verifies that for $\varepsilon \to 0$, the value of u^* coincides with the ε-expansion of the fixed point value for N large.

For a regularized propagator of the form

$$\tilde{\Delta}_\Lambda(k) = \frac{1}{k^2 D(k^2/\Lambda^2) + m^2}$$

The constant $a(d)$, which depends explicitly on the regularization, that is on the way large momenta are cut, is given by

$$a(d) = N_d \times \begin{cases} \displaystyle\int_0^\infty k^{d-5} \mathrm{d}k \left(1 - \frac{1}{D^2(k^2)}\right) & \text{for } d < 4, \\ \displaystyle -\int_0^\infty \frac{k^{d-5} \mathrm{d}k}{D^2(k^2)} & \text{for } d > 4. \end{cases}$$

It is not necessarily positive, except near $d = 4$ since for $\varepsilon = 4 - d \to 0$ it satisfies

$$a(d) \underset{\varepsilon = 4 - d \to 0}{\sim} 1/(8\pi^2 \varepsilon).$$

RG functions

For a more detailed verification of the consistency between the large N results and RG predictions, we now calculate RG functions at leading order for $N \to \infty$. We set

$$u = Ng\Lambda^{\varepsilon}, \quad g^* = u^* \Lambda^{-\varepsilon}/N = 6/(Na)\,,$$

where the constant $a(d)$ behaves for $\varepsilon = 4 - d \to 0$ like $a(d) \sim 1/(8\pi^2 \varepsilon)$.

One then verifies that m satisfies asymptotically for Λ large an equation that expresses that it is RG invariant:

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta(g)\frac{\partial}{\partial g} - \eta_2(g)\tau \frac{\partial}{\partial \tau}\right) m(\tau, g, \Lambda) = 0\,,$$

where in the r.h.s. contributions of order $1/\Lambda^2$ have been neglected. The RG functions $\beta(g)$ and $\eta_2(g)$ are then given by

$$\beta(g) = -\varepsilon g(1 - g/g^*),$$

$$\nu^{-1}(g) = 2 + \eta_2(g) = 2 - \varepsilon g/g^*.$$

When $a(d)$ is positive (but this not true for all regularizations, see the discussion below), one finds an IR fixed point g^*, as well as exponents $\omega = \varepsilon$, and $\nu^{-1} = d - 2$. In the framework of the ε-expansion, ω is associated to the leading corrections to scaling. In the large N limit ω remains smaller than 2 for $\varepsilon < 2$, and this extends the property established near $d = 4$ to all dimensions $2 \le d \le 4$.

Finally, applying the RG equations to the propagator, one finds $\eta(g) = 0$, a result consistent with the value found for $\eta = \eta(g^*)$.

The sign of $a(d)$. It is generally assumed that $a(d)$ is positive for $2 < d < 4$, as indeed one finds in the simplest regularizations, for example, when the function $D(k^2)$ is an increasing function. Moreover, $a(d)$ is always positive for $d \to 4$ since

$$a(d) \underset{d\to 4}{\sim} \frac{1}{8\pi^2 \varepsilon}.$$

Then, for $2 < d < 4$ there exists an IR fixed point, corresponding to a non-trivial zero u^* of the β-function. For the value $u = u^*$, the leading corrections to scaling vanish.

However, for $d < 4$ fixed the positivity of $a(d)$ is not assured. For example, in the case of simple lattice regularizations it has been shown that in $d = 3$ the sign is arbitrary.

When $a(d)$ is negative, the RG method for large N (at least in the perturbative framework) is confronted with a serious difficulty. Indeed, the coupling flows in the IR limit to large values where the large N expansion is no longer reliable.

It is not known whether this signals a real pathology of the model in the RG sense, or is just an artifact of the large N limit.

Another way of viewing the problem is to examine directly the relation between bare and renormalized coupling constant. Calling $g_{\rm r} m^{4-d}$ the renormalized four-point function at zero momentum, we find

$$m^{4-d} g_{\rm r} = \frac{\Lambda^{4-d} g}{1 + \Lambda^{4-d} g N B_\Lambda(0,m)/6}.$$

In the limit $m \ll \Lambda$, the relation can be written as

$$\frac{1}{g_{\rm r}} = \frac{1}{g_{\rm r}^*} + \left(\frac{m}{\Lambda}\right)^{4-d} \left(\frac{1}{g} - \frac{N a(d)}{6}\right), \quad \frac{1}{g_{\rm r}^*} = \frac{(d-2) N K(d)}{12}.$$

We see that when $a(d) < 0$, the limiting value $g_{\rm r} = g_{\rm r}^*$ for $m/\Lambda = 0$ cannot be reached by varying g when m/Λ is small but finite (since $g > 0$). In the same way, leading corrections to scaling can no longer be canceled.

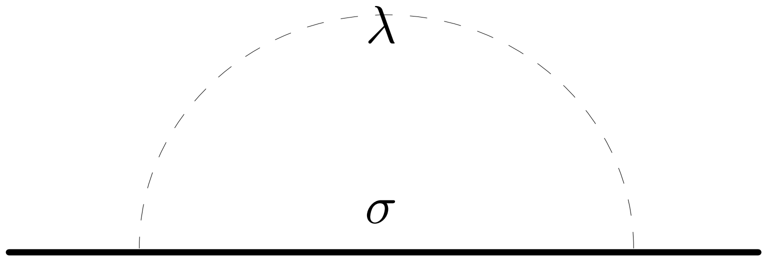


Fig. 6 The diagram contributing to $\Gamma^{(2)}_{\sigma\sigma}$ at order $1/N$.

The $1/N$ *correction to the* σ *two-point function*

The large technique, as explained here, allows generating a systematic $1/N$ expansion. The $1/N$ correction to the $\langle\sigma\sigma\rangle$ correlation function involves only one diagram (Fig. 6), containing two $\lambda^2\sigma$ vertices. After mass renormalization, in the large cut-off limit, one finds

$$\Gamma^{(2)}_{\sigma\sigma}(p) = p^2 + \frac{2}{N(2\pi)^d}\int \frac{\mathrm{d}^d q}{(6/u) + b(d)q^{-\varepsilon}}\left(\frac{1}{(p+q)^2} - \frac{1}{q^2}\right) + O\left(\frac{1}{N^2}\right).$$

This expression can be used to determine the $1/N$ correction to η.

Other methods

The large N limit can be obtained by several other algebraic methods. Without being exhaustive, let us list a few. Schwinger–Dyson equations for N large lead to a self-consistent equation for the two-point function. Some versions of the Hartree–Fock approximation or variational methods also yield the large N result. Variational methods may give additional insight on the large N result, but unlike the large N method, cannot be systematically improved. The functional (also called exact) RG takes the form of partial differential equations in the large N limit.

From the point of view of stochastic quantization or critical dynamics the Langevin equation also becomes linear and self-consistent for N large, because the fluctuations of $\boldsymbol{\phi}^2(x,t)$ are small. As a byproduct the large N expansion of the equilibrium equal-time correlation functions is recovered.

More general vector field theories

We have shown how the large N-expansion can be generated for a general function $NU(\boldsymbol{\phi}^2/N)$. We now briefly explain how the algebraic method of previous sections can be generalized to $O(N)$ symmetric actions that depend on several vector fields. Again, the composite fields which are expected to have small fluctuations, are the $O(N)$ scalars constructed from all $O(N)$ vectors. One thus introduces pairs of fields and Lagrange multipliers for all independent $O(N)$ invariant scalar products constructed from the many-component fields.

Let us illustrate the idea with the example of two fields $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$, and a symmetric interaction arbitrary function of the three scalar invariants $\rho_{12} = \boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_2/N$, $\rho_{11} = \boldsymbol{\phi}_1^2/N$ and $\rho_{22} = \boldsymbol{\phi}_2^2/N$:

$$\mathcal{S}(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2) = \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \left[\partial_\mu \boldsymbol{\phi}_1(x)\right]^2 + \tfrac{1}{2} \left[\partial_\mu \boldsymbol{\phi}_2(x)\right]^2 + NU(\rho_{11}, \rho_{12}, \rho_{22}) \right\}.$$

One then introduces three pairs of fields $\rho_{ij}(x)$ and $\lambda_{ij}(x)$ and uses the identity

$$\exp\left[-\int \mathrm{d}^d x\, NU(\boldsymbol{\phi}_1^2/N, \boldsymbol{\phi}_1\cdot\boldsymbol{\phi}_2/N, \boldsymbol{\phi}_2^2/N)\right] \propto \int [\mathrm{d}\rho_{ij}(x)\,\mathrm{d}\lambda_{ij}(x)]$$
$$\times \exp\left\{-\int \mathrm{d}^d x\left[\sum_{i,j}\tfrac{1}{2}\lambda_{ij}\left(\boldsymbol{\phi}_i\cdot\boldsymbol{\phi}_j - N\rho_{ij}\right) + NU(\rho_{11},\rho_{12},\rho_{22})\right]\right\}.$$

The identity transforms the action into a quadratic form in $\boldsymbol{\phi}_i$, the integration over $\boldsymbol{\phi}$ can thus be performed. The large N limit is then again obtained by the steepest descent method. In the special case in which $U(\rho)$ is a quadratic function, the integral over all ρ's can also be performed. If the action is a general $O(N)$ invariant function of p fields $\boldsymbol{\phi}_i$, it is necessary to introduce $p(p-1)/2$ pairs of ρ and λ fields.