Weak-coupling expansion of p_{QCD}

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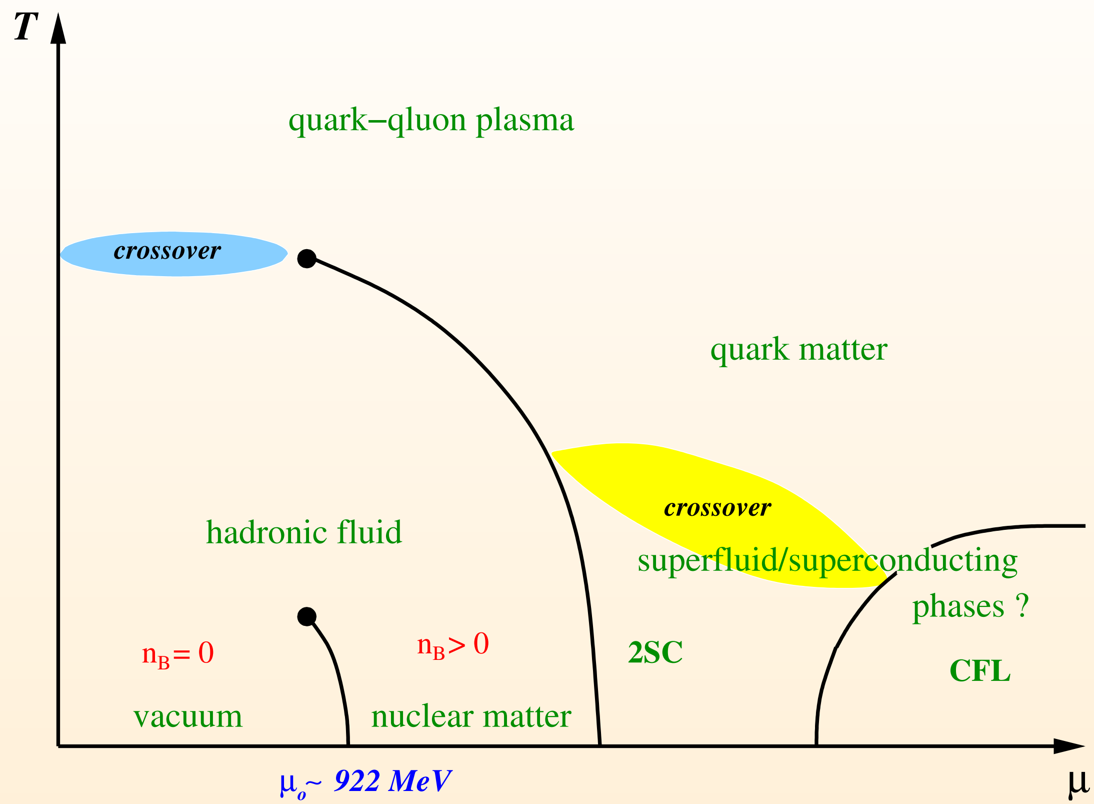
The QCD (equilibrium) phase diagram

$\mathcal{L}(\psi\,,\,A_\mu) \to p\,,\,n$ quasiparticles of the vacuum; complicated !

squeeze (μ_B) / heat (T) matter $\to$ simpler ?

new phases ? "closer" to $\mathcal{L}$?

The QCD (equilibrium) phase diagram



nature: early univ, μ tiny ($\sim \frac{\#baryons}{entropy}$), $T_c \sim 170 MeV \sim 10 \mu s$
neutron/quark stars

lab expt.: SPS / RHIC $\mu_B \sim \frac{\#baryons}{pions} \sim 45 MeV$ / LHC / GSI

$p(T)$ in cosmology

important for cosmology: cooling rate of the universe

$$\partial_t T = -\frac{\sqrt{24\pi}}{m_{\text{Pl}}} \frac{\sqrt{e(T)}}{\partial_T \ln s(T)}$$

with entropy $s = \partial_T p$ and energy density $e = Ts - p$

$\Rightarrow$ cosmol. relics (dark matter, background radiation etc.) originate when an interaction rate $\tau(T)$ gets larger than the age of the universe $t(T)$.

- Ex.: WIMPs of mass m decouple at $T_{\text{f}} \sim m/25$
  for $m = 10 \dots 1000$ GeV we then have $T_{\text{f}} \sim 0.4 \dots 40$ GeV
  there, QCD dominates the partition function [Hindmarsh, Philipsen 05]

- Ex.: "sterile" ν_R with $m_\nu \sim$ keV can be warm dark matter,
  and decouple around $T \sim 150$ MeV [Abazajian, Fuller 02; Asaka, Shaposhnikov 05]

$p(T)$ in cosmology

future CMB-experiments determine Ω_{DM} up to a few %
$\Rightarrow$ need to push theory-uncertainty to same level!

- WIMPs: 10% error in EoS at $T \sim T_c$ corresponds to 1% error in Ω

p in principle visible in gravitational wave background (generated during inflation)

- SM has trace anomaly $T^{\mu}_{\mu} \neq 0$
- influences gravity-related cosmol scenarios [e.g. Steinhardt et al, 04/05]

$p(T)$ in heavy ion collisions

expansion rate (after thermalization) given by

$$\partial_\mu T^{\mu\nu} = 0 \quad , \quad T^{\mu\nu} = [p(T) + e(T)]\, u^\mu u^\nu - p(T) g^{\mu\nu}$$

with flow velocity $u^\mu(t, x)$

- hydrodynamic expansion: hadronization at $T \sim 100 - 150$ MeV
  $\Rightarrow$ observed hadron spectrum depends (indirectly) on $p(T)$
- Ex.: "elliptic flow" sensitive on early stage $T \gtrsim 200$ MeV

prediction of susceptibilities $\Rightarrow$ determine fluctuations

- Ex.: charge-fluctuations $\langle(\delta Q)^2\rangle = -T\partial^2_{\mu_Q^2} F = -VT\chi_{\mathrm{Q}}$
  χ diverges $\Rightarrow$ fluctuation grows near crit. point

Theoretical treatment

asymptotic freedom $g_s^2(T \to \infty) \to 0$, perturbative expansion ?!

QCD at $T \gg 200$ MeV

- ideal gas of weakly interacting partons

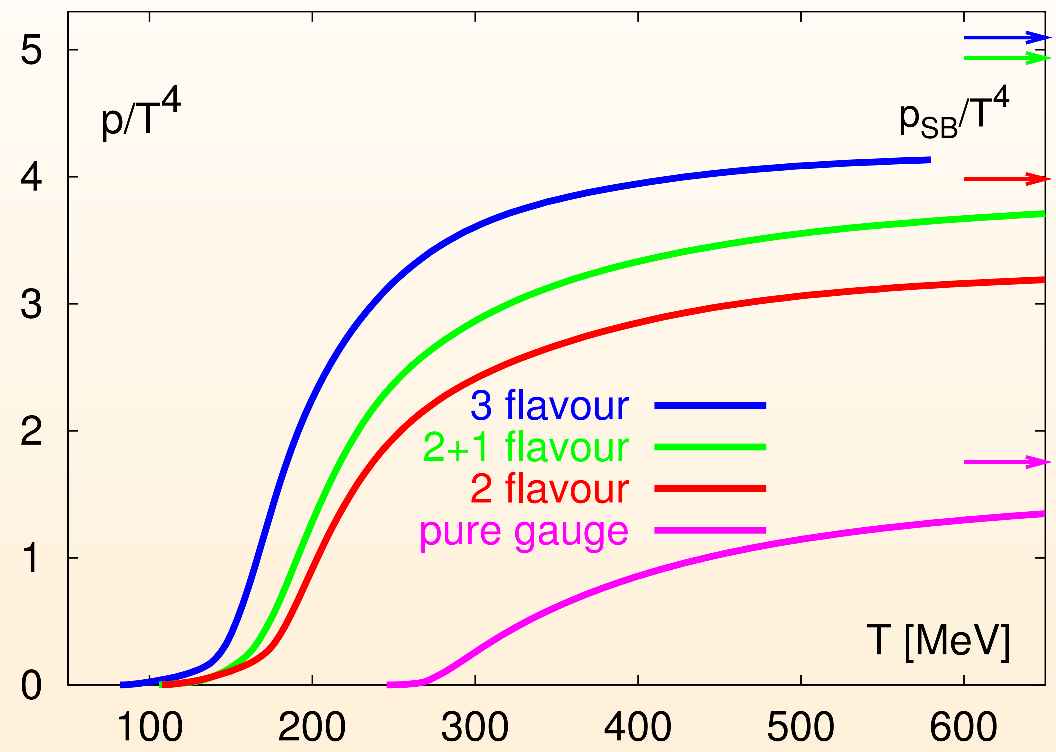
$$p_{\mathsf{SB}}(T) \quad = \quad \frac{\pi^2 T^4}{90}\left[2(N_c^2-1)+\frac{7}{2}N_c N_f\right] \approx 5.2 T^4$$

- + corrections ?

$$p_{\mathsf{QCD}}({\color{red}T}) \quad \equiv \quad \lim_{V\to\infty}\frac{{\color{red}T}}{V}\ln\int\mathcal{D}[A_\mu^a,\psi,\bar\psi]\exp\left(-\frac{1}{\hbar}\int_0^{\hbar/{\color{red}T}}d\tau\int d^{3-2\epsilon}x\,\mathcal{L}_{\mathsf{QCD}}\right)$$

$$\mathcal{L}_{\mathsf{QCD}} \quad = \quad \frac{1}{4g^2}\sum_{a=1}^{N_c^2-1}F_{\mu\nu}^a F_{\mu\nu}^a + \sum_{i=1}^{N_f}\bar\psi(\gamma_\mu D_\mu + m_i - \gamma_0\mu_i)\psi + \mathcal{L}_{\mathsf{GF}} + \mathcal{L}_{\mathsf{FP}}$$

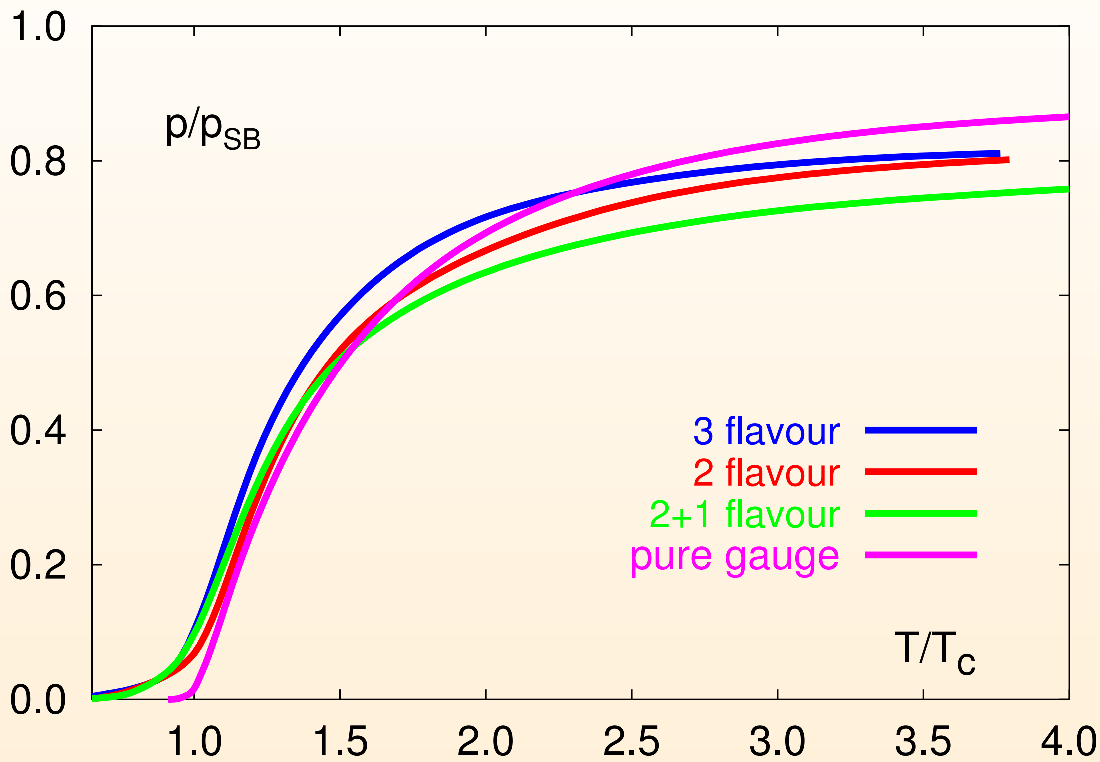
$p(T)$ on the lattice ($\mu_B = 0$)



[lattice data from Karsch et.al.]

at $T \to \infty$, expect ideal gas: $p_{\text{SB}} = \left(16 + \frac{21}{2} N_f\right) \frac{\pi^2 T^4}{90}$

$p(T)$ on the lattice ($\mu_B = 0$)



[lattice data from Karsch et.al.]

at $T \to \infty$, expect ideal gas: $p_{\text{SB}} = \left(16 + \frac{21}{2} N_f\right) \frac{\pi^2 T^4}{90}$

Corrections?

need to explain 20% ..

structure of pert series is non-trivial !

- Ex.: $p(T) = g^2 + g^3 + g^4 \ln g + g^4 + g^5 + g^6 \ln g + \ldots$

reason: interactions make QCD a multiscale system

dynamically generated scales ($|k| \sim \pi T$ is called "hard"):
color-electric screening at $|k| \sim m_{\mathsf{E}} \sim gT$ ("soft")
color-magnetic screening at $|k| \sim g^2 T$ ("ultrasoft")

expansion parameter

$$g^2\, n_b(|k|) = \frac{g^2}{e^{|k|/T} - 1} \overset{|k| \lesssim T}{\approx} \frac{g^2 T}{|k|}$$

treatment of a multiscale system: effective field theory !

Effective theory setup: QCD $\to$ EQCD

high T: QCD dynamics contained in 3d EQCD

integrate out $|p| \gtrsim 2\pi T$: ψ, $A_\mu(n \neq 0)$

$$p_{\rm QCD}(T) \equiv p_{\rm E}(T) + \frac{T}{V}\ln\int \mathcal{D}[A_k^a, A_0^a]\exp\Big(-\int d^dx\,\mathcal{L}_{\rm E}\Big)$$

$$\mathcal{L}_{\rm E} = \frac{1}{2}Tr\,F_{kl}^2 + Tr\,[D_k, A_0]^2 + m_{\rm E}^2 Tr\,A_0^2 + \lambda_{\rm E}^{(1)}(Tr\,A_0^2)^2 + \lambda_{\rm E}^{(2)}Tr\,A_0^4 + \dots$$

five matching coefficients [E. Braaten, A. Nieto, 95; KLRS 02; M. Laine, YS, 05]

$$p_{\rm E} = T^4\left[\# + \#g^2 + \#g^4 + \#g^6 + \dots\right],\; m_{\rm E}^2 = T^2\left[\#g^2 + \#g^4 + \dots\right],$$

$$g_{\rm E}^2 = T\left[g^2 + \#g^4 + \#g^6 + \dots\right],\; \lambda_{\rm E}^{(1),(2)} = T\left[\#g^4 + \dots\right].$$

higher order operators do not (yet) contribute [S. Chapman, 94; Kajantie et al, 97, 02]

$$\frac{\delta p_{\rm QCD}(T)}{T} \sim \delta\mathcal{L}_{\rm E} \sim g^2\frac{D_kD_l}{(2\pi T)^2}\mathcal{L}_{\rm E} \sim g^2\frac{(gT)^2}{(2\pi T)^2}(gT)^3 \sim g^7T^3$$

Effective theory setup: QCD → EQCD → MQCD

the IR of 3d EQCD is contained in 3d MQCD

integrate out $|p| \gtrsim gT$: A_0

$$p_{\mathsf{QCD}}(T) \equiv p_{\mathsf{E}}(T) + p_{\mathsf{M}}(T) + \frac{T}{V} \ln \int \mathcal{D}[A_k^a] \exp\Big(-\int d^d x \, \mathcal{L}_{\mathsf{M}}\Big)$$

$$\mathcal{L}_{\mathsf{M}} = \frac{1}{2} Tr \, F_{kl}^2 + \ldots$$

two matching coefficients [KLRS 03; P. Giovannangeli 04, M. Laine/YS 05]

$$p_{\mathsf{M}} = T m_{\mathsf{E}}^3 \left[\# + \# \frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}} + \# \frac{g_{\mathsf{E}}^4}{m_{\mathsf{E}}^2} + \# \frac{g_{\mathsf{E}}^6}{m_{\mathsf{E}}^3} + \ldots\right], \quad g_{\mathsf{M}}^2 = g_{\mathsf{E}}^2 \left[1 + \# \frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}} + \# \frac{g_{\mathsf{E}}^4}{m_{\mathsf{E}}^2} + \ldots\right].$$

higher order operators could contribute

$$\frac{\delta p_{\mathsf{QCD}}(T)}{T} \sim \delta \mathcal{L}_{\mathsf{M}} \sim g_{\mathsf{E}}^2 \frac{D_k D_l}{m_{\mathsf{E}}^3} \mathcal{L}_{\mathsf{M}} \sim g_{\mathsf{E}}^2 \frac{(g^2 T)^2}{m_{\mathsf{E}}^3} (g^2 T)^3 \sim g^9 T^3$$

Effective theory prediction for $p(T)$

$\mathcal{L}_{\mathsf{M}}$ only has one (dimensionful) parameter

$$p_{\mathsf{G}}(T) \equiv \frac{T}{V}\ln\int\mathcal{D}[A_k^a]\exp\Big(-S_{\mathsf{M}}\Big) = T\#\, g_{\mathsf{M}}^6$$

coefficient is non-perturbative, but computable!

$$\frac{p_{\mathsf{QCD}}(T)}{p_{\mathsf{SB}}} = \frac{p_{\mathsf{E}}(T)}{p_{\mathsf{SB}}}+\frac{p_{\mathsf{M}}(T)}{p_{\mathsf{SB}}}+\frac{p_{\mathsf{G}}(T)}{p_{\mathsf{SB}}} \quad , \quad p_{\mathsf{SB}} = \left(16+\frac{21}{2}N_f\right)\frac{\pi^2T^4}{90}$$

$$= 1+g^2 + g^4 + g^6 + \dots \quad \Leftarrow \text{4d QCD}$$

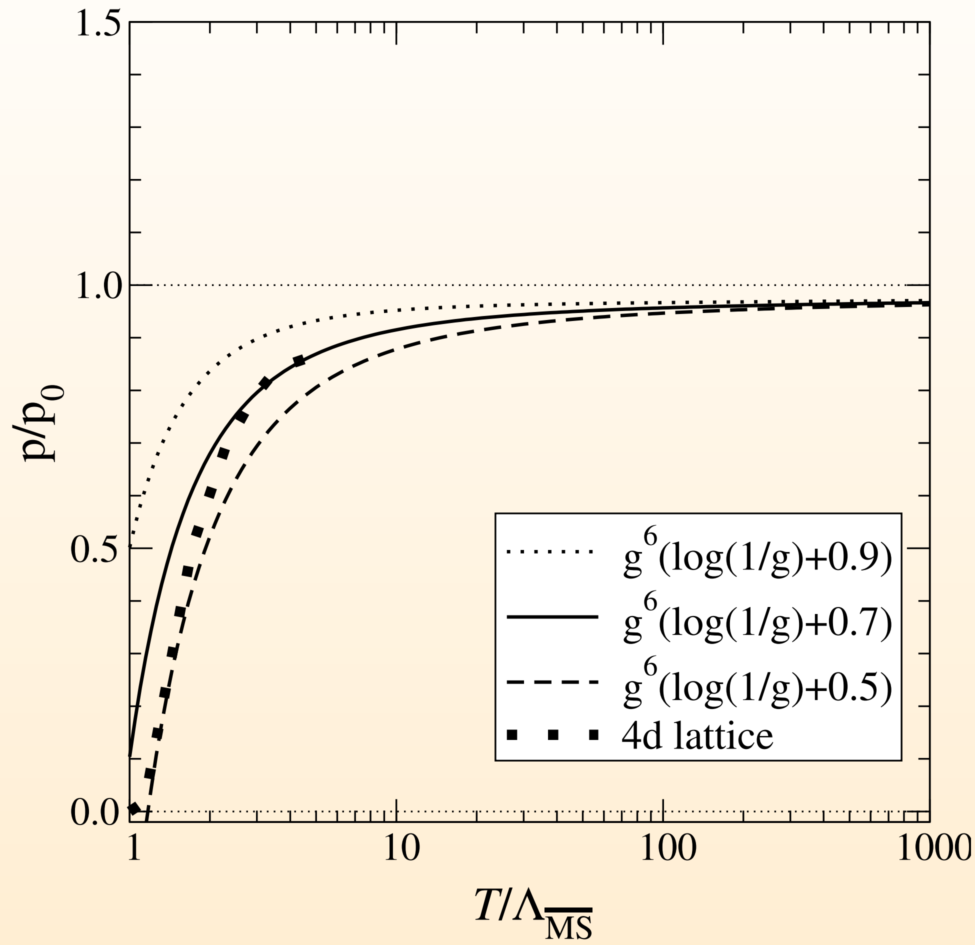
$$+g^3+g^4+g^5+g^6+\dots \quad \Leftarrow \text{3d adj H}$$

$$+\frac{1}{p_{\mathsf{SB}}}\frac{T}{V}\int\mathcal{D}[A_k^a]\exp\Big(-S_{\mathsf{M}}\Big) \quad \Leftarrow \text{3d YM}$$

$$= c_0+c_2g^2+c_3g^3+(c_4'\ln g+c_4)g^4+c_5g^5+(c_6'\ln g+c_6)g^6+\mathcal{O}(g^7)$$

[c_2 Shuryak 78, c_3 Kapusta 79, c_4' Toimela 83, c_4 Arnold/Zhai 94, c_5 Zhai/Kastening 95, Braaten/Nieto 96, c_6' KLRS 03]

Thermal pressure $p(T)$: 4d vs 3d $(N_f = 0)$



depencence on g^6 constant

this non-perturbative contribution is unknown, but computable!

Shopping list for c_6

$\ldots + g^6$

- 4-loop sum-integrals needed, const term
- DOABLE?! manpower OR brainpower? [YS/AV 07?]

matching coeffs

- 2-loop ϵ-terms for m_{E}^2, g_{E}^2 DONE. ML/YS 05

$\ldots + g^6$

- 4-loop integrals needed DONE. KLRS 03: reduction, master ints

match $\overline{\text{MS}}$/LAT

- 4-loop const in LAT reg via NSPT DONE. LMRST 06 [to be published]

$\ldots + g^6$

- measure $<$ Plaquette $>$ in 3d SU(N) DONE. HKLRS 05

Parametric behavior of some observables

pressure, energy density, ..

- $\frac{p}{T^4} \sim 1 + g^2 + g^3 + g^4 \ln g + g^4 + g^5 + g^6(\ln g + [\text{np}]) + \ldots$

correlation lengths $\xi = m_{\mathsf{E}}^{-1}$ für $Tr\, F_{0i}F_{jk}$, $Tr\,$Pol etc.

- $m_{\mathsf{E}} \sim gT + g^2T(\ln g + [\text{np}]) + \ldots$

correlation lengths $\xi = m_{\mathsf{G}}^{-1}$ für $Tr\, F_{ij}^2$

- $m_{\mathsf{G}} \sim [\text{np}] \times g^2T + \ldots$

spatial string tension

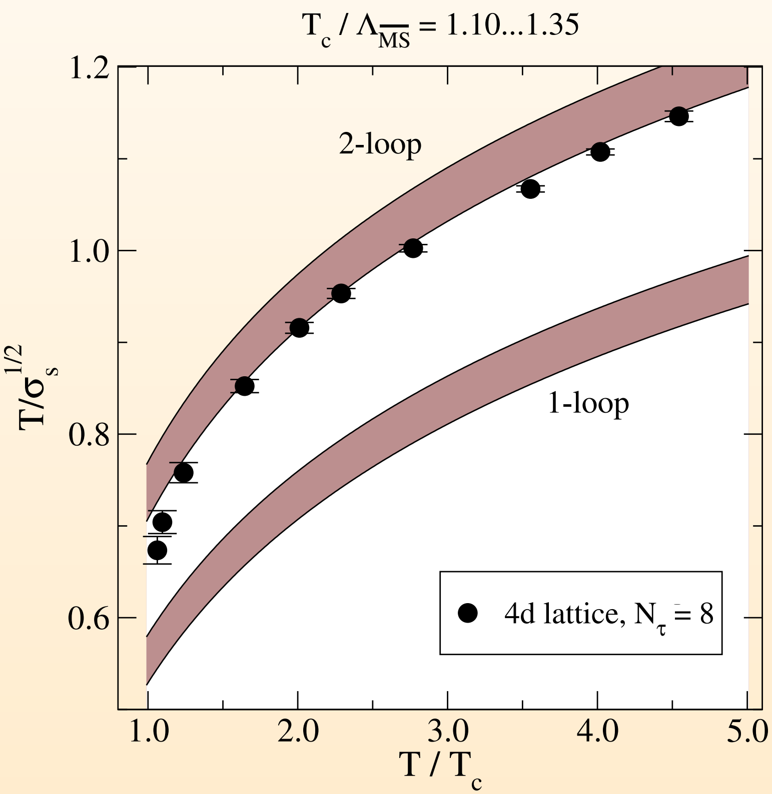
- $\sqrt{\sigma_s} \sim [\text{np}] \times g^2T + \ldots$

$\Rightarrow$ use these quantities e.g. as precision test of eff. th. setup

Spatial string tension: $W_s(R_1,R_2) = \exp(-\sigma_s R_1 R_2)$ at large R_1, R_2

SU(3), 4d lat: $\frac{\sqrt{\sigma_s}}{T} = \text{fct}\left(\frac{T}{T_c}\right)$; $T_c \approx 1.2\Lambda_{\overline{\text{MS}}}$

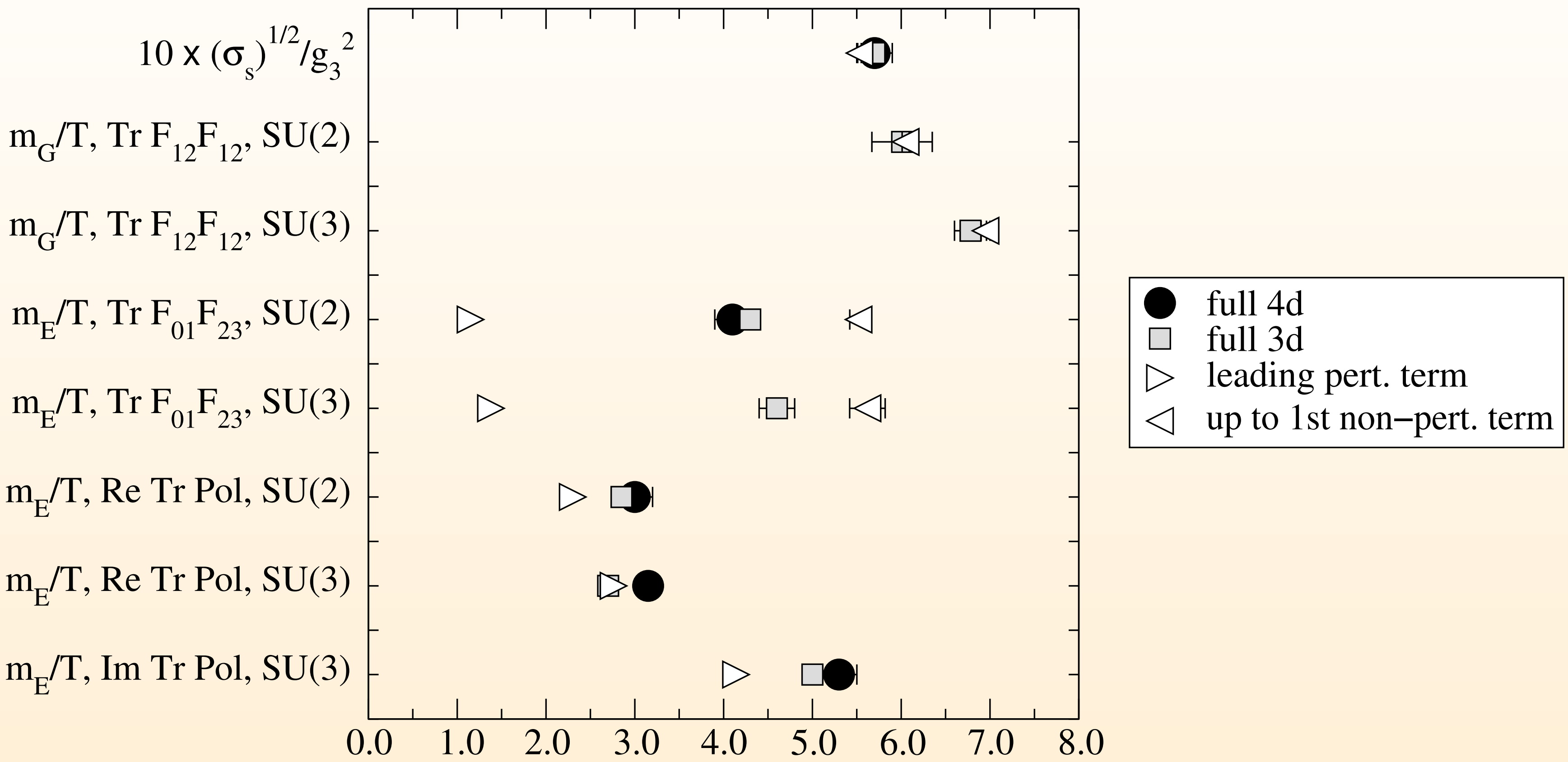
SU(3), 3d MQCD: $\frac{\sqrt{\sigma_s}}{T} = \# \frac{g_M^2}{g_E^2}\frac{g_E^2}{T} = \text{fct}\left(\frac{T}{\Lambda_{\overline{\text{MS}}}}\right)$; $\# = 0.553(1)$ [Teper, Lucini 02]



[4d lattice data from Boyd et al, 96] (cave: no cont. extrapolation)

parameter-free comparison; support for hard/soft+ultrasoft picture

String tension and inverse correlation lengths



[lattice date from: Hart et al 00; Boyd et al 96; Kaczmarek at al 00; Teper 98; Laine et al 01; Datta et al 02]

''full 4d'': 4d lattice Monte Carlo

''full 3d'': 3d lattice, couplings(g^2,T)

Conclusions / Outlook

- QCD contains an extremely rich structure
- thermodynamic quantities of QCD are relevant for cosmology and heavy ion collisons
- these quantities can be determined numerically at $T \sim 200$ MeV, and via effective field theory at $T \gg 200$ MeV
- effective field theory opens up tremendous opportunities: analytic treatment of fermions, universality, superrenormalizabilty
- for precise results, sometimes need very deep expansions
- techniques are ''state of the art'': can be transferred e.g. to (cold) collider physics, LHC phenomenology (ρ parameter, decoupling constants, $R(s)$, ..)

Outlook: 06 → 08 → 10

$$\frac{p_{\mathsf{G}}}{p_{\mathsf{SB}}} = \#_{(6)}\left(\frac{g_{\mathsf{M}}^2}{T}\right)^3 + [\delta\mathcal{L}_{\mathsf{M}}]_{(9)}$$

$$g_{\mathsf{M}}^2 = g_{\mathsf{E}}^2\left[1 + \#_{(7)}\frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}} + \left(\frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}}\right)^2\left(\#_{(8)} + \#_{(10)}\frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2}\right) + \ldots_{(9)}\right]$$

$$\frac{p_{\mathsf{M}}}{p_{\mathsf{SB}}} = \left[\#_{(3)} + \frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}}\left(\#_{(4)} + \#_{(6)}\frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2}\right) + \left(\frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}}\right)^2\left(\#_{(5)} + \#_{(7)}\frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2} + \#_{(9)}\left(\frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2}\right)^2\right)\right.$$
$$+ \left(\frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}}\right)^3\left(\#_{(6)} + \#_{(8)}\frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2} + \#_{(10)}\left(\frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2}\right)^2 + \#_{(12)}\left(\frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2}\right)^3\right)$$
$$\left. + [\text{3d 5loop 0pt}]_{(7)} + [\delta\mathcal{L}_{\mathsf{E}}]_{(7)} + [\text{3d 6loop 0pt}]_{(8)} + \ldots_{(9)}\right]$$

$$m_{\mathsf{E}}^2 = T^2\left[\#_{(3)}g^2 + \#_{(5)}g^4 + [\text{4d 3loop 2pt}]_{(7)} + \ldots_{(9)}\right]$$

$$\lambda_{\mathsf{E}} = T\left[\#_{(6)}g^4 + \#_{(8)}g^6 + \ldots_{(10)}\right]$$

$$g_{\mathsf{E}}^2 = T\left[g^2 + \#_{(6)}g^4 + \#_{(8)}g^6 + \ldots_{(10)}\right]$$

$$\frac{p_{\mathsf{E}}}{p_{\mathsf{SB}}} = \#_{(0)} + \#_{(2)}g^2 + \#_{(2)}g^4 + \#_{(6)}g^6 + [\text{4d 5loop 0pt}]_{(8)} + \ldots_{(10)}$$

notation: $\#_{(n)}$ enters p_{QCD} at g^n

[cave: no $\frac{1}{\epsilon} + 1 + \epsilon$ and no IR/UV shown above]