Computer-algebraic methods
at finite temperature

York Schröder

(Uni Bielefeld)

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Outline

Physics problem: QCD pressure

- motivation, effective theory setup, status $\longrightarrow$ Mikko
- systematic control, higher order op's $\longrightarrow$ Mikko; Pierre

Methods

- Diagram generation
- classification
- reduction
- integration $\longrightarrow$ Aleksi?
- (lattice MC) $\longrightarrow$ Ari
- lattice: perturbative $\longrightarrow$ Antonio; Francesco, Christian

Outlook

General setup

RHIC $\longrightarrow$ QCD at $T \gtrsim$ (a few) 100 MeV

asymptotic freedom $\longrightarrow$ weak coupling expansion

slow convergence, non-trivial structure

problematic dof's are identified

- soft modes $p \sim gT$ $\longrightarrow$ odd powers in g
- ultrasoft modes $p \sim g^2T$ $\longrightarrow$ non-pert coeffs

general picture

- perturbation theory OK for parametrically hard scales $p \sim 2\pi T$
- soft and ultrasoft scales need improved analytic schemes, or non-pert treatment
- starting point: dim red eff. theory, or HTL eff. theory

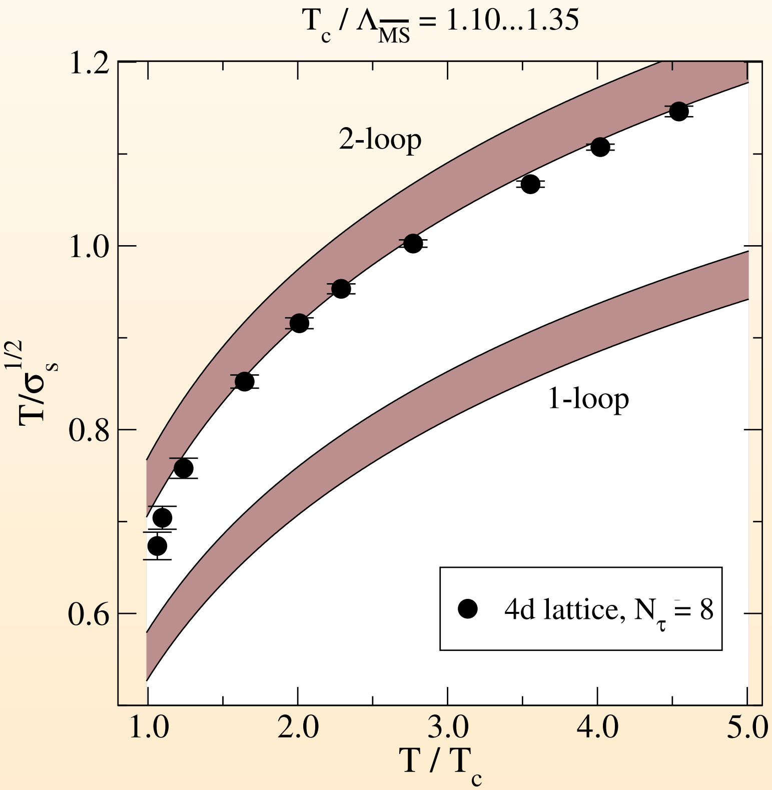
quantitative evidence:

- pick some simple observables
- compare 4d lattice vs soft/ultrasoft eff. theory
- e.g. static correlation lengths, string tensions $\longrightarrow$ agreement down to $T \sim 2T_c$

Spatial string tension: $W_s(R_1, R_2) = \exp(-\sigma_s R_1 R_2)$ at large R_1, R_2

SU(3), 4d lat: $\frac{\sqrt{\sigma_s}}{T} = \text{fct}\left(\frac{T}{T_c}\right)$; $T_c \approx 1.2\Lambda_{\overline{\text{MS}}}$

SU(3), 3d MQCD: $\frac{\sqrt{\sigma_s}}{T} = \# \frac{g_M^2}{g_E^2} \frac{g_E^2}{T} = \text{fct}\left(\frac{T}{\Lambda_{\overline{\text{MS}}}}\right)$; $\# = 0.553(1)$ [Teper, Lucini 02]



[4d lattice data from Boyd et al, 96] (cave: no cont. extrapolation)

parameter-free comparison; support for hard/soft+ultrasoft picture

The pressure of thermal QCD

want to compute the QCD pressure ($\mu_B \equiv 0$ here)

$$p_{\mathsf{QCD}}(T) \equiv \lim_{V\to\infty} \frac{T}{V} \ln \int \mathcal{D}[A^a_\mu, \psi, \bar\psi] \exp\Big(-\frac{1}{\hbar} S_{\mathsf{QCD}}\Big)$$

$$S_{\mathsf{QCD}} = \int_0^{\hbar/T} d\tau \int d^dx\, \mathcal{L}_{\mathsf{QCD}} \quad , \quad d = 3 - 2\epsilon$$

$$\mathcal{L}_{\mathsf{QCD}} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar\psi \gamma_\mu D_\mu \psi + \mathcal{L}_{\mathsf{GF}} + \mathcal{L}_{\mathsf{FP}}$$

$p_{\mathsf{QCD}}(T)$ renormalised such that it vanishes at $T = 0$.

asymptotically, expect ideal gas: $p_{\mathsf{QCD}}(T\to\infty) \equiv p_{\mathsf{SB}} = \big(16 + \frac{21}{2} N_f\big) \frac{\pi^2 T^4}{90}$

Effective theory setup: QCD $\to$ EQCD

high T: QCD dynamics contained in 3d EQCD

integrate out $|p| \gtrsim 2\pi T$: ψ, $A_\mu(n \neq 0)$

$$p_{\mathsf{QCD}}(T) \equiv p_{\mathsf{E}}(T) + \frac{T}{V} \ln \int \mathcal{D}[A_k^a, A_0^a] \exp\Big(-\int d^dx\, \mathcal{L}_{\mathsf{E}}\Big)$$

$$\mathcal{L}_{\mathsf{E}} = \frac{1}{2}\mathrm{Tr}\, F_{kl}^2 + \mathrm{Tr}\,[D_k, A_0]^2 + m_{\mathsf{E}}^2 \mathrm{Tr}\, A_0^2 + \lambda_{\mathsf{E}}^{(1)} (\mathrm{Tr}\, A_0^2)^2 + \lambda_{\mathsf{E}}^{(2)} \mathrm{Tr}\, A_0^4 + \ldots$$

five matching coefficients [E. Braaten, A. Nieto, 95; KLRS 02; M. Laine, YS, 05]

$$p_{\mathsf{E}} = T^4 \left[\# + \# g^2 + \# g^4 + \# g^6 + \ldots\right], \quad m_{\mathsf{E}}^2 = T^2 \left[\# g^2 + \# g^4 + \ldots\right],$$

$$g_{\mathsf{E}}^2 = T \left[g^2 + \# g^4 + \# g^6 + \ldots\right], \quad \lambda_{\mathsf{E}}^{(1/2)} = T \left[\# g^4 + \ldots\right].$$

higher order operators do not (yet) contribute [S. Chapman, 94; Kajantie et al, 97, 02]

$$\frac{\delta p_{\mathsf{QCD}}(T)}{T} \sim \delta \mathcal{L}_{\mathsf{E}} \sim g^2 \frac{D_k D_l}{(2\pi T)^2} \mathcal{L}_{\mathsf{E}} \sim g^2 \frac{(gT)^2}{(2\pi T)^2} (gT)^3 \sim g^7 T^3$$

Effective theory setup: QCD $\to$ EQCD $\to$ MQCD

the IR of 3d EQCD is contained in 3d MQCD

integrate out $|p| \gtrsim gT$: A_0

$$p_{\mathsf{QCD}}(T) \equiv p_{\mathsf{E}}(T) + p_{\mathsf{M}}(T) + \frac{T}{V}\ln\int\mathcal{D}[A_k^a]\exp\Big(-\int d^dx\,\mathcal{L}_{\mathsf{M}}\Big)$$

$$\mathcal{L}_{\mathsf{M}} = \frac{1}{2}\mathrm{Tr}\,F_{kl}^2 + \ldots$$

two matching coefficients [KLRS 03; P. Giovannangeli 04, M. Laine/YS 05]

$$p_{\mathsf{M}} = Tm_{\mathsf{E}}^3\left[\# + \#\frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}} + \#\frac{g_{\mathsf{E}}^4}{m_{\mathsf{E}}^2} + \#\frac{g_{\mathsf{E}}^6}{m_{\mathsf{E}}^3} + \ldots\right],\quad g_{\mathsf{M}}^2 = g_{\mathsf{E}}^2\left[1 + \#\frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}} + \#\frac{g_{\mathsf{E}}^4}{m_{\mathsf{E}}^2} + \ldots\right].$$

higher order operators could contribute

$$\frac{\delta p_{\mathsf{QCD}}(T)}{T} \sim \delta\mathcal{L}_{\mathsf{M}} \sim g_{\mathsf{E}}^2\frac{D_kD_l}{m_{\mathsf{E}}^3}\mathcal{L}_{\mathsf{M}} \sim g_{\mathsf{E}}^2\frac{(g^2T)^2}{m_{\mathsf{E}}^3}(g^2T)^3 \sim g^9T^3$$

Effective theory prediction for $p(T)$

$\mathcal{L}_{\mathsf{M}}$ only has one (dimensionful) parameter

$$p_{\mathsf{G}}(T) \equiv \frac{T}{V}\ln\int\mathcal{D}[A_k^a]\exp\Big(-S_{\mathsf{M}}\Big) = T\#\, g_{\mathsf{M}}^6$$

coefficient is non-perturbative!

$$\frac{p_{\mathsf{QCD}}(T)}{p_{\mathsf{SB}}} = \frac{p_{\mathsf{E}}(T)}{p_{\mathsf{SB}}} + \frac{p_{\mathsf{M}}(T)}{p_{\mathsf{SB}}} + \frac{p_{\mathsf{G}}(T)}{p_{\mathsf{SB}}} \quad , \quad p_{\mathsf{SB}} = \left(16 + \frac{21}{2}N_f\right)\frac{\pi^2 T^4}{90}$$

$$= 1 + g^2 + g^4 + g^6 + \ldots \qquad \Leftarrow \text{4d QCD}$$

$$+ g^3 + g^4 + g^5 + g^6 + \ldots \qquad \Leftarrow \text{3d adj H}$$

$$+ \frac{1}{p_{\mathsf{SB}}}\frac{T}{V}\int\mathcal{D}[A_k^a]\exp\Big(-S_{\mathsf{M}}\Big) \qquad \Leftarrow \text{3d YM}$$

$$= c_0 + c_2 g^2 + c_3 g^3 + (c_4' \ln g + c_4)g^4 + c_5 g^5 + (c_6' \ln g + c_6)g^6 + \mathcal{O}(g^7)$$

[c_2 Shuryak 78, c_3 Kapusta 79, c_4' Toimela 83, c_4 Arnold/Zhai 94, c_5 Zhai/Kastening 95, Braaten/Nieto 96, c_6' KLRS 03]

shopping list for c_6

$\ldots + g^6$ $\Leftarrow$ 4d QCD

- 4-loop sum-integrals needed, const term
- DOABLE?! manpower OR brainpower?

matching coeffs

- 2-loop ϵ-terms for m_E^2, g_E^2 DONE. ML/YS 05

$\ldots + g^6$ $\Leftarrow$ 3d adj H

- 4-loop integrals needed DONE. KLRS 03: reduction, master ints

match $\overline{\text{MS}}$/LAT

- 4-loop const in LAT reg
- DOABLE?! Parma: NSPT; diaPT?

$\ldots + g^6$ $\Leftarrow$ 3d YM

- measure $<$ Plaquette $>$ in 3d SU(N) DONE. HKLRS 05

Methods I: diagram generation

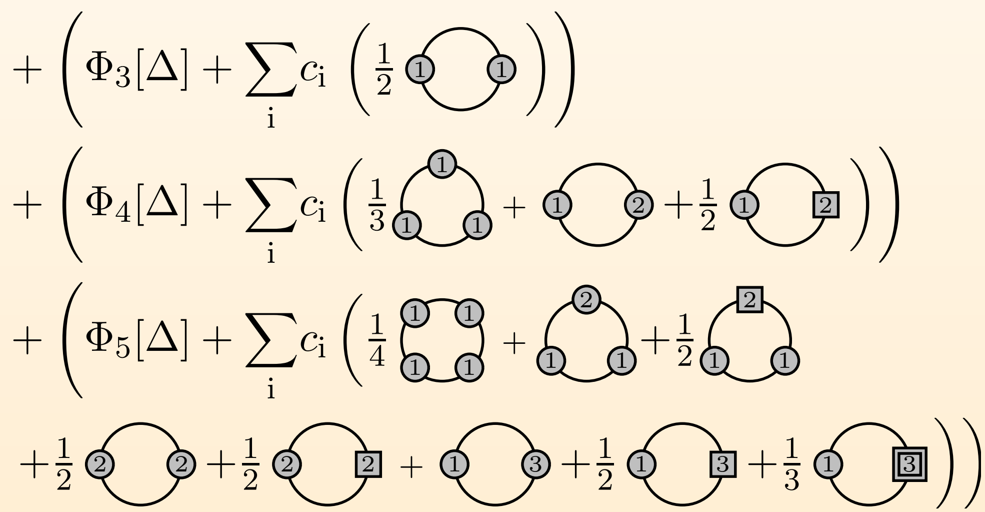
yet another generator? QGRAF [Nogueira], FeynArts [Denner/Hahn] n/a for 0-pt fcts.

skeleton (2PI) expansion [Luttinger/Ward, Baym, ...]

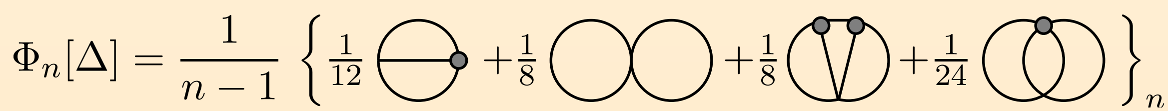
$$F[D] = \sum_i c_\mathrm{i} \left(\mathrm{Tr}\,\ln D_\mathrm{i}^{-1} + \mathrm{Tr}\,\Pi_\mathrm{i}[D]D_\mathrm{i}\right) - \Phi[D]$$

extremal property of partition function $\Rightarrow\ \delta_{D_\mathrm{i}}\Phi[D] \;=\; c_\mathrm{i}\Pi[D]$

$$-F \;=\; -F_0 + \Phi_2[\Delta]$$



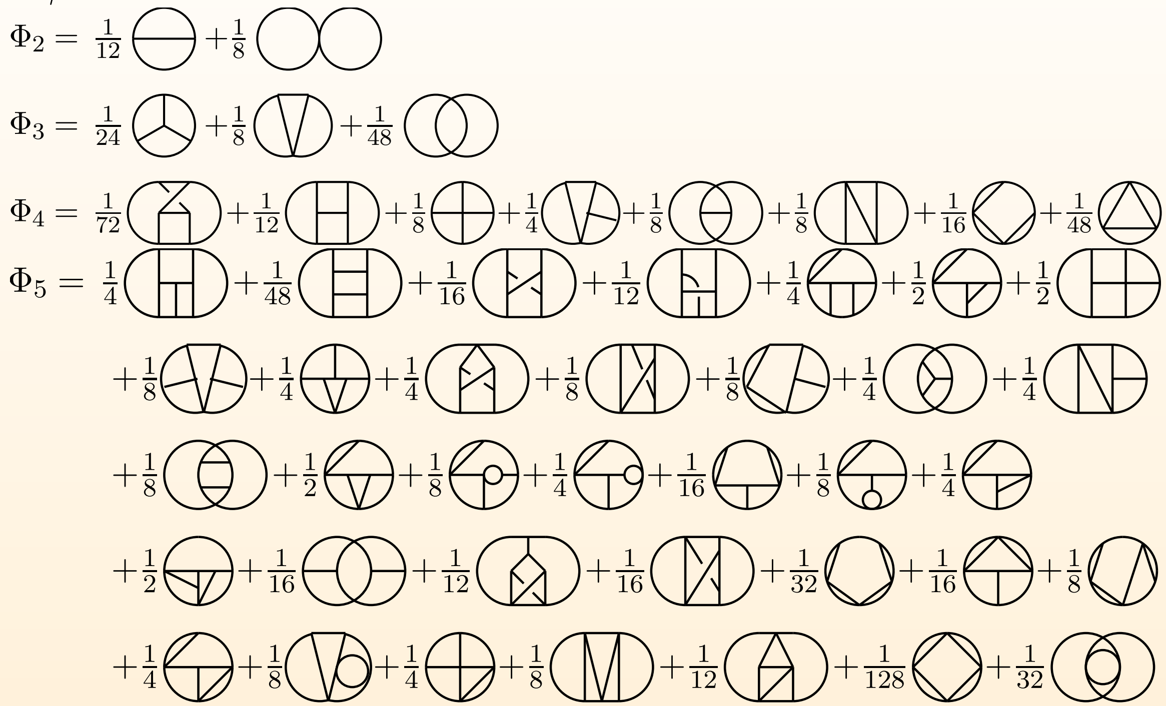
get skeletons from



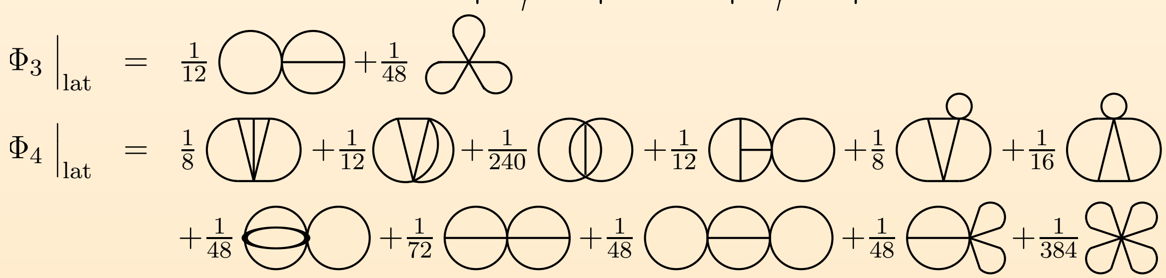
and SD eqs $\quad \Gamma_n^{1PI} = \delta_\phi^{n-1}\, S'[\phi + D[\phi]\delta_\phi]\Big|_{\phi=0}$

Methods I: diagram generation

generic $\phi^3 + \phi^4$ skeletons



LAT: additional skeletons $\ldots + \phi^5 + \ldots + \phi^8 + \ldots$



Methods II: classification

once you have a long list of Feynman integrals, need tools that replace human ‘staring’ at them

topology recognition

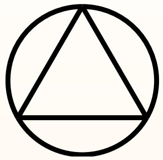
- ```
  #define mom3 "gl(k1,k2,k3,k1-k2,k1-k3,k1-k2-k3)"
  #define maxTopo3 "5"
  #define maxLines3 "6"
  *** format of sets: nrLines,nrReps,..binary reps..
  set t31: 3,16,7,11,14,21,22,25,26,28,35,37,38,41,44,49,50,56;
  set t32: 4,12,15,23,27,29,43,45,46,51,53,54,58,60;
  set t33: 4,3,30,39,57;
  set t34: 5,6,31,47,55,59,61,62;
  set t35: 6,1,63;
  ```

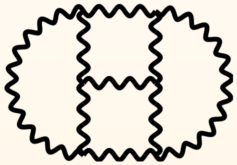
find symmetry relations

- ```
  id f(3,0,0,0,0,0,0)=0;
  [...]
  al f(3,0,f(?A2),f(?A3),f(?A4),f(?A5),f(?A6))=
  fsy(f(3,f(?A6),f(?A2),f(?A4),f(?A3),f(?A5),0),f(3,f(?A6),f(?A2),f(?A5),f(?A3),
  f(?A4),0),f(3,f(?A6),f(?A5),f(?A3),f(?A4),f(?A2),0),f(3,f(?A6),f(?A5),f(?A2),f(
  ?A4),f(?A3),0),f(3,f(?A6),f(?A4),f(?A3),f(?A5),f(?A2),0),f(3,f(?A6),f(?A4),f(
  ?A2),f(?A5),f(?A3),0),f(3,f(?A6),f(?A3),f(?A4),f(?A2),f(?A5),0),f(3,f(?A6),f(
  ?A3),f(?A5),f(?A2),f(?A4),0));
  ```

etc.

Methods III: reduction, IBP

can do 4-loop scalar theory on paper:  1 integral

for YM, need a computer:  25M integrals $(2^9 6^6)$

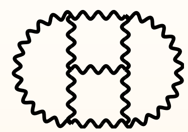
powerful method: integration by parts (IBP)

$\Rightarrow$ systematically use ($T = 0$ here) $\quad 0 = \int d^dk \, \partial_{k_\mu} f_\mu(k)$

key idea: lexicographic ordering among all loop integrals [Laporta 00]

arrive at rep in terms of irreducible ($\equiv$ master) integrals

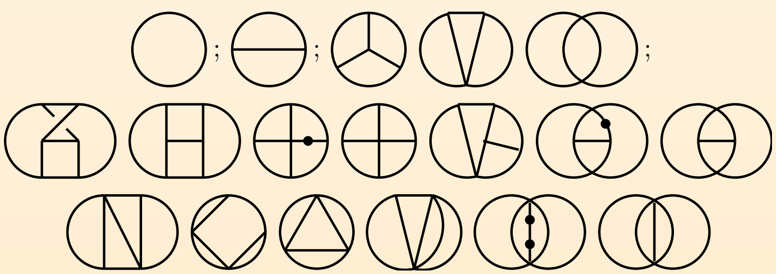
Methods III: reduction, IBP

in a nutshell, IBP reduces e.g. 

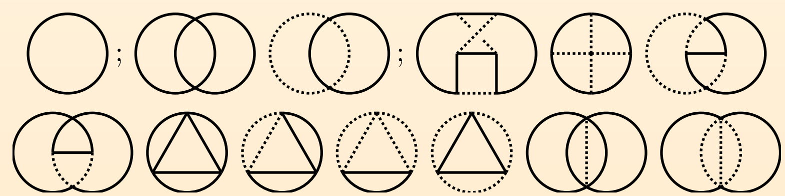
to

$$d_A C_A^3 \frac{g^6}{(4\pi)^4} \sum_i \frac{poly_i(d,\xi)}{poly_i(d)} \text{Master}_i(d)$$

18 fully massive master ints



13 ''QED'' type master ints



Methods III: reduction, IBP

one may try to copy the IBP method at $T > 0$

measure and propagators differ from $T = 0$ case:

$$\int d^d k \to T \sum_{n=-\infty}^{\infty} \int d^d k \qquad , \qquad \frac{1}{k^2 + m^2} \to \frac{1}{k^2 + (2\pi(n+\frac{1}{2})T)^2}$$

$\Rightarrow$ use IBP algorithm in integral; Matsubara frequencies are 'masses'!

delta fct at each vertex: 'masses' are linearly dependent

'practical criterion for irreducibility' wrt IBP [Baikov 05]:

- integral $\longrightarrow$ polynomial
- non-zero stable points $\longrightarrow$ irreducible (master)
- applied to sum-integrals $\longrightarrow$ reductions exist
- but might be hard to find, due to many mass-scales

Methods III: reduction, IBP

1-loop

- find reductions like $\sum\!\!\!\!\!\!\!\int_q \frac{q_0^{2i}}{[q_0^2+\vec{q}^2]^n} = \frac{2n-2-d}{2n-2} \sum\!\!\!\!\!\!\!\int_q \frac{q_0^{2i-2}}{[q_0^2+\vec{q}^2]^{n-1}}$
- find infinitely many master integrals!
- $\sum\!\!\!\!\!\!\!\int_{q_b} \frac{1}{[q_0^2+\vec{q}^2]^n} = \frac{2\pi^{d/2}T^{1+d}}{(2\pi T)^{2n}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \zeta(2n-d)$
- in practice: only a finite number can (and do) contribute

2-loop

- find reductions like $\sum\!\!\!\!\!\!\!\int_q \sum\!\!\!\!\!\!\!\int_r \frac{1}{[q_0^2+\vec{q}^2][r_0^2+\vec{r}^2][(q_0+r_0)^2+(\vec{q}+\vec{r})^2]} = 0$
- find no master integrals

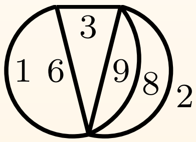
3-loop

- take the [Braaten/Nieto 96] calculation as 'benchmark'
- reproduced their Feynman gauge result
- show that all ξ-terms vanish (not finished yet)
- some example relations in /projects/sts/tabT/*.tab

Methods IV a: analytic Integration

3d, Euclidean, massive, dim. reg., $\overline{\text{MS}}$, x-space, ...

one 3d example [YS,AV]:



$$= \left(\frac{\bar{\mu}}{m_{316}}\right)^{8\epsilon} \frac{1}{32} \left[\frac{1}{\epsilon^2} + \frac{8}{\epsilon} + 4\, S\left(\frac{m_{316}}{m_{16289}}, \frac{2m_1}{m_{316}}, \frac{2m_3}{m_{316}} - 1\right) + \mathcal{O}(\epsilon)\right]$$

$$\text{where} \quad S(x,y,z) = 13 + \frac{7}{12}\pi^2 + 2\text{Li}_2(1-y) + 2\text{Li}_2(y+z) + 2\text{Li}_2(-z)$$
$$- 4\,(\ln x)^2 + 8\frac{1-x}{x(1+z)}\,\text{Li}_2(1-x)$$
$$+ 8\left(1 + \frac{1-x}{x(1+z)}\right)\left(\text{Li}_2(-xz) + \ln(x)\ln(1+xz) - \frac{\pi^2}{6}\right)$$

or semi-analytic approach (need a difference equation, see next slides)

- Harmonic Sums $S_{\vec{m}}(n)$ [Vermaseren 98]
- Harmonic PolyLogs $H_{\vec{m}}(x)$ [Remiddi/Vermaseren 00]
- .. wait for 3 slides ..

Methods IV b: numeric Integration, Deqs

very general setup [Laporta 00]

derive difference equation for generalized master $U(x) \equiv \int \frac{1}{D_1^x D_2 \dots D_N}$

$$\sum_{j=0}^{R} p_j(x) U(x+j) \quad = \quad F(x)$$

solve via factorial series $U(x) = U_0(x) + \sum_{j=1}^{R} U_j(x)$, where

$$U_j(x) = \mu_j^x \sum_{s=0}^{\infty} a_j(s) \frac{\Gamma(x+1)}{\Gamma(x+1+s-K_j)}$$

plug in, get μ, $K_j(d)$, and recursion rels for $a_j(s)$.

need bc for fixing, say, $a_j(0)$

Methods IV b: numeric Integration, Deqs

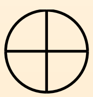
particularly simple bc at large x:

$$U(x) = \int \frac{1}{(p_1^2+1)^x}\, g(p_1)$$

$$\lim_{x\to\infty} U(x) = \left[\int \frac{1}{(p_1^2+1)^x}\right] \times \left[g(0)\right] \sim (1)^x x^{-d/2} g(0)$$

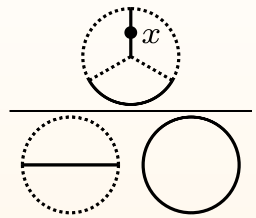
while factorial series behaves as $\sum_j \mu_j^x x^{K_j} a_j(0)$

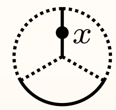
numerics: truncate sum. example:



$$= +1.27227054184989419939788 - 5.67991293994853579036683\epsilon$$
$$+17.6797238948173732343788\epsilon^2 - 46.5721846649543261864019\epsilon^3$$
$$+111.658522176214385363568\epsilon^4 - 252.463963901002177743236\epsilon^5$$
$$+549.301665961614269417 05\epsilon^6 - 1164.51205889715216235 46\epsilon^7 + \mathcal{O}(\epsilon^8)$$

Methods IV c: Deqs, concrete example

$M_h(x) \equiv$ 

$=$ 

$\dfrac{1}{J^3}\,\dfrac{2^{d-2}\Gamma(\frac{1}{2})}{\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2})}$

difference equation for M_h:

$$\begin{aligned} 0 \;=\;& -2(x+1)M_h(x+2) + 3(x+2-d/2)M_h(x+1) - (x+3-d)M_h(x) \\ &+ \frac{\Gamma(x+5-\frac{3d}{2})}{\Gamma(x+1)}\frac{3-d}{\Gamma(5-\frac{3d}{2})}M_h(0) + \frac{\Gamma(x+3-d)}{\Gamma(x+1)}\frac{1}{\Gamma(2-d)} \\ &- \frac{\Gamma(x+2-\frac{d}{2})}{\Gamma(x+1)}\frac{2}{\Gamma(1-\frac{d}{2})} + \frac{\Gamma(x+5-\frac{3d}{2})\Gamma(x+3-d)}{\Gamma(x)\Gamma(x+7-2d)}\frac{2}{\Gamma(1-\frac{d}{2})} \end{aligned}$$

2nd order deq $\rightarrow$ 2 boundary conditions

$$\begin{aligned} M_h(0) &= -\frac{\Gamma(\frac{3d}{2})\Gamma(1-\frac{3d}{2})\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{\Gamma(d)\Gamma(d-2)\Gamma(1-d)} \\ M_h(x \gg 1) &= \frac{\Gamma(x-\frac{d}{2})}{\Gamma(x)}\frac{(d-3)(d-6)}{\Gamma(1-\frac{d}{2})} \;\sim\; x^{-\frac{d}{2}}\,\frac{(d-3)(d-6)}{\Gamma(1-\frac{d}{2})} \end{aligned}$$

Methods IV c: harmonic sums $S_{\vec{m}}(n)$

'the language that Feynman integrals speak'? [J. Vermaseren]

nested sums $S_m(n) = \sum_{i=1}^n \frac{1}{i^m}$; $S_{m,\vec{m}}(n) = \sum_{i=1}^n \frac{1}{i^m} S_{\vec{m}}(i)$ [$S_m(\infty) = \zeta(m)$]

that satisfy an algebra $S_a(n)S_b(n) = S_{a,b}(n) + S_{b,a}(n) - S_{a+b}(n)$ etc.

usage: via some (not yet very short) detour, solve M_h

- Laplace trafo $M_h(x) = \int_0^1 dt\, t^{x-1}\, v(t)$
- solve differential Eqn via Harmonic PolyLogs [$H_{01}(x) = \text{Li}_2(x)$]
  $H_0(x) = \ln(x)$; $H_1(x) = -\ln(1-x)$; $H_{-1}(x) = \ln(1+x)$
  $H_{m,\vec{m}}(x) = \int_0^x dy\, f(m,y)\, H_{\vec{m}}(y)$; $f(\{0,1,-1\},x) = \{\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}\}$
- translate $H_{\vec{m}}(1) \longrightarrow S_{\vec{m}}(\infty)$
- express $S_{\vec{m}}(\infty)$ in term of known numbers (where possible)
- some example relations in /projects/harmsums/[h]table*.prc

```
Mh1 =
   + ep^2 * (  - 2*z3 )

   + ep^3 * ( 7/60*pi^4 - 16*li4half + 2/3*ln2^2*pi^2 - 2/3*ln2^4 )

   + ep^4 * (  - 16*li5half - 49/180*ln2*pi^4 - 2/9*ln2^3*pi^2 + 2/15*
     ln2^5 - 137/8*z5 - 2*z3 + 19/12*z3*pi^2 )

   + ep^5 * ( 7/60*pi^4 + 41/945*pi^6 + 110*s6 - 16*li6half - 16*li4half
      + 10/3*li4half*pi^2 + 2/3*ln2^2*pi^2 - 1/360*ln2^2*pi^4 - 2/3*ln2^4
      + 7/36*ln2^4*pi^2 - 1/45*ln2^6 - 4*z3 - 103/2*z3^2 )

   + ep^6 * ( 7/30*pi^4 - 816/7*s7b - 46/7*s7a - 16*li7half - 16*li5half
      + 10/3*li5half*pi^2 - 32*li4half - 49/180*ln2*pi^4 - 1709/3780*ln2*
     pi^6 + 46/7*ln2*s6 + 4/3*ln2^2*pi^2 - 2/9*ln2^3*pi^2 + 1/1080*ln2^3*
     pi^4 - 4/3*ln2^4 + 2/15*ln2^5 - 7/180*ln2^5*pi^2 + 1/315*ln2^7 -
     490507/448*z7 - 137/8*z5 + 41257/672*z5*pi^2 + 1705/16*z5*ln2^2 - 8*
     z3 + 19/12*z3*pi^2 + 671/126*z3*pi^4 - 156*z3*li4half + 13/2*z3*ln2^2
     *pi^2 - 13/2*z3*ln2^4 - 115/14*z3^2*ln2 )

   + ep^7 * ( 7/15*pi^4 + 41/945*pi^6 + 12041677/127008000*pi^8 - 46/7*s8d
      + 816/7*s8c + 13876/7*s8b + 389891/2240*s8a + 110*s6 - 461/42*s6*
     pi^2 - 16*li8half - 16*li6half + 10/3*li6half*pi^2 - 32*li5half - 64*
     li4half + 10/3*li4half*pi^2 + 6571/630*li4half*pi^4 - 49/90*ln2*pi^4
      + 8/3*ln2^2*pi^2 - 1/360*ln2^2*pi^4 - 2531/15120*ln2^2*pi^6 - 408/7*
     ln2^2*s6 - 4/9*ln2^3*pi^2 - 8/3*ln2^4 + 7/36*ln2^4*pi^2 + 2627/6048*
     ln2^4*pi^4 + 4/15*ln2^5 - 1/45*ln2^6 + 7/1080*ln2^6*pi^2 - 1/2520*
     ln2^8 - 137/4*z5 - 408/7*z5*ln2*pi^2 - 16*z3 + 19/6*z3*pi^2 - 1000/7*
     z3*li5half - 1531/504*z3*ln2*pi^4 - 125/63*z3*ln2^3*pi^2 + 25/21*z3*
     ln2^5 - 562693/448*z3*z5 - 103/2*z3^2 - 3505/168*z3^2*pi^2 - 459/28*
     z3^2*ln2^2 )
```

where `z3` $= \zeta(3)$, `li4half` $= \mathrm{Li}_4(1/2) = \sum_{k\geq 1} \frac{1}{k^4 2^k}$ etc.

Methods IV d: sum-integrals

sum-integrals are hard!

- 4d ϵ-expansion of (some relevant) 1-2-3-loop integrals exist, up to constant term
- derived by hand, case by case, with sweat ..
- 1-loop example: $\SumInt_{q_b} \frac{1}{[q_0^2+\vec{q}^2]^n} = \frac{2\pi^{d/2} T^{1+d}}{(2\pi T)^{2n}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \zeta(2n-d)$
- not a *single* 4-loop example is solved yet

$\longrightarrow$ new methods needed?

- again, one may try to copy the $T=0$ methods at $T>0$
- IBP, Deqs, numerics / HPLs, harmSums
- only *words* at this stage, nothing tested

Methods VI: Lattice perturbation theory

amusing: 1loop tadpole has elliptic integral in 3d [M.Shaposhnikov]

$$a^{2-d}\int_{-\pi}^{\pi}\frac{d^d\hat{k}}{(2\pi)^d}\frac{1}{\sum_{\mu=0}^{d-1}4\sin^2(\hat{k}_\mu/2)+\hat{m}^2}=\frac{1}{a}\sum_{n\geq 0}\hat{m}^{2n}\left(\{\Sigma,\xi\}+\{1\}\hat{m}\right)$$

where $\Sigma = 4\pi G(0) = \frac{8}{\pi}(18+2\sqrt{2}-10\sqrt{3}-7\sqrt{6})K^2((2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2)$

2loop example:

$$\kappa_5 = \frac{1}{\pi^4}\int_{-\pi/2}^{\pi/2} d^3x\, d^3y \frac{\sum_i \sin^2 x_i \sin^2(x_i+y_i)\sin^2 y_i}{\sum_i \sin^2 x_i \sum_i \sin^2(x_i+y_i)\sum_i \sin^2 y_i} = 1.013041(1)$$

$\longrightarrow$ classification? *very* little is known systematically.

1loop IBP + coordinate-space method [Lüscher/Weisz] [Becher/Melnikov]

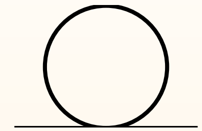
or

Numerical Stochastic Perturbation Theory [F. Di Renzo, V. Miccio, C. Torrero]

no diagrams!

Methods VI: Lattice perturbation theory

1loop IBP + coordinate-space method [Lüscher/Weisz] [Becher/Melnikov]

Example: 1-loop massive tadpole in 3d, $I(m) \equiv$ 

$$
\begin{aligned}
I(m) &= \int_{-\pi}^{\pi} \frac{d^3k}{(2\pi)^3} \frac{1}{\sum_{j=1}^{3} 4\sin^2(k_j/2) + m^2} \\
&= \frac{1}{4\pi a} \sum_{n\geq 0} (am)^{2n} \left[(a_n \Sigma + b_n \xi) + (am) c_n 1 \right]
\end{aligned}
$$

$$a_{0..6} = \frac{(-1)^{n+1}}{4^n (2n)!} \frac{8}{2^n} \{-1/8, 0, 1, 53, 13559/3, 612241, 124073817\}$$

$$b_{0..6} = \frac{(-1)^{n}}{4^n (2n)!} \frac{16}{2^n} \{0, 1, 17, 677, 155591/3, 6685249, 1321874313\}$$

$$c_{0..6} = \frac{(-1)^{n+1}}{4^n (2n)!} \frac{1}{2n+1} \{1, 3, 33, 843, 40257, 3152115, 370071585\}$$

where $\Sigma = 3.1759...$ and $\xi = 0.15285933...$ are 'master' lattice constants

Conclusions

Trento 05 → Trento 07 → Trento 09

$$\frac{p_G}{p_{SB}} = \#_{(6)} \left(\frac{g_M^2}{T}\right)^3 + [\delta\mathcal{L}_M]_{(9)}$$

$$g_M^2 = g_E^2 \left[1 + \#_{(7)} \frac{g_E^2}{m_E} + \left(\frac{g_E^2}{m_E}\right)^2 \left(\#_{(8)} + \#_{(10)} \frac{\lambda_E}{g_E^2}\right) + \ldots_{(9)}\right]$$

$$\frac{p_M}{p_{SB}} = \left[\#_{(3)} + \frac{g_E^2}{m_E}\left(\#_{(4)} + \#_{(6)}\frac{\lambda_E}{g_E^2}\right) + \left(\frac{g_E^2}{m_E}\right)^2 \left(\#_{(5)} + \#_{(7)}\frac{\lambda_E}{g_E^2} + \#_{(9)}\left(\frac{\lambda_E}{g_E^2}\right)^2\right)\right.$$
$$+ \left(\frac{g_E^2}{m_E}\right)^3 \left(\#_{(6)} + \#_{(8)}\frac{\lambda_E}{g_E^2} + \#_{(10)}\left(\frac{\lambda_E}{g_E^2}\right)^2 + \#_{(12)}\left(\frac{\lambda_E}{g_E^2}\right)^3\right)$$
$$\left. + [\text{3d 5loop 0pt}]_{(7)} + [\delta\mathcal{L}_E]_{(7)} + [\text{3d 6loop 0pt}]_{(8)} + \ldots_{(9)}\right]$$

$$m_E^2 = T^2\left[\#_{(3)} g^2 + \#_{(5)} g^4 + [\text{4d 3loop 2pt}]_{(7)} + \ldots_{(9)}\right]$$

$$\lambda_E = T\left[\#_{(6)} g^4 + \#_{(8)} g^6 + \ldots_{(10)}\right]$$

$$g_E^2 = T\left[g^2 + \#_{(6)} g^4 + \#_{(8)} g^6 + \ldots_{(10)}\right]$$

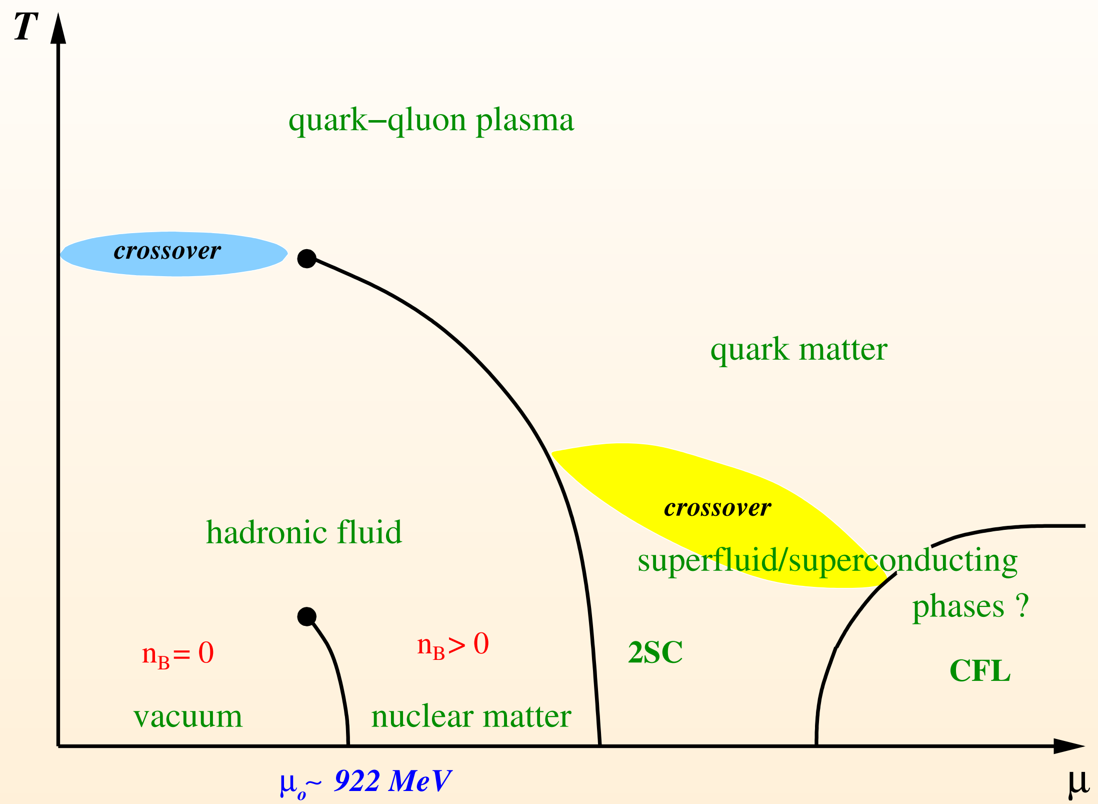
$$\frac{p_E}{p_{SB}} = \#_{(0)} + \#_{(2)} g^2 + \#_{(2)} g^4 + \#_{(6)} g^6 + [\text{4d 5loop 0pt}]_{(8)} + \ldots_{(10)}$$

notation: $\#_{(n)}$ **enters** p_{QCD} **at** g^n

[cave: no $\frac{1}{\epsilon} + 1 + \epsilon$ and no IR/UV shown above]

Backup slides

The QCD (equilibrium) phase diagram



nature: early univ, μ tiny ($\sim \frac{\#baryons}{entropy}$), $T_c \sim 170MeV \sim 10\mu s$

neutron/quark stars

lab expt.: SPS / RHIC $\mu_B \sim \frac{\#baryons}{pions} \sim 45MeV$ / LHC / GSI

The IR problem

from now on: high T, $\mu_B \equiv 0$

$\Rightarrow$ asymptotic freedom!

$$g(Q \sim T) \overset{ren}{\propto} \frac{1}{\ln \frac{T}{\Lambda_{\overline{MS}}}} \overset{T\to\infty}{\longrightarrow} 0$$

$\Rightarrow$ QGP = ideal gas of quarks/gluons?
+ perturbative corrections?

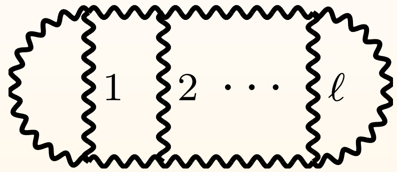
$T > 0$: $\int d^4p \to T \sum_{p_0} \int d^3p$

Matsubara frequencies: $p_0 = 2\pi n T$ (gluons, ghosts)
$p_0 = 2\pi(n + 1/2)T$ (fermions)

$\Rightarrow$ IR divergences from bosons, $n = 0$, *'zero mode'*

The IR problem

[Linde 1979; Gross/Pisarski/Yaffe 1981]



(ℓ+1) loops, 2ℓ vert, 3ℓ propags

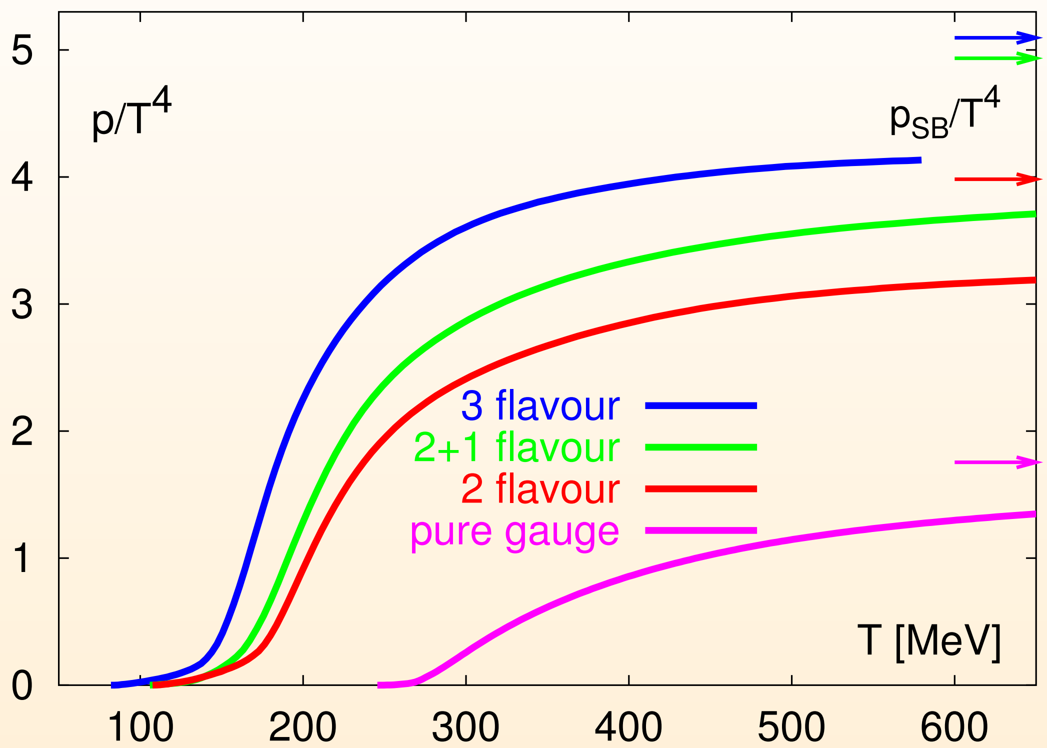
$$\sim \left(T \sum_n \int d^3p \right)^{\ell+1} \frac{(gp)^{2\ell}}{[(2\pi n T)^2 + \vec{p}^2 + \Pi(2\pi n T, \vec{p})]^{3\ell}}$$

IR power counting: n=0, define $\Pi(0, \vec{p} \to 0) \equiv m^2$

$$\sim T^{\ell+1} g^{2\ell} m^{3(\ell+1)+2\ell-6\ell} = g^6 T^4 \left(\frac{g^2 T}{m} \right)^{\ell-3}$$

- $\Pi_L(0, \vec{p}) \sim (gT)^2 \Leftarrow$ OK: get series in g
- $\Pi_T(0, \vec{p}) \sim (g^2T)^2 \Leftarrow$ all orders important!

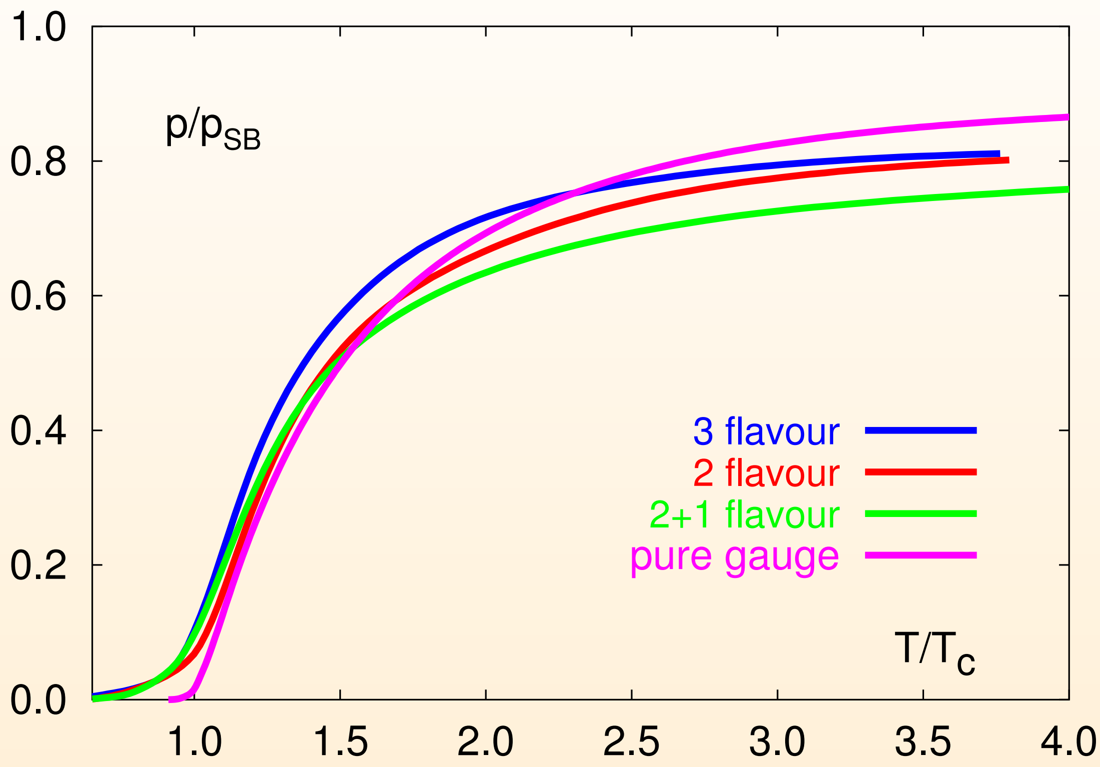
$p(T)$ on the lattice ($\mu_B = 0$)



[lattice data from Karsch et.al.]

at $T \to \infty$, expect ideal gas: $p_{\text{SB}} = \left(16 + \frac{21}{2} N_f\right) \frac{\pi^2 T^4}{90}$

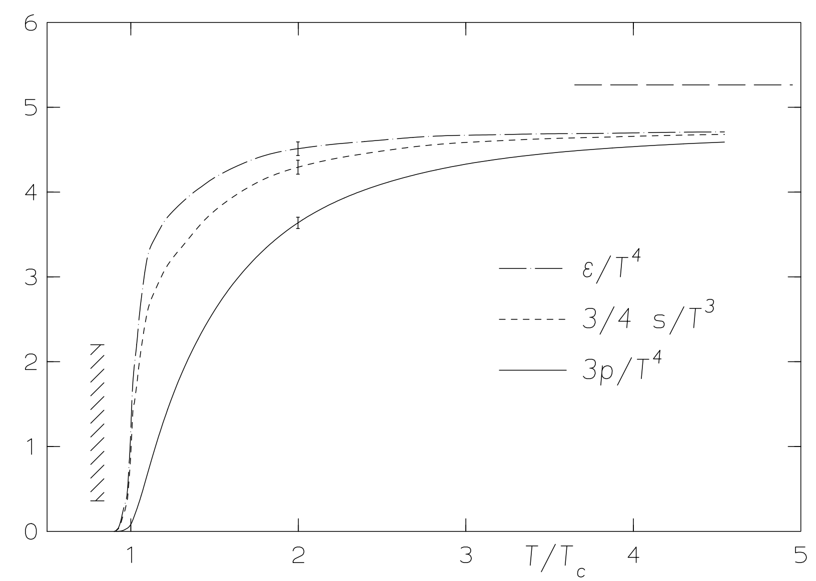
$p(T)$ on the lattice ($\mu_B = 0$)



[lattice data from Karsch et.al.]

at $T \to \infty$, expect ideal gas: $p_{\text{SB}} = \left(16 + \frac{21}{2} N_f\right) \frac{\pi^2 T^4}{90}$

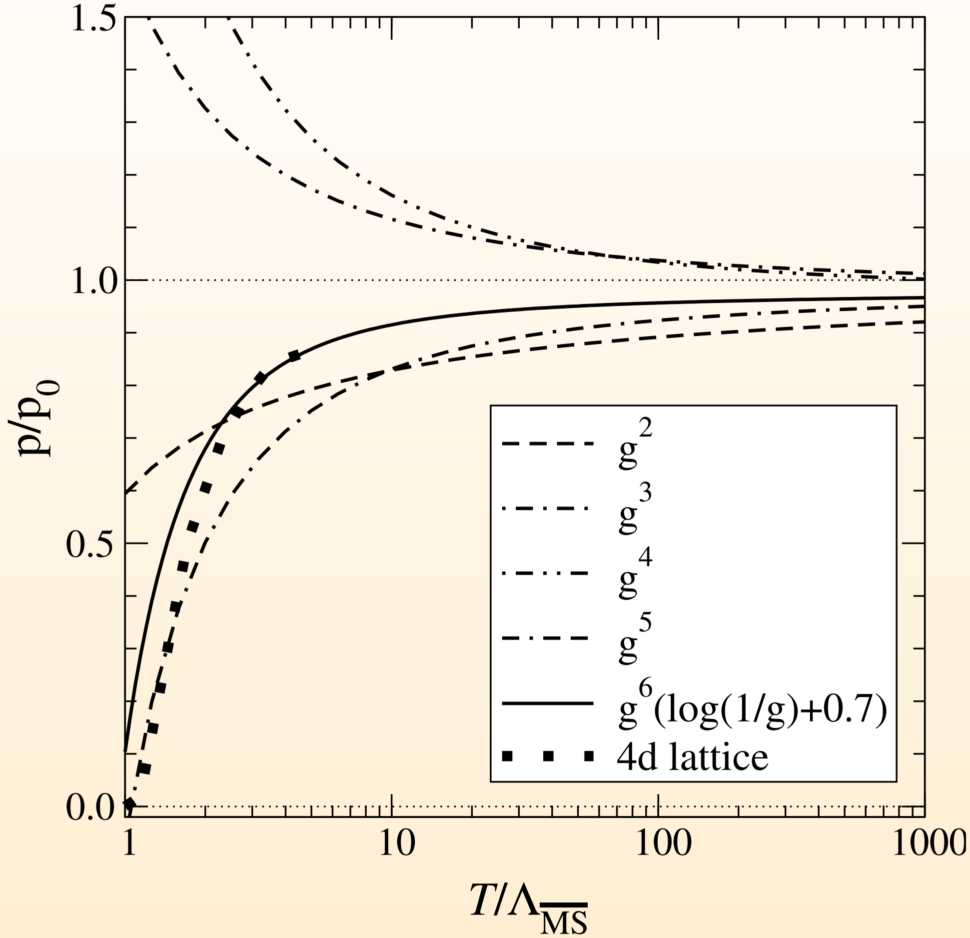
energy density in pure YM



pure YM, [Boyd/Engels/Karsch/Laermann/Legeland/Lütgemeier/Petersson 1995]

continuum extrapolated, $T_0 \approx 270 MeV$

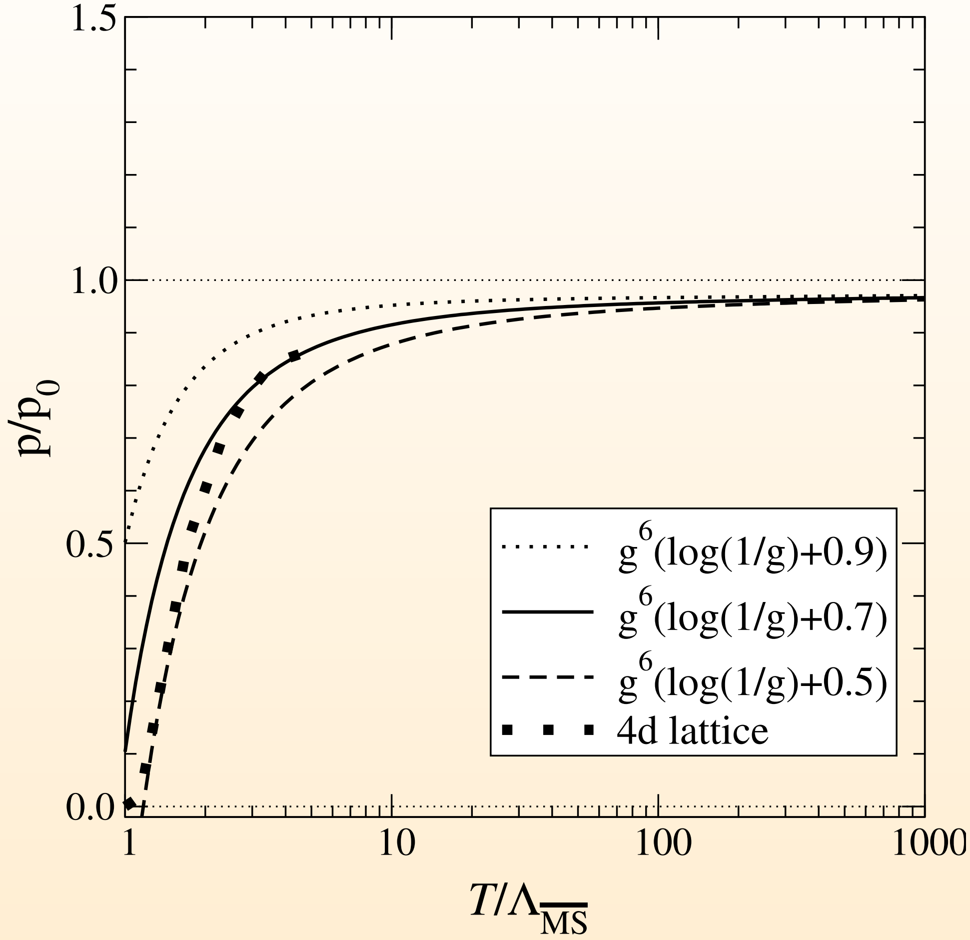
thermal pressure $p(T)$: 4d vs 3d ($N_f = 0$)



g^6 constant is a guess.

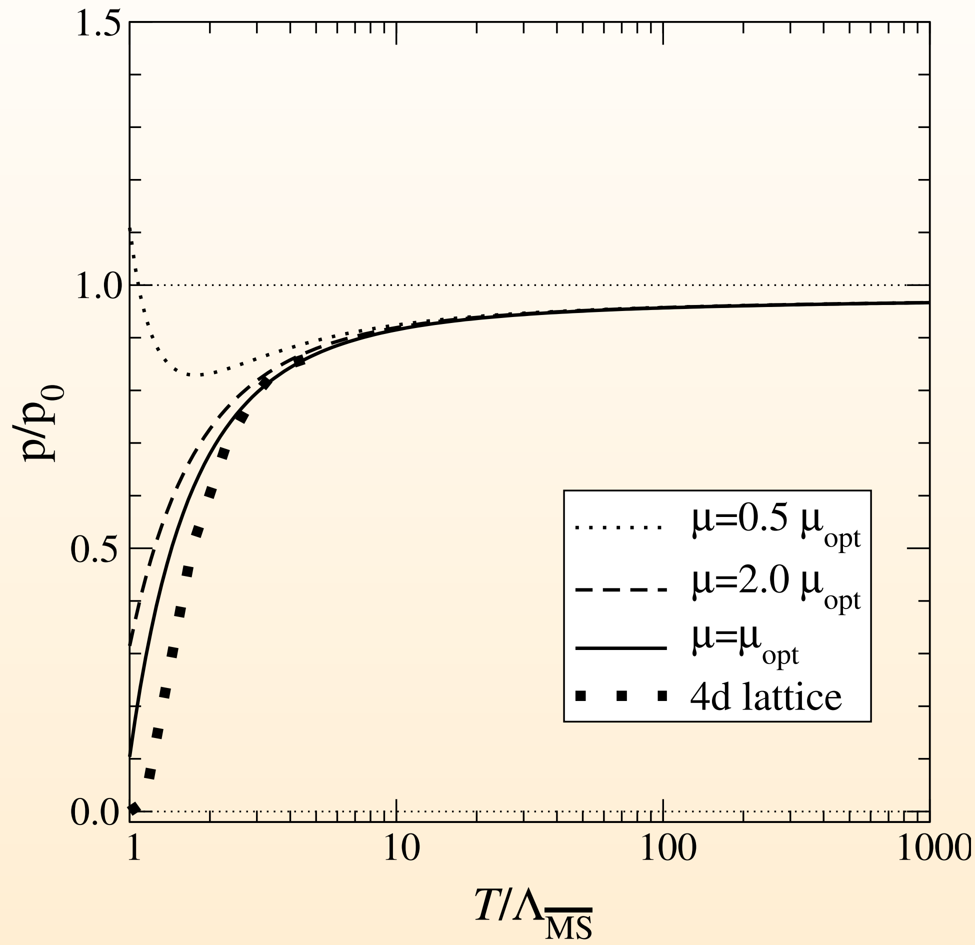
non-perturbative contrib not known, but computable!

thermal pressure $p(T)$: 4d vs 3d ($N_f = 0$)



depencence on g^6 constant

thermal pressure $p(T)$: 4d vs 3d ($N_f = 0$)



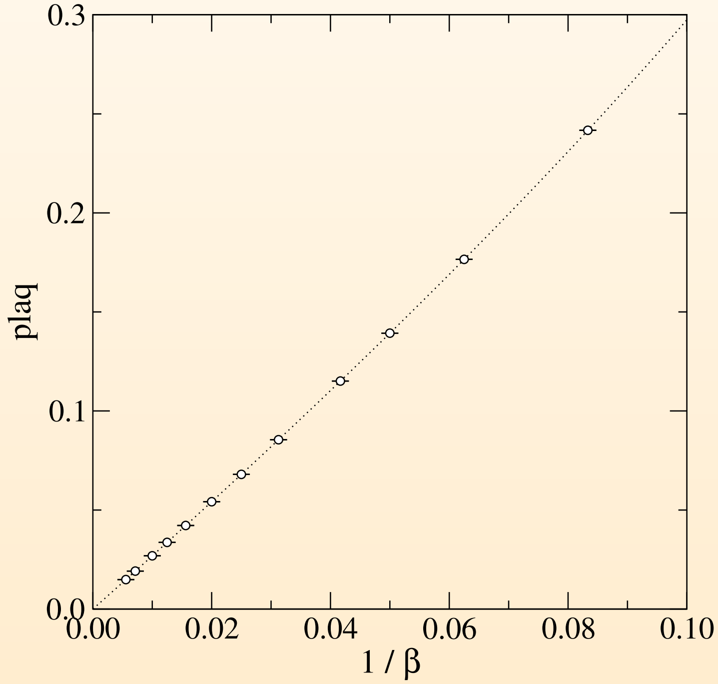
scale dependence

Methods V: Lattice MC

3d, finite box $(aL)^3$. infinite-volume ($L \to \infty$) and continuum ($\frac{1}{\beta} \equiv \frac{g_{\mathsf{M}}^2 a}{2N_c} \to 0$) limits

$$\frac{1}{2g_{\mathsf{M}}^2}\left\langle \mathrm{Tr}\,[F_{kl}^2]\right\rangle_{\overline{\mathsf{MS}}} \equiv g_{\mathsf{M}}^2 \frac{\partial}{\partial g_{\mathsf{M}}^2} p_{\mathsf{G},\overline{\mathsf{MS}}} = 3 g_{\mathsf{M}}^6 \frac{d_A C_A^3}{(4\pi)^4}\left[\alpha_{\mathsf{G}}\left(\ln\frac{\bar\mu}{2C_A g_{\mathsf{M}}^2} - \frac{1}{3}\right) + B_{\mathsf{G}} + \mathcal{O}(\epsilon)\right]$$

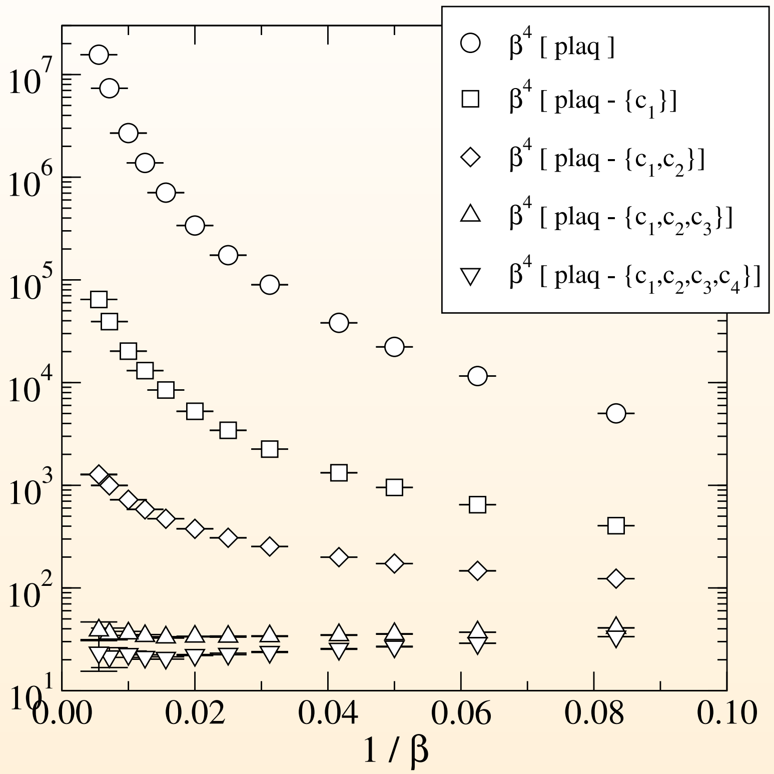
$$8\frac{d_A C_A^6}{(4\pi)^4} B_{\mathsf{G}} = \lim_{\beta\to\infty} \beta^4 \left\{\left\langle 1 - \frac{1}{C_A}\mathrm{Tr}\,[P_{12}]\right\rangle_a - \left[\frac{c_1}{\beta} + \frac{c_2}{\beta^2} + \frac{c_3}{\beta^3} + \frac{c_4}{\beta^4}\left(\ln\beta + c_4'\right)\right]\right\}$$

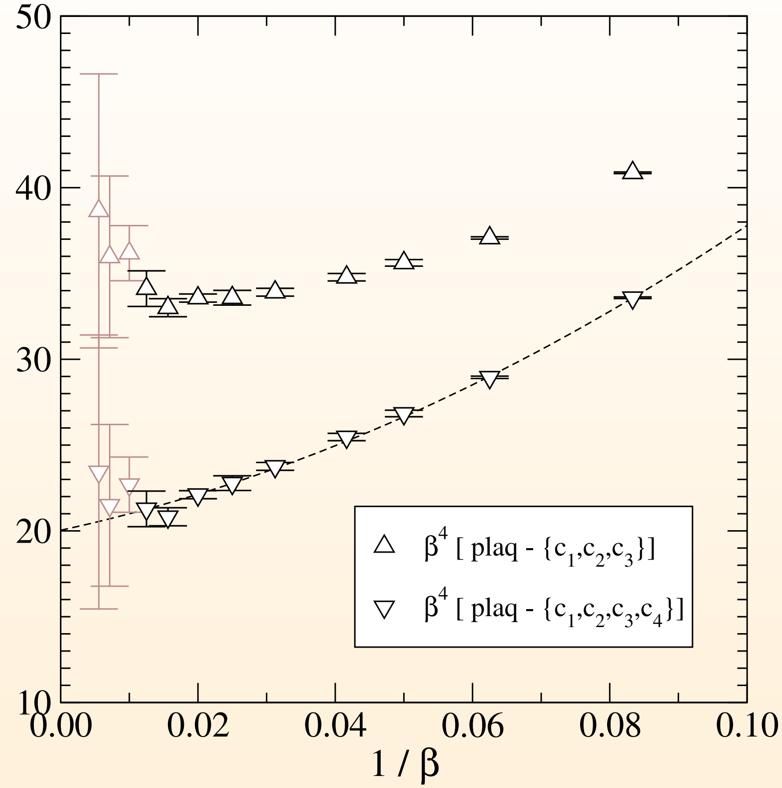


statistical errors are (much) smaller than the symbol sizes

Fit: $c_1/\beta + c_2/\beta^2 + c_3/\beta^3 + c_4 \ln\beta/\beta^4$ $+c_4'/\beta^4$ $+c_5/\beta^5 + c_6/\beta^6$

Methods V: Lattice MC





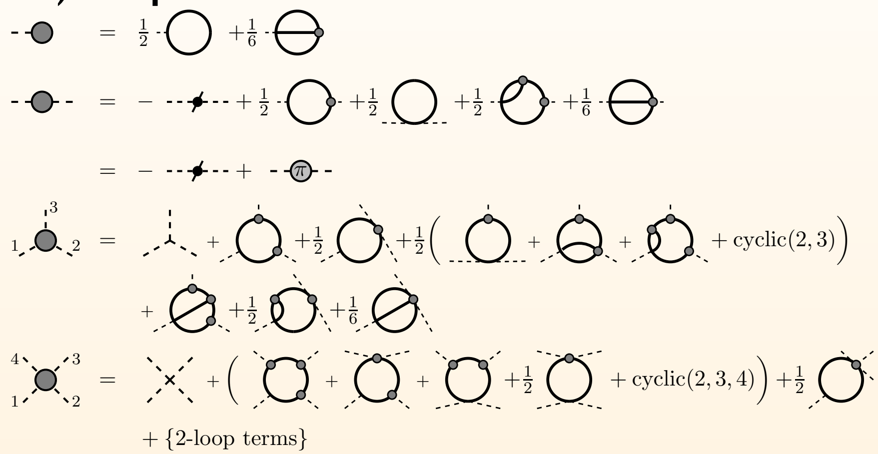
significance loss due to the UV subtractions

continum limit of infinite-volume extrapolated data

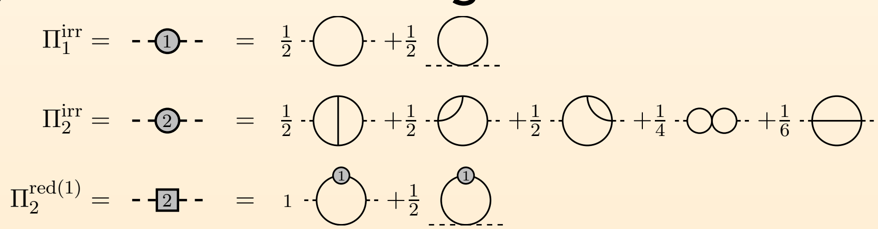
$$B_{\mathsf{G}} + \left(\frac{43}{12} - \frac{157}{768}\pi^2\right) c_4' \quad = \quad 10.7 \pm 0.4 \qquad (N_c = 3)$$

more on diagram generation

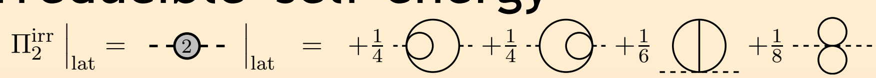
Schwiger-Dyson (SD) eqs



(2PI) skeletons generate self-energies. for bosonic particles ($c_i = \frac{1}{2}$):



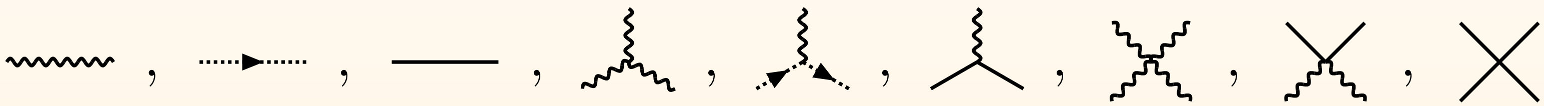
LAT: additional irreducible self-energy



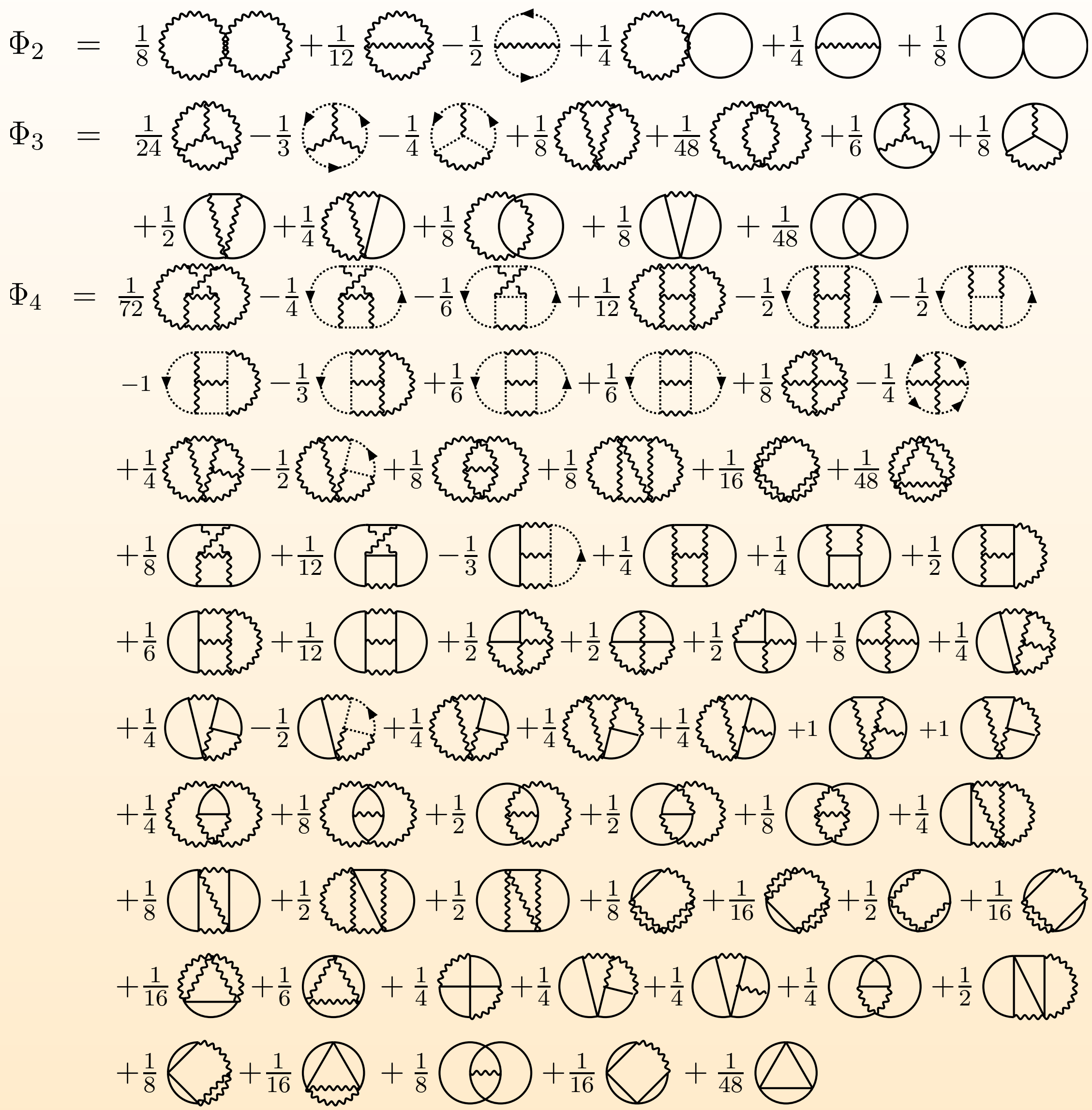
more on diagram generation

consider SU(N) gauge theory with fermions and a scalar field.

this class includes QCD, QED, EW, SQED

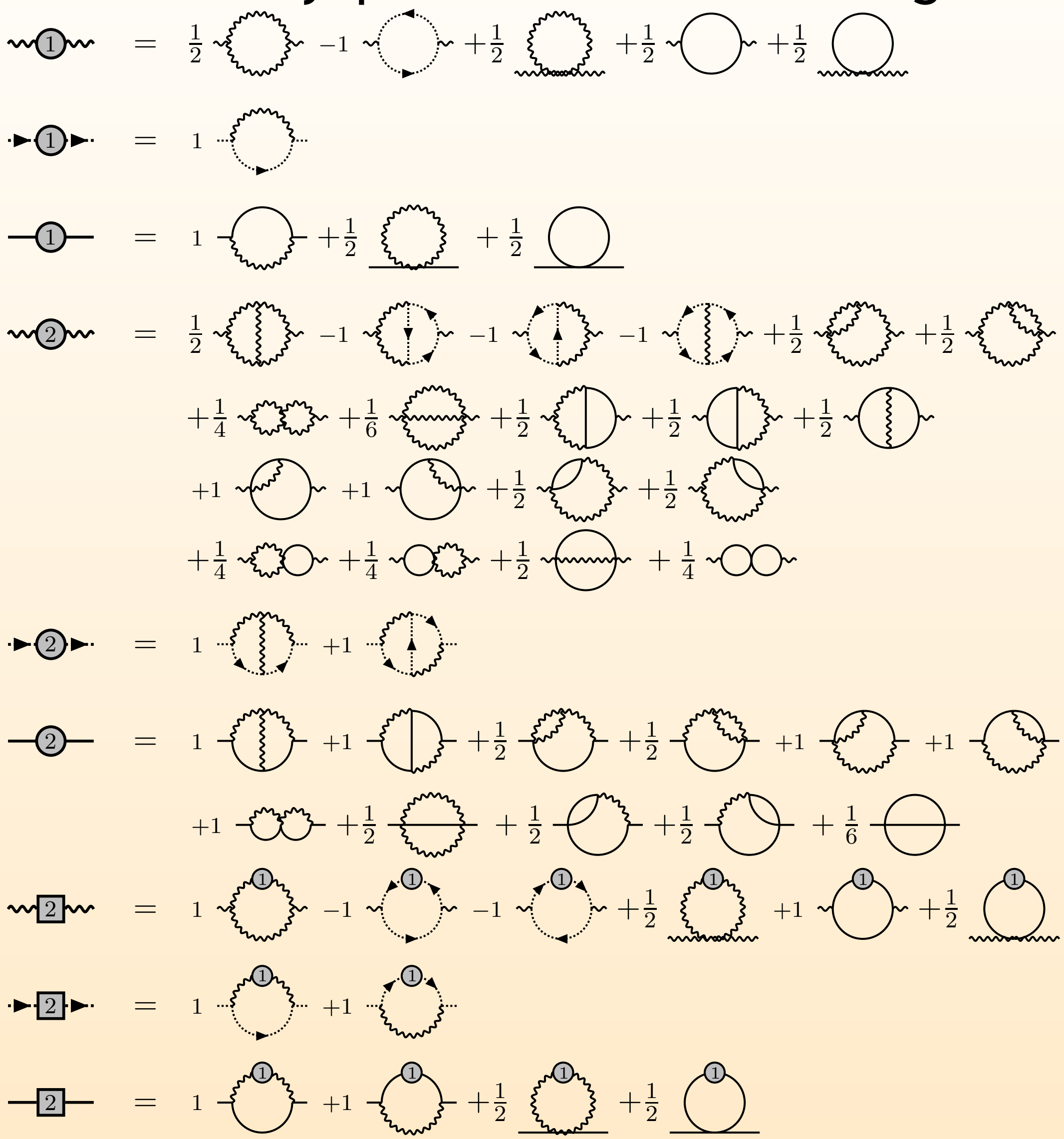


more on diagram generation



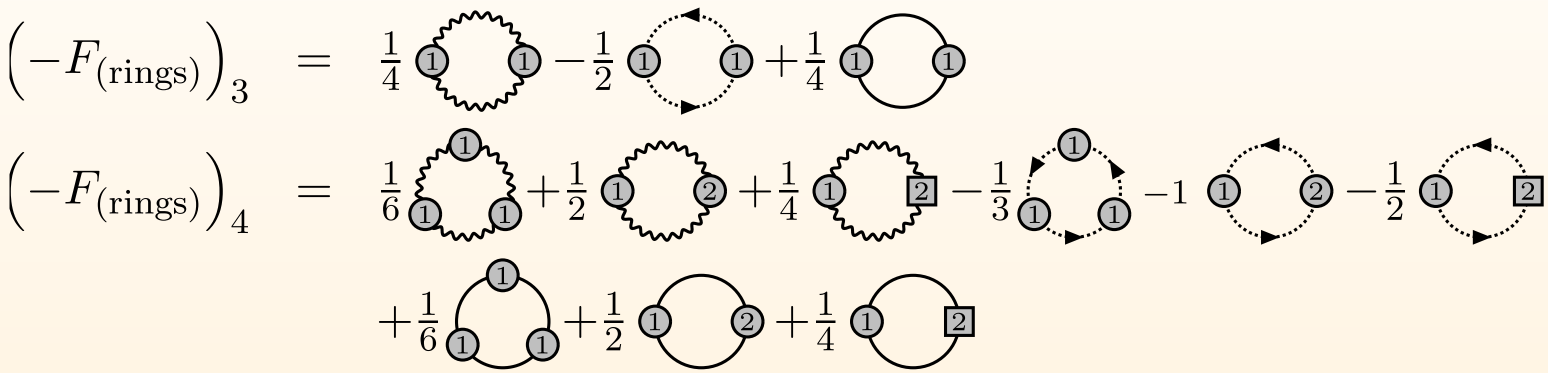
more on diagram generation

the skeletons immediately produce the self-energies of the model



more on diagram generation

ring diagrams for the model



extremely economic structure of the skeleton expansion:

the few ring diagrams above summarize

22 (276) 3-loop (4-loop) diagrams