Computation and applications of 4-loop bubbles

York Schröder

(Univ Bielefeld)

work with: Ervin Bejdakic, Aleksi Vuorinen

LL06, Eisenach, 26 Apr 2006

Outline

here: #loops $\gg 1$, #legs $\ll 1$

why vacuum diagrams (bubbles)?

- pressure $p(T)$ important for cosmology: determines cooling rate of the universe

$$\partial_t T = -\frac{\sqrt{24\pi}}{m_{\mathsf{Pl}}} \frac{\sqrt{e(T)}}{\partial_T \ln s(T)}$$

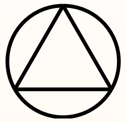
  with entropy $s = \partial_T \, p$ and energy density $e = T s - p$

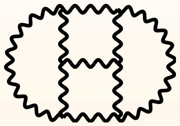
- map many other problems to calculating bubbles
  - ▷ *talks by Czakon, Steinhauser, Sturm*

Methods (''traditional''?)

- Diagram generation
- Classification, scalarization
- Reduction
- Integration

Reduction, IBP

can do 4-loop scalar theory on paper:  1 integral

for YM, need a computer:  25M integrals ($2^9 6^6$)

powerful method: integration by parts (IBP) [Chetyrkin/Tkachov 81]

$\Rightarrow$ systematically use $0 = \int d^d k \, \partial_{k_\mu} f_\mu(k)$

seen many instances already: Laporta, Baikov, Gröbner Chetyrkin, Smirnov

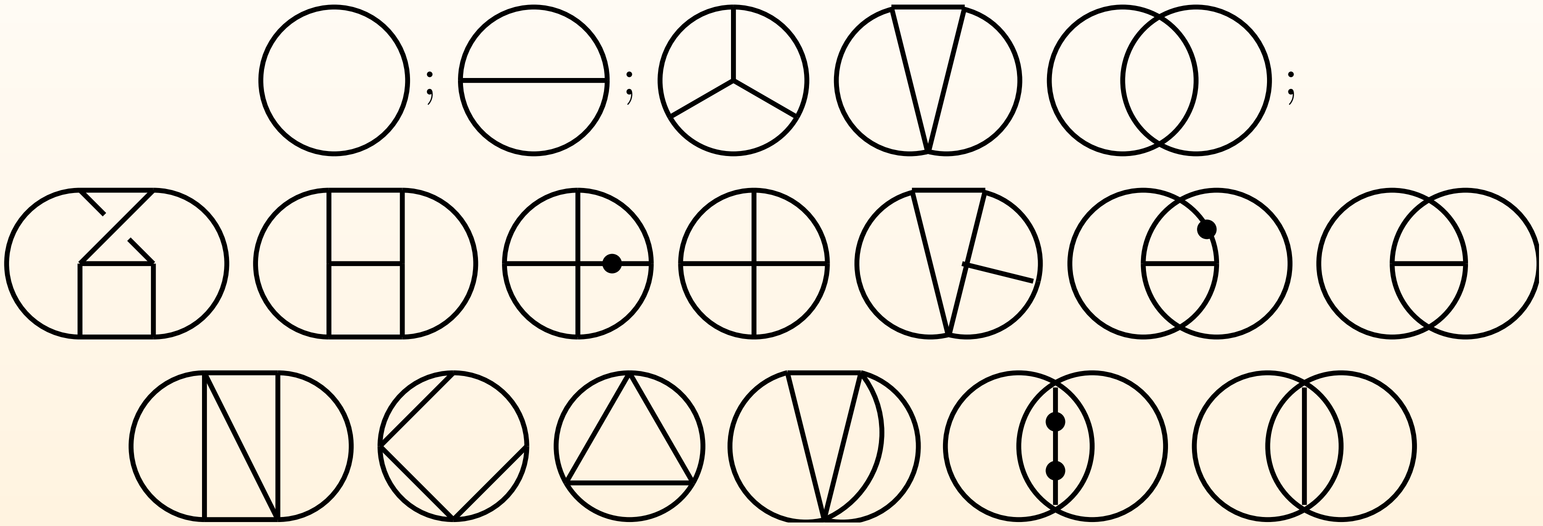
key idea: lexicographic ordering among all loop integrals [Laporta 00]

arrive at rep in terms of irreducible ($\equiv$ master) integrals

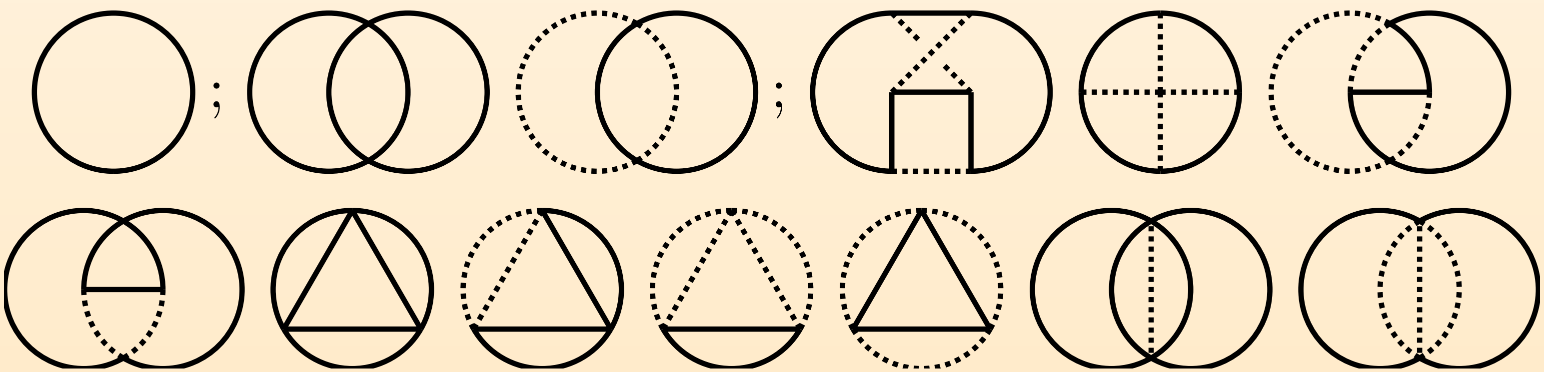
$$\sum_i \frac{\text{poly}_i(d,\xi)}{\text{poly}_i(d)} \text{Master}_i(d)$$

Classes of masters

18 fully massive master ints:



13 ''QED'' type master ints:



Integration

Evaluating Masters

- numerical integration; cave: precision (MC?)
- explicit integration; can be an ‘‘art’’
- difference equations
  - ▷ *solve directly*
  - ▷ *solve numerically*
  - ▷ *Laplace transform*
- differential equations

Mathematical structure

- interested in the coefficients of an ϵ expansion
- in many cases, these are from a generic class of functions/numbers
- e.g. harmonic polylogarithms HPL(x) [Remiddi/Vermaseren 00]
- e.g. harmonic sums $S(N)$ [Vermaseren 98]

Harmonic Sums to weight 9

$$S_{a,\underline{m}}(N) \equiv \sum_{i=1}^{N} \frac{[\mathrm{sgn}(a)]^i}{i^{|a|}} S_{\underline{m}}(i) \quad , \quad S(i \geq 0) = 1 \quad , \quad S(i < 0) = 0$$

$$\ln 2 = -S_{-1}(\infty)$$

$$\zeta_{n\geq 2} = S_n(\infty) = \zeta(n)$$

$$a_{n\geq 4} = -S_{-1,\underline{1}_{n-1}}(\infty) = Li_n\,(1/2)$$

$$s_6 = S_{-5,-1}(\infty) \approx +0.98744142640329971377165000804182021413602714 89$$

$$s_{7a} = S_{-5,1,1}(\infty) \approx -0.95296007575629860340865215892596050767328040 17$$

$$s_{7b} = S_{5,-1,-1}(\infty) \approx +1.02912126296432453422440408804384184300201671 26$$

$$s_{8a} = S_{5,3}(\infty) \approx +1.04178502918279188338999002080231238008156211 01$$

$$s_{8b} = S_{-7,-1}(\infty) \approx +0.99644774839783766598087290122422927214404887 82$$

$$s_{8c} = S_{-5,-1,-1,-1}(\infty) \approx +0.98396667382173367092073025030655946912191093 09$$

$$s_{8d} = S_{-5,-1,1,1}(\infty) \approx +0.99996261346268344769671661371698277760950413 87$$

$$s_{9a} = S_{7,-1,-1}(\infty) \approx +1.00640196269235635900624639964476428233166947 26$$

$$s_{9b} = S_{-7,-1,1}(\infty) \approx +0.99842952512288855439558760159302751364219403 56$$

$$s_{9c} = S_{-6,-2,-1}(\infty) \approx -0.98747515763691535588078119527905528481850005 81$$

$$s_{9d} = S_{-5,-1,1,1,1}(\infty) \approx +1.00219817413397743629950024721465888358222751 73$$

$$s_{9e} = S_{-5,-1,-1,-1,1}(\infty) \approx +0.98591171955244547261592606241209220907857551 87$$

$$s_{9f} = S_{-5,-1,-1,1,-1}(\infty) \approx +0.97848117128116624247868774328701556375431608 57$$

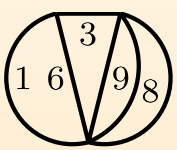
Explicit Integration

trivial integrals: $\Rightarrow$ solvable in terms of Γ's, e.g.

$$
\begin{aligned}
BB_2 &\equiv \frac{1}{J^4}\int_{p_{1..4}}^{(d)} \frac{1}{p_1^2+1}\,\frac{1}{p_2^2+1}\,\frac{1}{p_3^2}\,\frac{1}{p_4^2}\,\frac{1}{(p_1+p_2+p_3+p_4)^2} \\
&= 3(d-2)4^{d-3}\frac{\Gamma(5-2d)\Gamma(\frac{8-3d}{2})\Gamma(\frac{5-d}{2})\Gamma^2(\frac{d}{2})}{\Gamma(\frac{11-3d}{2})\Gamma^3(\frac{4-d}{2})}
\end{aligned}
$$

a more laborious example [YS,AV]:

3d, Euclidean, massive, dim. reg., $\overline{\text{MS}}$, x-space, ...



$$
{}_2 = \left(\frac{\bar{\mu}}{m_{316}}\right)^{8\epsilon}\frac{1}{32}\left[\frac{1}{\epsilon^2}+\frac{8}{\epsilon}+4\,S\left(\frac{m_{316}}{m_{16289}},\frac{2m_1}{m_{316}},\frac{2m_3}{m_{316}}-1\right)+\mathcal{O}(\epsilon)\right]
$$

$$
\begin{aligned}
\text{where}\quad S(x,y,z) &= 13+\frac{7}{12}\pi^2+2\mathrm{Li}_2(1-y)+2\mathrm{Li}_2(y+z)+2\mathrm{Li}_2(-z) \\
&\quad -4\,(\ln x)^2+8\frac{1-x}{x(1+z)}\,\mathrm{Li}_2(1-x) \\
&\quad +8\left(1+\frac{1-x}{x(1+z)}\right)\left(\mathrm{Li}_2(-xz)+\ln(x)\ln(1+xz)-\frac{\pi^2}{6}\right)
\end{aligned}
$$

Difference Equations

repeat reduction with symbolic power x on one line

derive difference equation for generalized master $U(x) \equiv \int \frac{1}{D_1^x D_2 \ldots D_N}$

$$\sum_{j=0}^{R} p_j(x) U(x+j) \quad = \quad F(x)$$

compute boundary conditions, e.g. at $x = 0$, $x \gg 1$

typically, want $U(1)$

solve the difference equation

- directly (if 1st order)
- numerically (very general setup)
- Laplace transform

Difference equation: direct solution

$$T_3(x) \equiv \frac{1}{J^4}\int_{p_{1,2,3,4}}^{(d)} \frac{1}{(p_1^2+1)^x}\frac{1}{(p_1+p_4)^2}\frac{1}{p_2^2+1}\frac{1}{(p_2+p_4)^2}\frac{1}{p_3^2+1}\frac{1}{(p_3+p_4)^2}$$

would like to know $T_3(1)$. First order difference eq:

$$T_3(x+1) = c(x)T_3(x) + G(x)$$

Difference equation: direct solution

$$T_3(x) \equiv \frac{1}{J^4}\int_{p_{1,2,3,4}}^{(d)} \frac{1}{(p_1^2+1)^x}\frac{1}{(p_1+p_4)^2}\frac{1}{p_2^2+1}\frac{1}{(p_2+p_4)^2}\frac{1}{p_3^2+1}\frac{1}{(p_3+p_4)^2}$$

would like to know $T_3(1)$. First order difference eq:

$$T_3(x+1) = c(x)T_3(x)+G(x) = T_3(x_0)\left(\prod_{i=x_0}^{x} c(i)\right)+\sum_{j=x_0}^{x} G(j)\left(\prod_{i=j+1}^{x} c(i)\right)$$

use bc $T_3(x \gg 1) \propto J(x \gg 1)$, since $c(x \ll 1) \sim 1/x$

$$\begin{aligned} T_3(1) &\overset{d\geq 2}{=} \frac{(2-d)(8-3d)}{8(3-d)^2} J(1)B_2(1) \times \\ &\times \left[{}_3F_2\left(5-\frac{3d}{2}, 9-3d, 1; 7-2d, 7-2d; 1\right) - \right. \\ &\left. -{}_3F_2\left(2-\frac{d}{2}, 9-3d, 1; 4-d, 7-2d; 1\right)\right] \end{aligned}$$

Expansion (in even dims; below: $d = 4 - 2\epsilon$) known

- Algorithm A [Moch Uwer Weinzierl 01]
- within FORM: XSummer [Moch Uwer 05] $\longrightarrow$ HPL(1)
- and: Summer [Vermaseren] $\longrightarrow$ minimal set of numbers

```
T3 = + 1/4

     + ep * ( 1/2 )

     + ep^3 * (  - 8 + 13/2*z3 )

     + ep^4 * (  - 241/4 - 5/8*pi^4 + 4*z3 )

     + ep^5 * (  - 669/2 - 1/5*pi^4 + 693/2*z5 - 36*z3 )

     + ep^6 * (  - 1636 + 21/5*pi^4 - 44/21*pi^6 + 72*z5 - 289*z3 + 241/2*
       z3^2 )

     + ep^7 * (  - 7472 + 589/20*pi^4 - 2/7*pi^6 + 45921/4*z7 - 2484*z5 -
       3061/2*z3 - 493/20*z3*pi^4 + 16*z3^2 )

     + ep^8 * (  - 32736 + 5857/40*pi^4 + 328/21*pi^6 - 509251/84000*pi^8 +
       5238/5*s8a + 996*z7 - 16677*z5 - 7059*z3 - 8/5*z3*pi^4 + 13977*z3*z5
        - 900*z3^2 )

     + ep^9 * (  - 139604 + 12903/20*pi^4 + 2146/21*pi^6 - 26/75*pi^8 +
       2094913/6*z9 - 87858*z7 - 162045/2*z5 - 3339/4*z5*pi^4 - 30584*z3 +
       954/5*z3*pi^4 - 600/7*z3*pi^6 + 576*z3*z5 - 5881*z3^2 + 4933/3*z3^3 )
```

where z3 $= \zeta(3)$ etc.

Difference eqs: numeric solution

very general setup [Laporta 00]

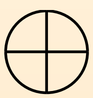
solve via factorial series $U(x) = U_0(x) + \sum_{j=1}^{R} U_j(x)$, where

$$U_j(x) = \mu_j^x \sum_{s=0}^{\infty} a_j(s) \frac{\Gamma(x+1)}{\Gamma(x+1+s-K_j)}$$

plug into difference eq, get μ, $K_j(d)$, and recursion rels for $a_j(s)$

need boundary condition for fixing, say, $a_j(0)$

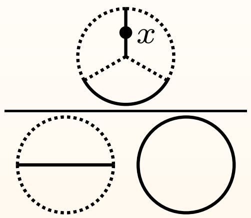
numerics: truncate sum. example:



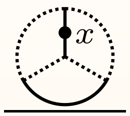
$$\begin{aligned} = \quad & +1.2722705418498941993978 8 - 5.67991293994853579036683\epsilon \\ & +17.6797238948173732343788\epsilon^2 - 46.5721846649543261864019\epsilon^3 \\ & +111.658522176214385363568\epsilon^4 - 252.463963901002177432 36\epsilon^5 \\ & +549.301665961614269417 05\epsilon^6 - 1164.51205889715216235 46\epsilon^7 + \mathcal{O}(\epsilon^8) \end{aligned}$$

Difference eqs: Laplace trafo

$$M_h(x) \quad \equiv$$



$$=$$



$$\frac{}{J^3}\;\frac{2^{d-2}\Gamma(\frac{1}{2})}{\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2})}$$

difference equation for M_h:

$$-2(x+1)M_h(x+2)+3(x+2-d/2)M_h(x+1)-(x+3-d)M_h(x)=F(x)$$

2nd order deq $\rightarrow$ 2 boundary conditions: $M_h(0)$, $M_h(x \gg 1)$

- Laplace trafo $M_h(x)=\int_0^1 dt\, t^{x-1}\, v(t)$
- solve differential Eqn via Harmonic PolyLogs [$H_{01}(x)=\mathrm{Li}_2(x)$]
  $H_0(x)=\ln(x) \quad ; \quad H_1(x)=-\ln(1-x) \quad ; \quad H_{-1}(x)=\ln(1+x)$
  $H_{m,\underline{m}}(x)=\int_0^x dy\, f(m,y)\, H_{\underline{m}}(y) \quad ; \quad f(\{0,1,-1\},x)=\{\frac{1}{x},\frac{1}{1-x},\frac{1}{1+x}\}$
- translate $H_{\underline{m}}(1) \longrightarrow S_{\underline{m}}(\infty)$
- express $S_{\underline{m}}(\infty)$ in term of a minimal basis of numbers (where possible)

```
Mh = + ep^2 * (  - 2*z3 )

     + ep^3 * ( 7/60*pi^4 - 16*li4half + 2/3*ln2^2*pi^2 - 2/3*ln2^4 )

     + ep^4 * (  - 16*li5half - 49/180*ln2*pi^4 - 2/9*ln2^3*pi^2 + 2/15*
       ln2^5 - 137/8*z5 - 2*z3 + 19/12*z3*pi^2 )

     + ep^5 * ( 7/60*pi^4 + 41/945*pi^6 + 110*s6 - 16*li6half - 16*li4half
        + 10/3*li4half*pi^2 + 2/3*ln2^2*pi^2 - 1/360*ln2^2*pi^4 - 2/3*ln2^4
        + 7/36*ln2^4*pi^2 - 1/45*ln2^6 - 4*z3 - 103/2*z3^2 )

     + ep^6 * ( 7/30*pi^4 - 816/7*s7b - 46/7*s7a - 16*li7half - 16*li5half
        + 10/3*li5half*pi^2 - 32*li4half - 49/180*ln2*pi^4 - 1709/3780*ln2*
       pi^6 + 46/7*ln2*s6 + 4/3*ln2^2*pi^2 - 2/9*ln2^3*pi^2 + 1/1080*ln2^3*
       pi^4 - 4/3*ln2^4 + 2/15*ln2^5 - 7/180*ln2^5*pi^2 + 1/315*ln2^7 -
       490507/448*z7 - 137/8*z5 + 41257/672*z5*pi^2 + 1705/16*z5*ln2^2 - 8*
       z3 + 19/12*z3*pi^2 + 671/126*z3*pi^4 - 156*z3*li4half + 13/2*z3*ln2^2
       *pi^2 - 13/2*z3*ln2^4 - 115/14*z3^2*ln2 )

     + ep^7 * ( 7/15*pi^4 + 41/945*pi^6 + 12041677/127008000*pi^8 - 46/7*s8d
        + 816/7*s8c + 13876/7*s8b + 389891/2240*s8a + 110*s6 - 461/42*s6*
       pi^2 - 16*li8half - 16*li6half + 10/3*li6half*pi^2 - 32*li5half - 64*
       li4half + 10/3*li4half*pi^2 + 6571/630*li4half*pi^4 - 49/90*ln2*pi^4
        + 8/3*ln2^2*pi^2 - 1/360*ln2^2*pi^4 - 2531/15120*ln2^2*pi^6 - 408/7*
       ln2^2*s6 - 4/9*ln2^3*pi^2 - 8/3*ln2^4 + 7/36*ln2^4*pi^2 + 2627/6048*
       ln2^4*pi^4 + 4/15*ln2^5 - 1/45*ln2^6 + 7/1080*ln2^6*pi^2 - 1/2520*
       ln2^8 - 137/4*z5 - 408/7*z5*ln2*pi^2 - 16*z3 + 19/6*z3*pi^2 - 1000/7*
       z3*li5half - 1531/504*z3*ln2*pi^4 - 125/63*z3*ln2^3*pi^2 + 25/21*z3*
       ln2^5 - 562693/448*z3*z5 - 103/2*z3^2 - 3505/168*z3^2*pi^2 - 459/28*
       z3^2*ln2^2 )
```

where $\texttt{z3} = \zeta(3)$, $\texttt{li4half} = \mathrm{Li}_4(1/2) = \sum_{k\geq 1} \frac{1}{k^4 2^k}$ etc.

Differential equations

repeat reduction with two masses: M, m

$\Rightarrow$ get differential eqn in $x = M/m$

use boundary values at $x = 0$ ($x = 1$)

use symmetry relations like $x \leftrightarrow 1/x$

typically, want the integral at $x = 1$ ($x = 0$)

show 4 examples where solution can be systematically constructed

Differential equations: 3-loop example

$$B_{24}(x) \equiv \frac{1}{J^3}\int_{p_{1..3}}^{(d)} \frac{1}{p_1^2+x^2}\,\frac{1}{p_2^2+x^2}\,\frac{1}{p_3^2+1}\,\frac{1}{(p_1+p_2+p_3)^2+1}$$

$$B_{24}(x) = x^{3d-8}\,B_{24}(1/x) \quad , \quad B_{24}(0) = 2^{d-3}\,\frac{\Gamma(\frac{8-3d}{2})\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{7-2d}{2})\Gamma(\frac{2-d}{2})}$$

$$\Big\{x(1-x^2)\partial_x^2 - 2(1-2x^2)(d-3)\partial_x - x(d-3)(3d-8)\Big\}\,B_{24}(x)$$
$$= (d-2)^2\,x^{d-3}\left(x^{d-2}-1\right)$$

solution standard, via variation of constants, in terms of HPL(x):

```
B24 = +ep^0*(  - 1/3*(1 + 4*z^2 + z^4) )

      +ep^1*(  - 1/6*(1 + 8*z^2 + z^4) + 2*(z^2)*(2 + z^2)*H(R(0),z) )

      +ep^2*(  + 5/12*(1 - 16/5*z^2 + z^4) - 12*(z^2)*(2/3 + z^2)*H(R(0,0),z) + (
           z^2)*(4 + z^2)*H(R(0),z) )

      +ep^3*(  - 8/3*( - 1 + z^2)*( - 1 + z^2)*z3 + 8*( - 1 + z^2)*( - 1 + z^2)*
           H(R(-1,0,0),z) - 8*( - 1 + z^2)*( - 1 + z^2)*H(R(1,0,0),z) + 79/24*(1
            + 48/79*z^2 + z^4) - 5/2*(z^2)*( - 8/5 + z^2)*H(R(0),z) + 56*(z^2)*(
           2/7 + z^2)*H(R(0,0,0),z) - 6*(z^2)*(z^2)*H(R(0,0),z) )
```

set $x = 1$ and use algebra of HPL(1) resp. $S(\infty)$

```
B4 = - 2

     + ep * (  - 5/3 )

     + ep^2 * (  - 1/2 )

     + ep^3 * ( 103/12 )

     + ep^4 * ( 1141/24 - 112/3*z3 )

     + ep^5 * ( 9055/48 + 136/45*pi^4 - 256*li4half + 32/3*ln2^2*pi^2 - 32/3
       *ln2^4 - 168*z3 )

     + ep^6 * ( 63517/96 + 68/5*pi^4 - 1536*li5half - 1152*li4half - 272/15*
       ln2*pi^4 + 48*ln2^2*pi^2 - 64/3*ln2^3*pi^2 - 48*ln2^4 + 64/5*ln2^5 +
       1240*z5 - 1876/3*z3 )

     + ep^7 * ( 418903/192 + 2278/45*pi^4 + 32/5*pi^6 - 3840*s6 - 9216*
       li6half - 6912*li5half - 4288*li4half - 408/5*ln2*pi^4 + 536/3*ln2^2*
       pi^2 + 272/5*ln2^2*pi^4 - 96*ln2^3*pi^2 - 536/3*ln2^4 + 32*ln2^4*pi^2
        + 288/5*ln2^5 - 64/5*ln2^6 + 5580*z5 - 6398/3*z3 + 4880/3*z3^2 )

     + ep^8 * ( 2667781/384 + 7769/45*pi^4 + 144/5*pi^6 + 87040/7*s7b -
       74240/7*s7a - 17280*s6 - 55296*li7half - 41472*li6half - 25728*
       li5half - 14624*li4half - 4556/15*ln2*pi^4 - 3824/135*ln2*pi^6 +
       74240/7*ln2*s6 + 1828/3*ln2^2*pi^2 + 1224/5*ln2^2*pi^4 - 1072/3*ln2^3
       *pi^2 - 544/5*ln2^3*pi^4 - 1828/3*ln2^4 + 144*ln2^4*pi^2 + 1072/5*
       ln2^5 - 192/5*ln2^5*pi^2 - 288/5*ln2^6 + 384/35*ln2^7 + 772868/7*z7
        + 20770*z5 - 130360/21*z5*pi^2 - 22320*z5*ln2^2 - 20797/3*z3 - 720/7
       *z3*pi^4 + 7320*z3^2 - 92800/7*z3^2*ln2 )
```

Differential equations: 4-loop example T_4

$$T_{24}(x) \equiv \frac{1}{J^4}\int_{p_{1..4}}^{(d)} \frac{1}{p_1^2+1}\,\frac{1}{(p_1+p_4)^2+1}\,\frac{1}{p_2^2+x^2}\,\frac{1}{(p_2+p_4)^2+x^2}\,\frac{1}{p_3^2}\,\frac{1}{(p_3+p_4)^2}$$

$T_{24}(0)$ is known. Symmetry relation:

$$T_{24}(x) = x^{4d-12}\,T_{24}(1/x)$$

The differential eqn is of second order

$$\begin{aligned}&\left\{x(1-x^2)\partial_x^2 + \left[10-3d+(5d-16)x^2\right]\partial_x - 4x(d-3)^2\right\}T_{24}(x)\\ &= \left(\frac{d}{2}-1\right)(3d-8)B_2\left(x^{d-3}-x^{3d-9}\right)\end{aligned}$$

Solve in terms of HPL(x), set $x=1$

```
T4 = + 2/3 + ep * ( 4/3 ) + ep^2 * ( 2/3 ) + ep^3 * (  - 44/3 + 16/3*z3 )
     + ep^4 * (  - 116 - 4/15*pi^4 + 200/3*z3 )
     + [...]
     + ep^10 * (  - 1067504 - 910432/45*pi^4 - 4208384/945*pi^6 - 39846143/
       47250*pi^8 - 2456/4455*pi^10 + 11599872/7*s9e + 9240576/7*s9d -
       90902528/21*s9c - 6750208*s9b + 114294784/7*s9a + 2605056*s8d +
       2899968*s8c + 28157952*s8b + 21695376/5*s8a - 13412352/7*s7b +
       35766272/49*s7b*pi^2 + 12048384/7*s7a - 25018368/49*s7a*pi^2 +
       2125824*s6 - 1327104*s6*pi^2 + 16777216*li9half + 14680064*li8half +
       9699328*li7half + 5668864*li6half + 3104768*li5half + 6397952/105*
       li5half*pi^4 + 1638400*li4half + 195584/15*li4half*pi^4 + 1831328/45*
       ln2*pi^4 + 5840672/945*ln2*pi^6 + 1479574556/165375*ln2*pi^8 -
       181805056/7*ln2*s8b - 144875904/35*ln2*s8a - 12048384/7*ln2*s6 +
       31784960/49*ln2*s6*pi^2 - 204800/3*ln2^2*pi^2 - 1671872/45*ln2^2*pi^4
        - 220544/135*ln2^2*pi^6 - 1449984*ln2^2*s6 + 388096/9*ln2^3*pi^2 +
       2860544/135*ln2^3*pi^4 + 120320/81*ln2^3*pi^6 + 1933312/7*ln2^3*s6 +
       204800/3*ln2^4 - 177152/9*ln2^4*pi^2 - 1009024/135*ln2^4*pi^4 -
       388096/15*ln2^5 + 303104/45*ln2^5*pi^2 + 6259712/4725*ln2^5*pi^4 +
       354304/45*ln2^6 - 229376/135*ln2^6*pi^2 - 606208/315*ln2^7 + 262144/
       945*ln2^7*pi^2 + 16384/45*ln2^8 - 131072/2835*ln2^9 + 7003834114/63*
       z9 - 124448588/7*z7 - 261111224/49*z7*pi^2 - 113401856/7*z7*ln2^2 -
       2348352*z5 + 6699072/7*z5*pi^2 - 2574721004/2205*z5*pi^4 + 283149312/
       7*z5*li4half - 1449984*z5*ln2*pi^2 + 3523584*z5*ln2^2 - 17310080/7*z5
       *ln2^2*pi^2 + 11797888/7*z5*ln2^4 + 6912*z5^2 + 627840*z3 + 4429772/
       315*z3*pi^4 - 131325728/6615*z3*pi^6 - 22769664*z3*s6 - 39845888/7*z3
       *li6half - 4980736*z3*li5half + 151552*z3*li4half - 7700480/7*z3*
       li4half*pi^2 - 4382176/45*z3*ln2*pi^4 - 18944/3*z3*ln2^2*pi^2 +
       24490624/315*z3*ln2^2*pi^4 - 622592/9*z3*ln2^3*pi^2 + 18944/3*z3*
       ln2^4 - 1642496/63*z3*ln2^4*pi^2 + 622592/15*z3*ln2^5 - 2490368/315*
       z3*ln2^6 + 10624*z3*z7 + 3168384*z3*z5 + 224415616/7*z3*z5*ln2 -
       868712*z3^2 - 539008*z3^2*pi^2 - 128/15*z3^2*pi^4 + 15060480/7*z3^2*
       ln2 - 29369856/49*z3^2*ln2*pi^2 - 407808*z3^2*ln2^2 + 543744/7*z3^2*
       ln2^3 + 48492880/9*z3^3 )
```

Differential equations: 4-loop example T_6

$$T_{36}(x) \quad \equiv \quad \frac{1}{J^4}\int_{p_{1..4}}^{(d)} \frac{1}{p_1^2+1}\,\frac{1}{(p_1+p_4)^2+x^2}\,\frac{1}{p_2^2+1}\,\frac{1}{(p_2+p_4)^2+x^2}\,\frac{1}{p_3^2+1}\,\frac{1}{(p_3+p_4)^2+x^2}$$

$T_{36}(0)$ is known. Symmetry relation:

$$T_{36}(x) \quad = \quad x^{4d-12}\,T_{36}(1/x)$$

The differential eqn is of second order

$$\begin{aligned} &\left\{x(1-x^2)\partial_x^2 + (9-3d+(5d-17)x^2)\partial_x - 4x(d-3)(d-4)\right\} T_{36}(x) \\ &= \frac{3}{2}(d-2)\left\{(1+x^{d-2})\partial_x - (3d-8)x^{d-3}\right\} B_{24}(x) \end{aligned}$$

Solve in terms of HPL(x), set $x=1$

```
T6 = + 3/2 + ep * ( 7/2 ) + ep^2 * ( 9/2 ) + ep^3 * (  - 39/2 - 3*z3 )
     + ep^4 * (  - 208 - 1/20*pi^4 + 109*z3 )
     + ep^5 * (  - 1254 - 547/60*pi^4 + 768*li4half - 32*ln2^2*pi^2 + 32*
       ln2^4 + 189*z5 + 855*z3 )
     + [...]
     + ep^9 * (  - 555600 - 42503/5*pi^4 - 11501/15*pi^6 - 9914311/176400*
       pi^8 - 124416*s9f - 311040*s9e + 62208*s9d + 775008*s9c + 1785024*s9b
        - 3409344*s9a + 1336320/7*s8d + 1566720/7*s8c + 14588928/7*s8b +
       2348061/7*s8a - 2079360/7*s7b - 502848/7*s7b*pi^2 + 1549440/7*s7a -
       212544/7*s7a*pi^2 + 483840*s6 - 687360/7*s6*pi^2 + 995328*li8half +
       1244160*li7half + 1161216*li6half + 951552*li5half - 4320*li5half*
       pi^4 + 725760*li4half + 19984/35*li4half*pi^4 + 248832*li4half*
       li5half + 20736*li4half^2 + 56168/5*ln2*pi^4 + 1870/3*ln2*pi^6 -
       1444551/1000*ln2*pi^8 + 4650048*ln2*s8b + 3705507/5*ln2*s8a - 1549440/
       7*ln2*s6 - 440640/7*ln2*s6*pi^2 + 53568/5*ln2*li4half*pi^4 - 30240*
       ln2^2*pi^2 - 34272/5*ln2^2*pi^4 - 1186/105*ln2^2*pi^6 - 62208*ln2^2*
       s7b - 783360/7*ln2^2*s6 - 10368*ln2^2*li5half*pi^2 - 1728*ln2^2*
       li4half*pi^2 + 13216*ln2^3*pi^2 + 2448*ln2^3*pi^4 - 726/5*ln2^3*pi^6
        - 72576*ln2^3*s6 + 3456*ln2^3*li4half*pi^2 + 30240*ln2^4 - 4032*
       ln2^4*pi^2 - 9026/21*ln2^4*pi^4 + 10368*ln2^4*li5half + 1728*ln2^4*
       li4half - 39648/5*ln2^5 + 864*ln2^5*pi^2 + 1692/5*ln2^5*pi^4 - 10368/
       5*ln2^5*li4half + 8064/5*ln2^6 - 936/5*ln2^6*pi^2 - 1728/7*ln2^7 +
       1152/5*ln2^7*pi^2 + 2124/35*ln2^8 - 432/5*ln2^9 - 585155695/16*z9 -
       18402885/7*z7 + 226180863/112*z7*pi^2 + 4418037/2*z7*ln2^2 - 821100*
       z5 + 1036920/7*z5*pi^2 + 120042813/560*z5*pi^4 - 7587000*z5*li4half
        - 783360/7*z5*ln2*pi^2 + 502200*z5*ln2^2 + 340425*z5*ln2^2*pi^2 -
       255861*z5*ln2^4 + 481632*z3 + 38287/14*z3*pi^4 + 2938237/420*z3*pi^6
        + 3784320*z3*s6 - 31104*z3*li6half - 2926656/7*z3*li5half - 30240*z3
       *li4half - 5184*z3*li4half*pi^2 - 274754/35*z3*ln2*pi^4 + 1260*z3*
       ln2^2*pi^2 + 14769/5*z3*ln2^2*pi^4 - 40648/7*z3*ln2^3*pi^2 - 1260*z3*
       ln2^4 - 108*z3*ln2^4*pi^2 + 121944/35*z3*ln2^5 - 216/5*z3*ln2^6 +
       2163963/7*z3*z5 - 5739903*z3*z5*ln2 - 180558*z3^2 - 288240/7*z3^2*
       pi^2 + 1936800/7*z3^2*ln2 - 333720/7*z3^2*ln2*pi^2 - 220320/7*z3^2*
       ln2^2 - 20412*z3^2*ln2^3 - 1703845/2*z3^3 )
```

Differential equations: 4-loop example BB_4

$$BB_{24}(x) \quad \equiv \quad \frac{1}{J^4}\int_{p_{1..4}}^{(d)} \frac{1}{p_1^2+1}\,\frac{1}{p_2^2+1}\,\frac{1}{p_3^2+x^2}\,\frac{1}{p_4^2+x^2}\,\frac{1}{(p_1+p_2+p_3+p_4)^2}$$

$T_{36}(0)$ is known. Symmetry relation:

$$BB_{24}(x) = x^{4d-10}\,BB_{24}(1/x)$$

The differential eqn is of third order [Mastrolia]

$$\begin{aligned}\Big\{x^2(1-x^2)\partial_x^3 - 4x(1-2x^2)(d-3)\partial_x^2 \\ + \left[3d-8-(19d-52)x^2\right](d-3)\partial_x \\ +2x(2d-5)(3d-8)(d-3)\Big\} \quad BB_{24}(x) \quad &= \quad (d-2)^3\,x^{2d-5}\end{aligned}$$

Solve in terms of HPL(x), set $x=1$

```
BB4 = - 1 + ep * (  - 1/2 ) + ep^2 * ( 17/36 ) + ep^3 * ( 1/216 )
      + [...]
      + ep^11 * (  - 1159758899155871/362797056 - 1280351464/6561*pi^4 -
        50573120/729*pi^6 - 12262093568/496125*pi^8 + 611844096/7*s9e +
        344457216/7*s9d - 13187219456/63*s9c - 1157496832/3*s9b + 16011624448/
        21*s9a + 2111832064/21*s8d + 2702311424/21*s8c + 23780687872/21*s8b
         + 57857778176/315*s8a - 12396216320/189*s7b + 1886519296/49*s7b*pi^2
         + 9687531520/189*s7a - 902692864/49*s7a*pi^2 + 1369364480/27*s6 -
        3315367936/63*s6*pi^2 + 679477248*li9half + 500170752*li8half +
        254935040*li7half + 995901440/9*li6half + 3560032256/81*li5half +
        111542272/35*li5half*pi^4 + 12050366720/729*li4half + 527958016/945*
        li4half*pi^4 + 1891267136/3645*ln2*pi^4 + 77775104/729*ln2*pi^6 +
        214880918272/496125*ln2*pi^8 - 26374438912/21*ln2*s8b - 7005710336/35
        *ln2*s8a - 9687531520/189*ln2*s6 + 1259601920/49*ln2*s6*pi^2 -
        1506295840/2187*ln2^2*pi^2 - 52907264/81*ln2^2*pi^4 + 49061888/2835*
        ln2^2*pi^6 - 1351155712/21*ln2^2*s6 + 445004032/729*ln2^3*pi^2 +
        13543424/27*ln2^3*pi^4 + 7122944/189*ln2^3*pi^6 + 101974016/7*ln2^3*
        s6 + 1506295840/2187*ln2^4 - 31121920/81*ln2^4*pi^2 - 631508992/2835*
        ln2^4*pi^4 - 445004032/1215*ln2^5 + 1593344/9*ln2^5*pi^2 + 21151744/
        525*ln2^5*pi^4 + 12448768/81*ln2^6 - 868352/15*ln2^6*pi^2 - 3186688/
        63*ln2^7 + 393216/35*ln2^7*pi^2 + 434176/35*ln2^8 - 65536/35*ln2^9 +
        380773366144/63*z9 - 35136347200/63*z7 - 39164256640/147*z7*pi^2 -
        15886533632/21*z7*ln2^2 - 9484148432/243*z5 + 18552001280/567*z5*pi^2
         - 395289197696/6615*z5*pi^4 + 13673512960/7*z5*li4half - 1351155712/
        21*z5*ln2*pi^2 + 339581440/3*z5*ln2^2 - 2408327168/21*z5*ln2^2*pi^2
         + 1709189120/21*z5*ln2^4 + 34442297596/6561*z3 + 33360640/63*z3*pi^4
         - 22994973184/19845*z3*pi^6 - 1117716480*z3*s6 - 1912602624/7*z3*
        li6half - 4223664128/21*z3*li5half - 287047680/7*z3*li4half*pi^2 -
        3827207168/945*z3*ln2*pi^4 + 336799744/105*z3*ln2^2*pi^4 - 527958016/
        189*z3*ln2^3*pi^2 - 15958016/21*z3*ln2^4*pi^2 + 527958016/315*z3*
        ln2^5 - 39845888/105*z3*ln2^6 + 10033915904/63*z3*z5 + 32555948032/21
        *z3*z5*ln2 - 4812226880/243*z3^2 - 1467352064/63*z3^2*pi^2 +
        12109414400/189*z3^2*ln2 - 1027985408/49*z3^2*ln2*pi^2 - 126670848/7*
        z3^2*ln2^2 + 28680192/7*z3^2*ln2^3 + 7109786624/27*z3^3 )
```

Summary

- need to calculate Feynman diagrams (Glover)
- many integrals at higher loop orders
- at the end of reduction, need to evaluate some integrals
- bubbles are an important class
- many methods in the *art* of integration
- numerically: need high precision
- analytically: HarmPolyLogs and HarmSums allow progress
- main vehicle: difference/differential eqs; don't ''feel'' #loops

Diagram generation

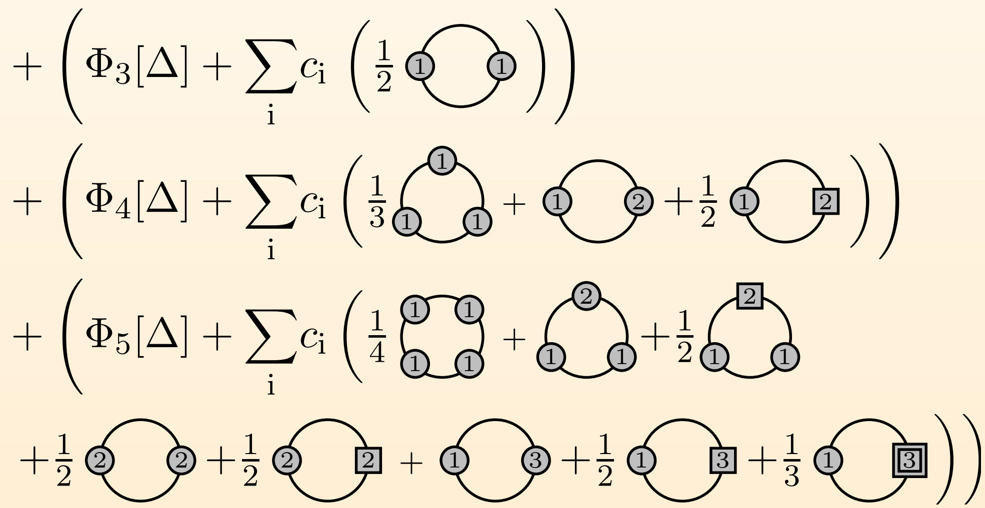
yet another generator? QGRAF [Nogueira], FeynArts [Denner/Hahn] n/a for 0-pt fcts.

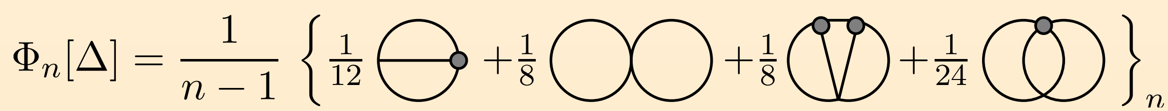
skeleton (2PI) expansion [Luttinger/Ward, Baym, ...]

$$F[D] = \sum_i c_{\rm i} \left(Tr \ln D_{\rm i}^{-1} + Tr\, \Pi_{\rm i}[D] D_{\rm i} \right) - \Phi[D]$$

extremal property of partition function $\Rightarrow$ $\delta_{D_{\rm i}} \Phi[D] \; = \; c_{\rm i} \Pi[D]$

$$-F \;=\; -F_0 + \Phi_2[\Delta]$$

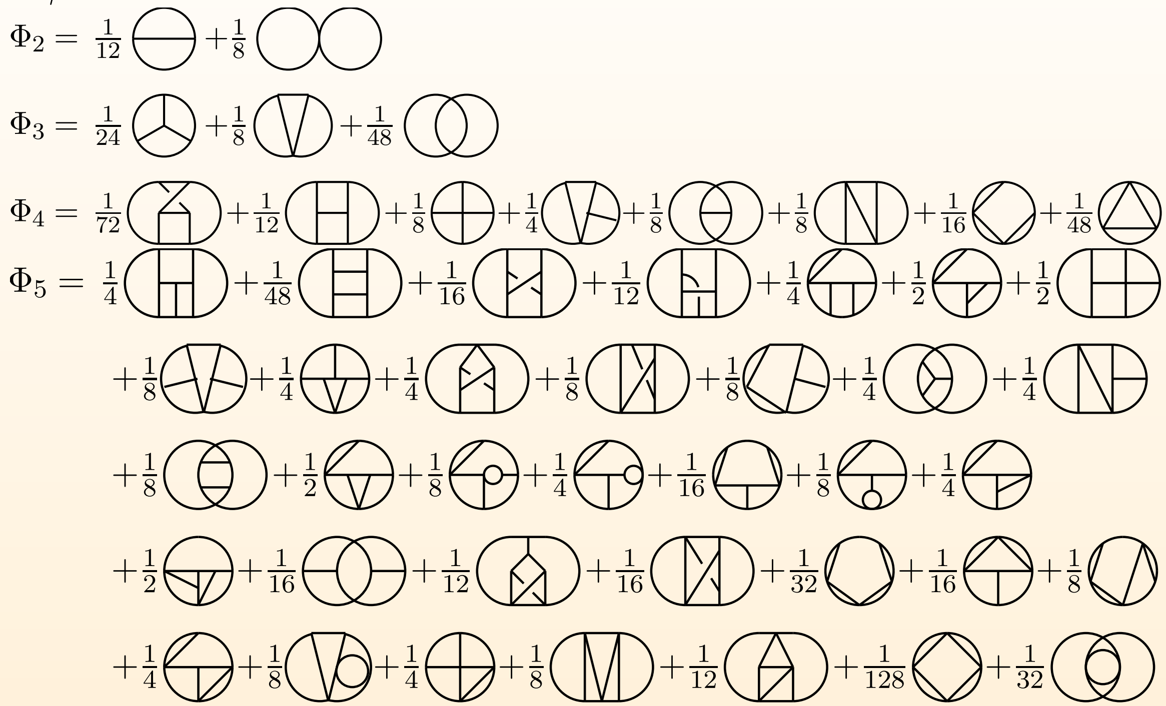


get skeletons from 

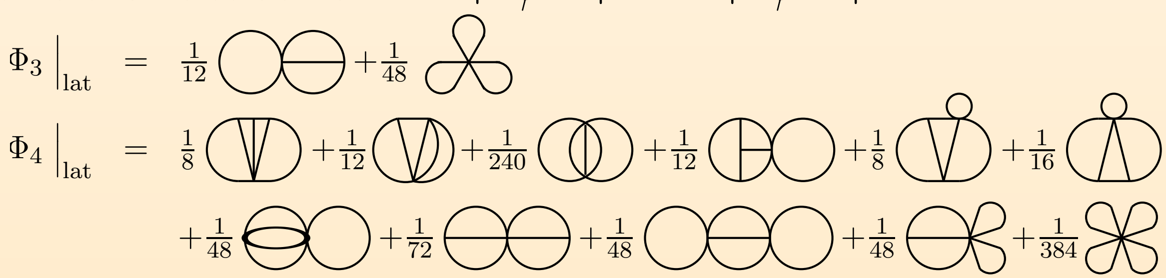
and SD eqs $\quad \Gamma_n^{1PI} = \delta_\phi^{n-1} S'[\phi + D[\phi]\delta_\phi] \Big|_{\phi=0}$

Diagram generation

generic $\phi^3 + \phi^4$ skeletons



LAT: additional skeletons $\ldots + \phi^5 + \ldots + \phi^8 + \ldots$



Classification

once you have a long list of Feynman integrals, need tools that replace human ‘staring’ at them

topology recognition

```
#define mom3 "gl(k1,k2,k3,k1-k2,k1-k3,k1-k2-k3)"
#define maxTopo3 "5"
#define maxLines3 "6"
*** format of sets: nrLines,nrReps,..binary reps..
set t31: 3,16,7,11,14,21,22,25,26,28,35,37,38,41,44,49,50,56;
set t32: 4,12,15,23,27,29,43,45,46,51,53,54,58,60;
set t33: 4,3,30,39,57;
set t34: 5,6,31,47,55,59,61,62;
set t35: 6,1,63;
```

find symmetry relations

```
id f(3,0,0,0,0,0,0)=0;
[...]
al f(3,0,f(?A2),f(?A3),f(?A4),f(?A5),f(?A6))=
fsy(f(3,f(?A6),f(?A2),f(?A4),f(?A3),f(?A5),0),f(3,f(?A6),f(?A2),f(?A5),f(?A3),
f(?A4),0),f(3,f(?A6),f(?A5),f(?A3),f(?A4),f(?A2),0),f(3,f(?A6),f(?A5),f(?A2),f(
?A4),f(?A3),0),f(3,f(?A6),f(?A4),f(?A3),f(?A5),f(?A2),0),f(3,f(?A6),f(?A4),f(
?A2),f(?A5),f(?A3),0),f(3,f(?A6),f(?A3),f(?A4),f(?A2),f(?A5),0),f(3,f(?A6),f(
?A3),f(?A5),f(?A2),f(?A4),0));
```

etc.

Explicit Integration in LatReg

1loop IBP + coordinate-space method [Lüscher/Weisz] [Becher/Melnikov]

amusing: 1loop tadpole has elliptic integral in 3d [M.Shaposhnikov]

$$J(m) = \int_{-\pi/a}^{\pi/a} \frac{d^3k}{(2\pi)^3} \frac{1}{\sum_{j=1}^{3} \frac{4}{a^2}\sin^2(ak_j/2) + m^2}$$

$$= \frac{1}{4\pi a} \sum_{n\geq 0} (am)^{2n} \left[(a_n \Sigma + b_n \xi) + (am) c_n 1 \right]$$

$$a_{0..6} = \frac{(-1)^{n+1}}{4^n (2n)!} \frac{8}{2^n} \{-1/8, 0, 1, 53, 13559/3, 612241, 124073817\}$$

$$b_{0..6} = \frac{(-1)^{n}}{4^n (2n)!} \frac{16}{2^n} \{0, 1, 17, 677, 155591/3, 6685249, 1321874313\}$$

$$c_{0..6} = \frac{(-1)^{n+1}}{4^n (2n)!} \frac{1}{2n+1} \{1, 3, 33, 843, 40257, 3152115, 370071585\}$$

where $\Sigma = \frac{8}{\pi}(18 + 2\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}) K^2((2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2) = 3.1759\ldots$ and $\xi = 0.15285933\ldots$ are 'master' lattice constants

For those who like numerics

$$
\begin{aligned}
[T3.4d] \quad = \quad & + 0.25000000000000000000000000000000000000000000000000 \\
& + 0.50000000000000000000000000000000000000000000000000\,\epsilon \\
& - 0.18663012946263714490170195017557506002758909978676\,\epsilon^3 \\
& - 116.32245428361314613117625528439489456801979587607\,\epsilon^4 \\
& - 37.960399563368202247860499294790882685706485051382\,\epsilon^5 \\
& - 3339.8413608179462488397630653491217320520738663275\,\epsilon^6 \\
& - 580.54049849788766311251957498973876017869237554386\,\epsilon^7 \\
& - 68729.911156005599119332697611795047235540437064886\,\epsilon^8 \\
& - 7274.2462825169856368639122222990743763077034333357\,\epsilon^9
\end{aligned}
$$

$$
\begin{aligned}
[M2.4d] \quad = \quad & -2.4041138063191885707994763230228999815299725846810\,\epsilon^2 \\
& +6.0920930219183236301259124208444904183141410820639\,\epsilon^3 \\
& -28.600718452293841661775505982296970214820644130964\,\epsilon^4 \\
& +86.468416459987399525230431979151207379526305101222\,\epsilon^5 \\
& -294.53750228628902992825553889029490440166233606\,\epsilon^6 \\
& +905.02035923688404278031626662635643896566081115\,\epsilon^7 \\
& -2836.7580169777789669838978426665538909833713686 4\,\epsilon^8
\end{aligned}
$$

$$
\begin{aligned}
[B4.4d] \quad = \quad & -\,2.0000000000000000000000000000000000000000000000000 \\
& -\,1.6666666666666666666666666666666666666666666666667\,\epsilon \\
& -\,0.50000000000000000000000000000000000000000000000000\,\epsilon^2 \\
& +\,8.5833333333333333333333333333333333333333333333333\,\epsilon^3 \\
& +\,2.6648756153751466784097753035725336781071784192880\,\epsilon^4 \\
& +\,196.73537825917304338585630537320304346641599253270\,\epsilon^5 \\
& +\,67.581864350250489693347939778082599454630184224763\,\epsilon^6 \\
& +\,2275.6648272888242761816138399011797559370085675269\,\epsilon^7 \\
& +\,794.912819148047186800906700344265406085130182\,\epsilon^8 \\
& +\,22561.10208658986263056699798573466902422834169\,\epsilon^9
\end{aligned}
$$

$$
\begin{aligned}
[BB4.4d] \quad = \quad & -1.0000000000000000000000000000000000000000000000000 \\
& -0.50000000000000000000000000000000000000000000000000\,\epsilon \\
& +0.47222222222222222222222222222222222222222222222222\,\epsilon^2 \\
& +0.0046296296296296296296296296296296296296296296296296\,\epsilon^3 \\
& -28.709104938271604938271604938271604938271604938272\,\epsilon^4 \\
& -14.897726533029588869214274870082319534267091487232\,\epsilon^5 \\
& -1678.8869291077729634030303102679175091514915038664\,\epsilon^6 \\
& -871.70403998569148157241455827839247031867740865866\,\epsilon^7 \\
& -42086.609721033389180857980860257534993579006104 57\,\epsilon^8 \\
& -22059.6786139055931764938109246501532927767084\,\epsilon^9 \\
& -828993.577841124168252946913835097736916711984\,\epsilon^{10} \\
& -436644.38416941367992038458156996656628847789\,\epsilon^{11}
\end{aligned}
$$

$$
\begin{aligned}
[T4.4d] \quad = \quad & + 0.66666666666666666666666666666666666666666666666667 \\
& + 1.3333333333333333333333333333333333333333333333333 \, \epsilon \\
& + 0.66666666666666666666666666666666666666666666666667 \, \epsilon^{2} \\
& - 8.2556965164821638112013964719389333825867397741840 \, \epsilon^{3} \\
& - 61.838630731761030903068211282891363615594936690016 \, \epsilon^{4} \\
& - 602.46544453826371568977931603360800753210616033689 \, \epsilon^{5} \\
& - 1514.0384214192179286481331117855494727387980472233 \, \epsilon^{6} \\
& - 14730.374776657815993478936423355221670957083986404 \, \epsilon^{7} \\
& - 28399.891055079438361606391148613328984530301481 \, \epsilon^{8} \\
& - 284976.00441287252393802030731751195775173999581 \, \epsilon^{9} \\
& - 489753.721233527563366286808399145239893025440 \, \epsilon^{10}
\end{aligned}
$$

$$
\begin{aligned}
[T6.4d] \quad = \quad & +1.500000000000000000000000000000000000000000000000 \\
& +3.500000000000000000000000000000000000000000000000\,\epsilon \\
& +4.500000000000000000000000000000000000000000000000\,\epsilon^2 \\
& -23.106170709478782856199214484534349972294958877021\,\epsilon^3 \\
& -81.846252107304344753250557029687206569102873418520\,\epsilon^4 \\
& -665.237770598104741767454761123145981440435874484 36\,\epsilon^5 \\
& -4145.552339618278334997816862825202259279629961516 0\,\epsilon^6 \\
& -11089.703457495763572046815324276090722389665790 0\,\epsilon^7 \\
& -97460.274268807531735322944455478838723813044253\,\epsilon^8 \\
& -172506.807732757266076052768462287560510724425 7\,\epsilon^9
\end{aligned}
$$