Weak-coupling expansions in hot QCD

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In this talk, I discuss known results for the weak-coupling expansion of the pure Yang-Mills pressure, including its first non-perturbative coefficient. I give an update on the values of all known coefficients, display them in compact numerical form, and speculate on the value of the last unknown parameter that contributes to the four-loop pressure.

1. Introduction

The purpose of this talk is to summarize the information that has been obtained for the weak-coupling expansion of the pure Yang-Mills pressure. For details of the setup, which solves the notorious infrared problem of hot gauge theories [1] via effective theories and dimensional reduction [2], see e.g. [3].

I concentrate on the pure Yang-Mills sector (i.e. QCD at $N_f = 0$) here for two reasons: First, as has been demonstrated by scaling behavior of lattice data (see Fig. 4 of [4]), the quark sector does not seem to play a major role for the qualitative features of the pressure. Second, for $N_f > 0$ lattice data (which one would like to use for a phenomenologigal fit later on) is not yet available with controlled systematic errors, due to the well-known challenges that light fermions pose in lattice Monte-Carlo simulations.

2. Ingredients

Let me now collect all information that is known about the weak-coupling expansion of the thermodynamic Yang-Mills pressure, following in parts the presentation in [5]. I will use re-scaled couplings, in order to make expressions compact. Although all formulae are valid for a gauge group SU(N), the expression for the pressure, at least up to the parametric order g^6 (where non-perturbative information first enters), will depend on N only via a trivial overall factor, which is made manifest by the re-scaled variables.

The framework of dimensional reduction allows to organize contributions to the pressure into three groups, corresponding to the three distinct physical scales involved in hot QCD:

$$p_{\rm QCD} = \frac{(N^2-1)\pi^2 T^4}{45}\,\{p_h + p_s + p_{us}\}\ , \qquad (1)$$

where T is the temperature and p_h, p_s and p_{us} are dimensionless functions.

At large T, these functions (representing the contributions from the three physical scales) can be determined via weak-coupling expansions. They depend on the parameters (couplings and, in general, masses) of the effective actions governing the respective

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physical scales. Their detailed functional dependence on those couplings is known to be:

$$
\begin{aligned}
p_h &= 1+\tilde{\bar{g}}^2\left(-\frac{1}{5}\right)+\tilde{\bar{g}}^4\frac{2}{1875}\left(174+30\gamma_0+370Z_1-95Z_3\right)\\
&\quad+\tilde{\bar{g}}^6\left[\widetilde{\#}_0-\frac{2}{3125}\left(4+11\pi^2+24Z_1+8\gamma_0\left(3+22Z_1\right)-176\gamma_1\right)\right]+\mathcal{O}\left(\tilde{\bar{g}}^8\right), \qquad (2)\\
p_s &= \tilde{m}_{\rm E}^3\frac{16}{25\sqrt{3}}+\tilde{g}_{\rm E}^2\tilde{m}_{\rm E}^2\frac{48}{125}\ln\left(\frac{2}{5\sqrt{3}}e^{-3/4}\tilde{m}_{\rm E}\right)+\tilde{g}_{\rm E}^4\tilde{m}_{\rm E}\left(-\frac{2\sqrt{3}}{625}\right)\left(89+4\pi^2-44\ln 2\right)\\
&\quad+\tilde{g}_{\rm E}^6\left(-\frac{3}{50000}\right)\left(8256-491\pi^2\right)\ln\left(\frac{2}{5\sqrt{3}}e^{\frac{768}{8256-491\pi^2}(1.391512)}\tilde{m}_{\rm E}\right)\\
&\quad+\tilde{m}_{\rm E}^2\left(-\frac{48}{3125}\right)\left[\left(1+\frac{1}{N^2}\right)\tilde{\lambda}_{\rm E}^{(1)}+\left(\frac{2}{3}-\frac{1}{N^2}\right)\tilde{\lambda}_{\rm E}^{(2)}\right]+\mathcal{O}\left(\tilde{g}_{\rm E}^8/\tilde{m}_{\rm E},\lambda^2\tilde{m}_{\rm E}\right), \qquad (3)\\
p_{us} &= \tilde{g}_{\rm M}^6\left(-\frac{3}{50000}\right)\left(2752-157\pi^2\right)\ln\left(c_N\tilde{g}_{\rm M}^2\right), \qquad (4)
\end{aligned}
$$

where constants $Z_n{=}\zeta'(-n)/\zeta(-n)$ and γ_n from $\zeta(1{-}\epsilon){=}{-}1/\epsilon{+}\gamma_0{+}\epsilon\gamma_1{+}\ldots$ were used.

The only unknown coefficient to this order is $\widetilde{\#}_0$. It is purely perturbative and entails a four-loop computation of all connected vacuum diagrams involving gluons and ghosts. The best that can be done before it is evaluated is to fix it phenomenologically, by fitting to lattice data for the pressure. This will be attempted in Sec. 3.

Note that p_{us} is a non-perturbative coefficient, and there are no perturbative corrections. It has taken great efforts to determine the constant c_N under the logarithm. These efforts combine an interesting mix of computing higher-order perturbative coefficients by continuum-calculations (e.g. [6]) and by numerical stochastic lattice methods [7], as well as non-perturbative numerical lattice Monte-Carlo simulations [8]. The result is

$$
\begin{aligned}
c_N &= 4\pi/\exp\left[2\ln(5N){+}1{+}\frac{768}{2752{-}157\pi^2}\left(\frac{1}{3}+[\text{pert}]-[\text{nspt}]_N+[\text{non–pert}]_N\right)\right], \qquad (5)\\
c_3 &= 4\pi/\exp\left[2\ln(15){+}1{+}\frac{768}{2752{-}157\pi^2}\left(\frac{1}{3}-3.73-13.8(4)+11.0(3)\right)\right] = 1.1(4)\,. \qquad (6)
\end{aligned}
$$

While the result of the numerical stochastic methods, $[\text{nspt}]_N$, is currently known for $N=3$ only, very recently the large-N estimate $[\text{non–pert}]_N = 15.9(2)-44(2)/N^2$ became available [9]. The numbers in brackets above estimate statistical and systematic errors.

As usual in effective theories, their parameters have to be matched in order to describe the same underlying physics. This step is perturbative, and the matching coefficients read

$$
\begin{aligned}
\tilde{g}_{\rm E}^2 &= \tilde{\bar{g}}^2+\tilde{\bar{g}}^4\frac{1}{75}\left(1+22\gamma_0\right)+\tilde{\bar{g}}^6\frac{1}{5625}\left(484\gamma_0^2+248\gamma_0+\frac{343}{2}-10\zeta(3)\right)+\mathcal{O}\left(\tilde{\bar{g}}^8\right), \qquad (7)\\
\tilde{m}_{\rm E}^2 &= \tilde{\bar{g}}^2+\tilde{\bar{g}}^4\frac{1}{75}\left(5+22\gamma_0\right)+\mathcal{O}\left(\tilde{\bar{g}}^6\right), \qquad (8)\\
\tilde{\lambda}_{\rm E}^{(1)} &= \tilde{\bar{g}}^4+\mathcal{O}\left(\tilde{\bar{g}}^6\right), \qquad \tilde{\lambda}_{\rm E}^{(2)} = \tilde{\bar{g}}^4+\mathcal{O}\left(\tilde{\bar{g}}^6\right), \qquad (9)\\
\tilde{g}_{\rm M}^2 &= \tilde{g}_{\rm E}^2\left\{1-\frac{1}{20\sqrt{3}}\frac{\tilde{g}_{\rm E}^2}{\tilde{m}_{\rm E}}-\frac{17}{2400}\frac{\tilde{g}_{\rm E}^4}{\tilde{m}_{\rm E}^2}-\frac{1}{1250}\left[\left(1+\frac{1}{N^2}\right)\tilde{\lambda}_{\rm E}^{(1)}+\left(\frac{2}{3}-\frac{1}{N^2}\right)\tilde{\lambda}_{\rm E}^{(2)}\right]\frac{\tilde{g}_{\rm E}^2}{\tilde{m}_{\rm E}^2}\right\}\\
&\quad+\mathcal{O}\left(\tilde{g}_{\rm E}^8/\tilde{m}_{\rm E}^3,\lambda^2\tilde{g}_{\rm E}^4/\tilde{m}_{\rm E}^3\right). \qquad (10)
\end{aligned}
$$

In the above, I have expressed all expansions in terms of $\tilde{\hat{g}}$, which essentially is the 4d running gauge coupling $g(\bar{\mu})$. The exact relation is along the following lines:

$$\tilde{\hat{g}}^2 \equiv \hat{g}^2 + \hat{g}^4 \frac{22}{75} L + \hat{g}^6 \frac{4}{5625} \left(121\, L^2 + 51\, L\right) + \mathcal{O}\left(\hat{g}^8\right) , \tag{11}$$

$$\hat{g}^2 \equiv \frac{25 N g^2(\bar{\mu})}{16\pi^2} = -\frac{275/34}{1 + W_{-1}\left[-\frac{121}{102} \exp\left(-1 - \frac{121}{51} \ln \frac{\bar{\mu}}{\Lambda_{\overline{\rm MS}}}\right)\right]} , \tag{12}$$

where $L = \ln\left(\bar{\mu}/4\pi T\right)$. In the last line I have taken the running 4d coupling as a solution of the 2-loop RGE equation, in terms of the negative real branch of the Lambert W function. Note that the construction $\tilde{\hat{g}}$ is renormalization-scale independent to the order we are working, $\partial_{\ln \bar{\mu}^2} \tilde{\hat{g}}^2 = \mathcal{O}(\hat{g}^8)$. In practice, however, one needs to fix this higher-order $\bar{\mu}$-dependence, a procedure that is somewhat arbitrary. Following [10], I choose the scale $\bar{\mu}$ by the principle of minimal sensitivity applied to the 1-loop result for $\tilde{g}_{\rm E}^2$. This fixes the optimal scale $\bar{\mu} = \bar{\mu}_{opt}$ as a function of T, resulting in $L = -\gamma_0 - \frac{1}{22}$. Finally, to ease comparison with existing 4d lattice data for the pressure, one can translate the $\overline{\rm MS}$ scale to a typical physical scale (taken to be the critical temperature T_c) via $T_c/\Lambda_{\overline{\rm MS}} = 1.22$. Note that both of these steps (optimizing the renormalization scale, and fixing $T_c/\Lambda_{\overline{\rm MS}}$) are associated with uncertainties, which are discussed in more detail e.g. in [5].

3. Numerical values

Taking the set of formulae of Sec. 2 and plugging in numerical values for all constants (using $Z_1 = 1.98505..$, $Z_3 = 0.645429..$, $\gamma_0 = 0.577216..$, $\gamma_1 = -0.0728158..$), one obtains

$$p_h = 1.000 - 0.2000\, \tilde{\hat{g}}^2 + 0.9221\, \tilde{\hat{g}}^4 + \left[\#_0 - 0.2487\right] \tilde{\hat{g}}^6 + \mathcal{O}\left(\tilde{\hat{g}}^8\right) , \tag{13}$$

$$\begin{aligned} p_s = {} & 0.3695\, \tilde{m}_{\rm E}^3 + 0.3840\, \tilde{g}_{\rm E}^2 \tilde{m}_{\rm E}^2 \ln\left(0.1091\, \tilde{m}_{\rm E}\right) - 0.5431\, \tilde{g}_{\rm E}^4 \tilde{m}_{\rm E} \\ & - 0.2046\, \tilde{g}_{\rm E}^6 \ln\left(0.3159\, \tilde{m}_{\rm E}\right) - 0.0256\, \tilde{m}_{\rm E}^2 \tilde{\hat{g}}^4 + \mathcal{O}\left(\tilde{g}_{\rm E}^8/\tilde{m}_{\rm E}, \tilde{\hat{g}}^8 \tilde{m}_{\rm E}\right) , \end{aligned} \tag{14}$$

$$p_{us} = -0.07215\, \tilde{g}_{\rm M}^6 \ln\left(1.1(4)\, \tilde{g}_{\rm M}^2\right) . \tag{15}$$

The matching coefficients then read

$$\tilde{g}_{\rm E}^2 = \tilde{\hat{g}}^2 + 0.18265\, \tilde{\hat{g}}^4 + 0.08247\, \tilde{\hat{g}}^6 + \mathcal{O}\left(\tilde{\hat{g}}^8\right) , \tag{16}$$

$$\tilde{m}_{\rm E}^2 = \tilde{\hat{g}}^2 + 0.2360\, \tilde{\hat{g}}^4 + \mathcal{O}\left(\tilde{\hat{g}}^6\right) , \tag{17}$$

$$\begin{aligned} \tilde{g}_{\rm M}^2 = {} & \tilde{g}_{\rm E}^2 \left\{1 - 0.02887\, \frac{\tilde{g}_{\rm E}^2}{\tilde{m}_{\rm E}} - 0.007083\, \frac{\tilde{g}_{\rm E}^4}{\tilde{m}_{\rm E}^2} - 0.001333\, \tilde{\hat{g}}^4 \frac{\tilde{g}_{\rm E}^2}{\tilde{m}_{\rm E}^2}\right\} \\ & + \mathcal{O}\left(\tilde{g}_{\rm E}^8/\tilde{m}_{\rm E}^3, \tilde{\hat{g}}^8 \tilde{g}_{\rm E}^4/\tilde{m}_{\rm E}^3\right) , \end{aligned} \tag{18}$$

while the (scaled and scale-optimized) effective gauge coupling reads

$$\tilde{\hat{g}}^2 = \hat{g}^2 - 0.18265\, \hat{g}^4 + 0.01078\, \hat{g}^6 + \mathcal{O}\left(\hat{g}^8\right) , \tag{19}$$

$$\hat{g}^2 = -\frac{8.088}{1 + W_{-1}\left[-1.186 \exp\left(-5.999 - 2.373 \ln \frac{T}{T_c}\right)\right]} . \tag{20}$$

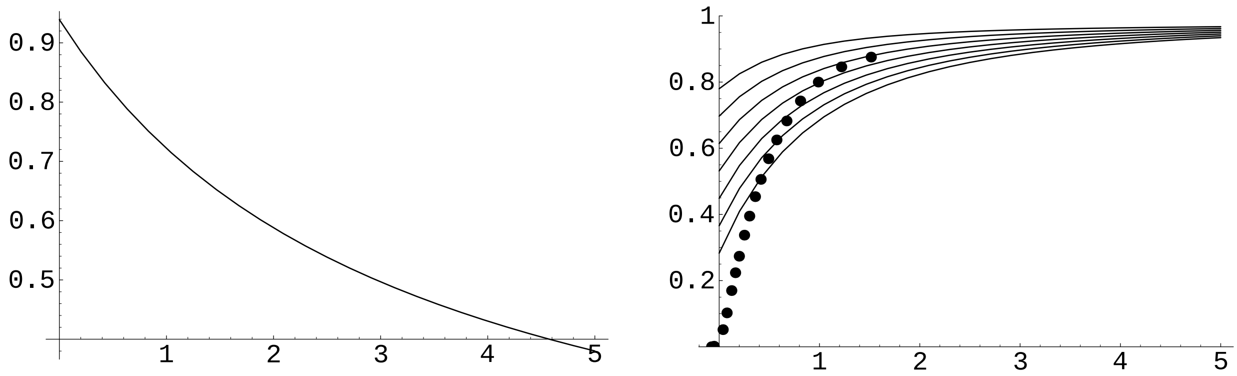

Figure 1. *Left panel:* The effective gauge coupling $\tilde{\tilde{g}}^2$ from Eq. (19), plotted versus $\ln\frac{T}{T_c}$. *Right panel:* The normalized pressure $p_{\rm QCD}/p_{{\rm SB}}$ at $N_f = 0$ plotted versus $\ln\frac{T}{T_c}$. The black dots correspond to lattice data from [11]. The $\tilde{\tilde{g}}^6$ coefficient depends on an unknown parameter $\widetilde{\#}_0$ as defined in Eq. (2), and the different curves correspond to choosing $\widetilde{\#}_0 = 0.0$ (lowest curve) to $\widetilde{\#}_0 = 0.6$, in steps of 0.1.

All the above is plotted in Fig. 1, for different choices of the unknown parameter $\widetilde{\#}_0$. In the right-hand panel, data from 4d lattice Monte-Carlo simulations has been included. Keeping the uncertainties discussed above in mind – and noting that in the overlap region ($T \sim$3-5 T_c) the weak-coupling expansion might be stretched to (or beyond) its limit of applicability – this comparison suggests a value of the order $\widetilde{\#}_0 \sim 0.4$. Whether this estimate is sensible can only be checked by a direct perturbative computation of $\widetilde{\#}_0$.

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