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The Free Energy of Hot QCD

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The free energy of QCD is a most fundamental quantity, studied intensively with a variety of approaches. Lattice data is available up to a few times the critical temperature T_c. Perturbation theory, even at very high temperatures, has serious convergence problems. Using a combined analytical and 3d numerical method, we show that it is possible to compute the QCD free energy from about $2T_c$ to infinity in a well-defined framework.

1. Introduction

The properties of QCD matter are expected to change above the critical temperature of the order of 200 MeV. While the low-temperature phase is governed by bound states, such as mesons, the high-temperature phase should, due to asymptotic freedom, look more like a gas of free quarks and gluons. Any observable witnessing this change is therefore a potential candidate to consider for measurements in heavy-ion collision experiments.

One such observable clearly is the free energy density of the system. The rough picture is that it is, according to the Stefan-Boltzmann law, proportional to the number of effective degrees of freedom (in reality the Stefan-Boltzmann law, which describes the ideal-gas limit, gets modified due to interactions). For vanishing baryon density $\mu_b = 0$ and at temperature T, the free energy density of QCD is simply given by the functional integral

$$f = -\frac{T}{V} \ln \int \mathcal{D}[A^a_\mu \bar\psi_f \psi_f] \exp\left(-\int_0^{1/T} d\tau \int_V d^3x \left[\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_f \bar\psi_f \gamma_\mu D_\mu \psi_f \right] \right) . \quad (1)$$

Note that, in the thermodynamic limit of infinite volume V, the pressure p of the plasma is given directly by $p = -f$. Below, we will choose to display results for the pressure.

The most direct way to evaluate this integral would now be to measure it numerically on the lattice. In fact, this has been done by a number of groups. While the results for $N_f = 0$ are rather complete, they are rapidly developing for a finite number of fermion flavours N_f [1,2]. The general picture emerging from these lattice simulations is the following: Normalizing the pressure to zero below T_c (in practice, one can only measure derivatives of the free energy, leaving an integration constant to be fixed), it rises sharply in the interval $(1-2)T_c$, to level off at a few times T_c. At the highest temperatures used in the simulations, typically $(4-5)T_c$, the deviation to the Stefan-Boltzmann limit is of the order of 15%. This general picture is surprisingly stable with respect to different values of N_f. At even higher temperatures, the pressure is then expected to asymptotically

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approach the ideal-gas limit $p_0(T) = (\pi^2T^4/45)(N_c^2-1+(7/4)N_cN_f)$, where N_c denotes the number of colours.

It turns out that this deviation of 15% is too big to be understood in terms of ordinary perturbation theory. In a series of impressive works, the expansion has been driven to 5th order in the gauge coupling g [3], which required the evaluation of a set of 3-loop vacuum diagrams in the framework of finite-temperature perturbation theory, i.e. including the so-called hard thermal loop effects which lead to a series nonanalytic in g^2:

$$\frac{p(T)}{p_0(T)} = 1+c_2g^2+c_3g^3+(c_4'\ln g+c_4)g^4+c_5g^5+\mathcal{O}(g^6\ln g, g^6) \tag{2}$$

While the coefficients are known analytically, convergence properties are extremely poor, cf. Fig. (1a), at least at all physically relevant temperatures.

Facing the poor convergence of the perturbative series, in the past few years a lot of effort has gone into refined and/or alternative approaches, in order to gain an analytic understanding of the high-temperature behaviour of the pressure. The spectrum ranging from Padé-Borel resummations over using effective masses to employing selfconsistent approximations [4], a general feature of these works is the suppression of infrared (long-distance) effects. While this suppression does not seem to be crucial in the computation of the pressure (which appears to be a short-distance dominated observable)[2], the aim of the remaining part of this paper is to outline a new approach [5], which sets up a framework to resum the long-distance contributions to the pressure to all orders.

2. New approach

A way to understand the poor convergence of the ordinary perturbative expansion is the observation that at small gauge coupling g, the system undergoes dimensional reduction (see e.g. [6] and references therein). The crucial point is that there is the scale hierarchy $gT \ll \pi T$, which allows to perturbatively construct an effective theory for the "soft" modes (momenta $\propto gT$) by integrating out the "hard" modes ($\propto T$). In the case of QCD (and actually for a much wider class of theories), this effective theory is a 3d SU(N_c) + adjoint Higgs model:

$$\mathcal{L}_{3\mathrm{d}} = \frac{1}{4}F_{ij}^2+\frac{1}{2}[D_i,A_0]^2+\frac{1}{2}m_D^2A_0^2+\frac{1}{4}\lambda_AA_0^4+\delta\mathcal{L}_{3\mathrm{d}}\,. \tag{3}$$

While the last term represents higher-order operators, which we do not take into account here, the parameters of the first terms (g_3, m_D^2, λ_A) are related to the physical parameters of the full 4d theory (T, $\Lambda_{\overline{\mathrm{MS}}}$). Using "fastest apparent convergence" optimized next-to-leading order perturbation theory [7] (let us introduce dimensionless parameters[3] x, y and set $N_f=0$, $N_c=3$ here), they read:

$$\frac{g_3^2}{T}=\frac{8\pi^2/11}{\ln(6.742T/\Lambda_{\overline{\mathrm{MS}}})}\quad,\quad x=\frac{\lambda_A}{g_3^2}=\frac{3/11}{\ln(5.371T/\Lambda_{\overline{\mathrm{MS}}})}\quad,\quad y=\frac{m_D^2}{g_3^4}=\frac{3}{8\pi^2x}+\frac{9}{16\pi^2} \tag{4}$$

Two comments are now in order. First, $\mathcal{L}_{3\mathrm{d}}$ can be used to reproduce Eq. (2) in a technically much simpler way [6], namely by evaluating a set of 3-loop QCD diagrams in

[2] In fact, the last method mentioned connects to the lattice data available quite nicely from above.
[3] The 3d gauge coupling g_3^2 has the dimension of a mass.

a "naive" (i.e. no hard-thermal-loop treatment; using dimensional regulation to regulate ultraviolet as well as infrared divergences) 4d calculation and adding the corresponding 3-loop set which includes the adjoint scalar, computed in 3d. This shows that the bad convergence is due precisely to the soft degrees of freedom, and it provides a clearer understanding of the contributions from separate physical scales.

Second, the effective theory is confining, hence non-perturbative [8]. This fact directly leads us to the conclusion that the only way to systematically include the long-distance contributions to the pressure is to treat $\mathcal{L}_{3\mathrm{d}}$ on the lattice[4].

To proceed, we rewrite the pressure, up to hard-scale g^6 contributions, as

$$\frac{p(T)}{p_0(T)} = 1 - \frac{5x}{2} - \frac{45}{8\pi^2}\left(\frac{g_3^2}{T}\right)^3 \left[\mathcal{F}_{\overline{\mathrm{MS}}}(x,y) - \frac{24y}{(4\pi)^2}\left(\ln\frac{\bar{\mu}_{3\mathrm{d}}}{T} + \delta\right)\right] , \quad (5)$$

where $\delta \sim 10^{-4}$ and the dependence on the scale $\mu_{3\mathrm{d}}$, which originates from an infrared divergence of the 4d part, cancels against a similar ultraviolet term[5] in the dimensionless 3d free energy density $\mathcal{F}_{\overline{\mathrm{MS}}}(x,y)$,

$$\mathcal{F}_{\overline{\mathrm{MS}}} = -\frac{1}{Vg_3^6}\ln\int\mathcal{D}A\exp\left(-\int d^3x\mathcal{L}_{3\mathrm{d}}\right) = \mathcal{F}_{\overline{\mathrm{MS}}}(x_0,y_0) + \int_{y_0}^{y}dy\left(\frac{\partial\mathcal{F}_{\overline{\mathrm{MS}}}}{\partial y} + \frac{dx}{dy}\frac{\partial\mathcal{F}_{\overline{\mathrm{MS}}}}{\partial x}\right) , \quad (6)$$

which should be measured on the lattice. This requires a measurement of the quadratic as well as quartic adjoint Higgs field correlators (which determine the partial derivatives under the integral), as well as a perturbative computation in lattice regularization, to match to the $\overline{\mathrm{MS}}$ scheme. Additionally, we choose to fix the integration constant perturbatively at extremely high temperatures ($10^{11}T_c$; T_c is of the order of $\Lambda_{\overline{\mathrm{MS}}}$, the coefficient can be measured on the lattice), where one is assured that the expansion converges.

While the setup is straightforward, in practice all these steps are hampered by considerable (but, as we think, surmountable) difficulties. The actual measurement of the correlators in the continuum- and infinite-volume limits as well as the numerical back-integration are quite involved (though simpler than the full 4d case), and on the perturbative side one would have to tackle up to 4-loop vacuum diagrams (ultimately even in lattice regularization), just to name a few of these difficulties.

3. Discussion

As a first result obtained along the strategy outlined above, Fig. (1b) shows the normalized pressure. The integration constant has been fixed perturbatively on the 3-loop level, allowing for an additional constant e_0, which represents an (up to now) unknown g_3^6 contribution. In principle, this constant can be determined in a setup equivalent to the above, after splitting off its perturbative part (starting at 4-loop): A further reduction step relates e_0 to the free energy of 3d pure gauge theory, which is at the core of the famous non-perturbative g^6 term, but can nevertheless be determined on the lattice. On the lattice side, we have only included the quadratic scalar condensate, while at temperatures closer to T_c, the quartic one will become important also.

[4]Note that simulating a 3d theory has great advantages over the full 4d computation, not only in terms of memory and runtime, but also because the effect of fermions is absorbed into the coefficients now.

[5]This is precisely the way the effective theory is set up: dependence on a matching scale has to cancel.

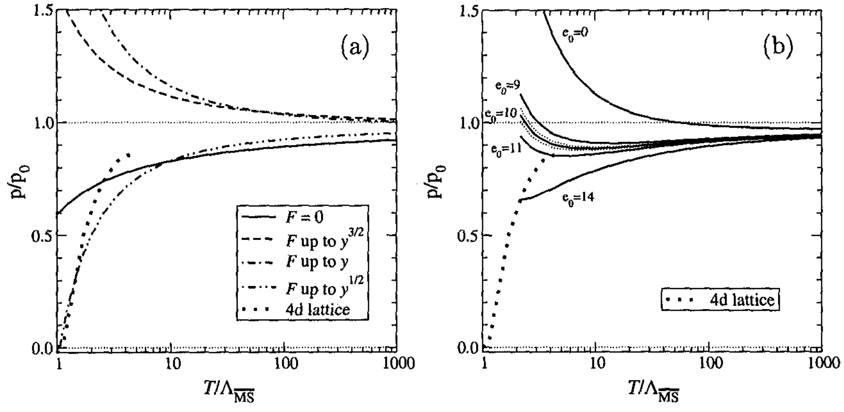

Figure 1. (a) Comparison of the perturbative pressure and lattice data from [1] ($N_f = 0$). The notation is like in Eq. (5), corresponding to p up to g^2, g^3, g^4 and g^5. (b) The pressure after inclusion of the long-distance part according to Eq. (6). Statistical errors are shown only for $e_0 = 10$.

While more work is required (and in progress), we wish to point out two important trends seen in Fig. (1): First, the outcome is sensitive to the non-perturbative parameter e_0, which in principle can be determined by additional computations. Clearly, there exists a range for that parameter (say, $\mathcal{O}(10)$)[6], which leads to a sensible result.

Second, at $T > 30\Lambda_{\overline{MS}}$ the curves for g^5 ($y^{\frac{1}{2}}$) and $e_0 = 10$ fall almost on top of each other, signalling a cancellation of all higher-order terms (determined here by the quadratic Higgs condensate) against the large non-perturbative g^6 contribution. Hence, in this temperature range the pressure is indeed dominated by short-distance effects.

While we have mostly presented results for QCD at $N_f = 0$ and zero baryon number, inclusion of N_f fermion flavours as well as a baryon chemical potential μ_b pose no further complications, and hence provide for a natural extension of this investigation.

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[6]This seems reasonable, since non-perturbative constants tend to be large, like e.g. in the Debye mass.