THE STATIC POTENTIAL IN QCD$_3$ AT ONE LOOP

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The static potential is evaluated perturbatively for massless QCD at one loop. In 2+1 dimensions the emerging linear part of the potential is compared with recent lattice calculations.

The static potential in a gauge theory is a fundamental concept. In QCD it is the interaction energy of an infinitely massive quark–antiquark system and constitutes the non–Abelian analogue of the Coulomb potential of electrodynamics.

In the 3+1 dimensional theory, the $q\bar{q}$ potential was first discussed by Susskind [1] in connection with asymptotic freedom in Yang–Mills theories. Two–loop corrections were calculated by Fischler [2]. Billoire [3] included the effect of fermion loops on the one–loop level, while the two–loop fermion contribution was calculated only recently by Peter [4].

In this letter, we will investigate the static potential in 2+1 dimensions. Understanding the spectrum of the 3D theory is a problem which arises in hot field theory, because of dimensional reduction [5]. In particular, the quest to analytically understand massive states seen on the lattice [6] has led to bound state models [7], for which the knowledge of the static potential is an important ingredient.

Since in 3D the gauge coupling g^2 acquires the dimension of a mass, one expects the perturbative expansion of the potential to be of the form $V_{\rm pert}^{3D}(r) = c_1 g^2 \ln(g^2 r) + c_2 g^4 r + \dots$, where the c_i denote pure numbers. This is in contrast to the behaviour in 4D, where $V_{\rm pert}^{4D}(r) = c_0 g^2/r * (1 + c_1 g^2 \ln(r\mu) + \dots)$ leads to the well–known effective (running) coupling.

The static potential is defined in a manifestly gauge invariant way via the vacuum expectation value of a Wilson loop [9,10] (in fundamental representation),

$$V(r) = \lim_{T\to\infty} \frac{1}{T} \ln \left\langle {\rm tr}\mathcal{P} \exp\left(g \oint_\Gamma dx_\mu A_\mu \right) \right\rangle \; . \tag{1}$$

Here, Γ is taken as a rectangular path with time extension T and spatial extension r.

In the limit of large time extension T, the spatial components of the gauge fields $A_i(\boldsymbol{r}, \pm T/2)$ reduce to pure gauge terms. Hence, since in the perturbative approach we restrict ourselves to the zero instanton sector of the theory, they can be gauged away. The definition (1) reduces to [a]

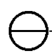
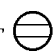
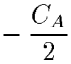
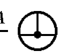
$$V_{\rm pert}(r) = \lim_{T\to\infty} \frac{1}{T} \ln \left\langle {\rm tr}\mathcal{T} \exp\left(- \int_x J^a_\mu A^a_\mu \right) \right\rangle \ , \qquad (2)$$

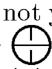
where $\mathcal{T}$ means time ordering and the static sources separated by a distance $r = |\boldsymbol{r} - \boldsymbol{r}\,'|$ are given by

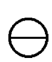
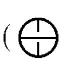
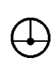
$$J^a_\mu(x) = g\,\delta_{\mu 0} T^a\ [\delta(\boldsymbol{x} - \boldsymbol{r}) - \delta(\boldsymbol{x} - \boldsymbol{r}\,')]\ . \qquad (3)$$

T^a are the generators of the fundamental representation. In the case of QCD we have $SU(3)$, of course, but we will do the calculation for an arbitrary compact semi-simple Lie group with structure constants defined by the Lie algebra $[T^a, T^b] = if^{abc}T^c$. The Casimir operators of the fundamental and adjoint representation are $T^aT^a = C_F$ and $f^{acd}f^{bcd} = C_A\delta^{ab}$. $tr(T^aT^b) = T_f\delta^{ab}$ is the trace normalization, while n_f denotes the number of massless quarks.

Expanding the expression in Eq. (2) perturbatively, one encounters (in addition to the usual Feynman rules) the source–gluon vertex $g\delta_{\mu 0}T^a$, which gets an additional sign for the antisource. Furthermore, the time–ordering presciption generates step functions, which can be viewed as source propagators and provide an analogy to heavy–quark effective theory (HQET) [11]. After working out the color traces, one obtains

$\langle ... \rangle = 1 + g^2 C_F$  $+ g^4 C_F \Big[C_F$  $+ (C_F - \frac{C_A}{2})$  $- \frac{C_A}{2}$  $+$  $\Big] + \ldots$.

Our diagrammatic notation is that of Susskind [1]. The outer loop stands for the source, while the inner lines are gluons, ghosts and (light) quarks. The blob in the last diagram denotes a (one–loop) self–energy insertion. Diagrams are conveniently collected in topological classes, from which the actual 2→2 amplitudes are generated by cutting the source loop twice. Note that the 4–gluon vertex does not yet contribute. After expanding the logarithm and using the relation  +  =  $^2/2$ (which can most easily be checked in position space due to the trivial identity $\theta(t) + \theta(-t) = 1$), only the non–Abelian parts of the diagrams remain:

$\ln\langle ... \rangle = g^2 C_F$  $+ g^4 C_F \Big[-\frac{C_A}{2}($  $+$  $) +$  $\Big] + \ldots$. (4)

[a] Note that in Abelian gauge theories all gauge transformations are continuously connected with the identity, so that the definition remains an exact one.

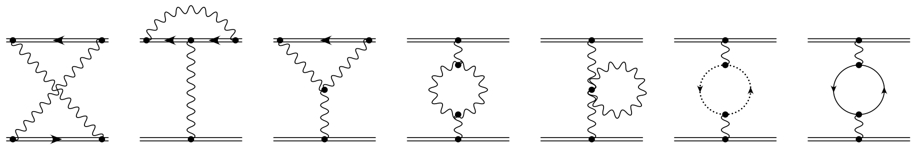


Figure 1: Exchange diagrams contributing to the static potential through 1–loop. Double, wiggly, dotted and solid lines denote source, gluon, ghost and quark propagators, respectively. Diagrams b and c include those obtained by a 180 degree rotation.

By definition, only pure exchange diagrams have to be considered in the calculation of $V_{pert}(r)$, since the others (self–energy corrections of the sources) are independent of r. Hence, the set of diagrams to be considered for the one–loop correction is reduced to the diagrams a–g in Fig. 1.

The calculation can be performed keeping the number of dimensions (D) arbitrary. Here, we work in Euclidean space and general covariant gauges.

It can be shown that the diagrams a–e form a gauge–invariant and IR–finite set, while f and g are gauge–invariant and IR–finite individually. While gauge–invariance was expected from the very definition of the static potential via a Wilson loop, it is more interesting to see how the exact cancellation of infrared divergencies takes place. We have constructed IR finite sets of diagrams first and then calculated the integrals in dimensional regularisation. In our opinion this is the cleanest way to prove IR finiteness, since cancellations take place already on an algebraic level and before specifying a (UV-) regularization scheme.

The result reads $V_{\rm pert}(r) = \int_{\boldsymbol{q}} \exp(i\boldsymbol{q}\boldsymbol{r}) V(\boldsymbol{q}^2)$, with

$$V(\boldsymbol{q}^2) = -\frac{g^2 C_F}{\boldsymbol{q}^2}\left\{ 1 \quad + \quad g^2\,[(4D-5)C_A - 4T_f n_f\,]\,\mu^{2\varepsilon} \times \frac{(\boldsymbol{q}^2)^{\frac{D-4}{2}}}{(16\pi)^{\frac{D-1}{2}}} \frac{\Gamma\left(\frac{4-D}{2}\right)\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D+1}{2}\right)} \; + \; ... \right\} . \qquad (5)$$

One clearly sees that the dimensionally regularized potential is perfectly finite in $D = 3$ dimensions.

As a check, one can evaluate (5) at $D = 4 - 2\varepsilon$. Adding the counterterms arising from gluon–wavefunction and coupling constant renormalization [b], one

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[b] Writing $A = Z_3^{1/2} A_R$ and $g = Z_1 Z_3^{-3/2} g_R$, one gets $V_{ct} = -\frac{g^2 C_F}{q^2}\{2\delta\,(Z_1/Z_3) - \delta Z_3\}$, as easily read off from (4).

recovers the known results [2,3]. Setting $D = 3$, no renormalization is needed and one obtains the new result

$$V^{3D}(\boldsymbol{q}^2) = -\frac{g^2 C_F}{\boldsymbol{q}^2}\left\{1 + \frac{g^2}{32|\boldsymbol{q}|}(7C_A - 4T_f n_f) + \ldots\right\} . \qquad (6)$$

In two (spatial) dimensions the Fourier transforms, which we need, read $-2\pi/\boldsymbol{q}^2 \rightarrow \ln(r/r_0)$ and $-2\pi/|\boldsymbol{q}|^3 \rightarrow r$. They can be derived from Fourier transform in the sense of a distribution, since the potential is understood as an operator acting on wavefunctions which fall off fast enough. The scale r_0 introduced in the above is purely arbitrary. Since the potential is defined only up to a constant, a change in r_0 is irrelevant. Hence, let us choose the only natural scale at hand, $r_0 = 1/g^2$, to obtain for the potential in coordinate space

$$V^{3D}_{\text{pert}}(r) = \frac{g^2 C_F}{2\pi}\ln(g^2 r) + \sigma r + \mathcal{O}(g^4 r^2) , \qquad (7)$$

$$\text{with } \sigma = \frac{g^4 C_F}{64\pi}(7C_A - 4T_f n_f) . \qquad (8)$$

The first term constitutes the well–known Coulomb potential in two space–dimensions, while the linear part is the new information added by this investigation.

The "string tension" σ can be compared with lattice results [8]. There, $\sqrt{\sigma}/g^2$ is measured in the 3D pure $SU(N_c)$ theory for $N_c = 2, 3, 4$ using smeared Polyakov loops. Specializing our result (8) to this case ($T_f = 1/2$, $C_F C_A = (N_c^2 - 1)/2$, $n_f = 0$), it reads

$$\text{pure } SU(N_c) : \qquad \sqrt{\sigma}/g^2 = \sqrt{\frac{7(N_c^2-1)}{128\pi}} \approx 0.132\sqrt{N_c^2 - 1} . \qquad (9)$$

The comparison (see Table 1) shows qualitative agreement, especially concerning N_c–dependence.

N_c	$\sqrt{\sigma}/g^2\|_{lat}$	$\sqrt{\sigma}/g^2\|_{pert}$	lat/pert
2	.34	.23	1.47
3	.55	.37	1.48
4	.76	.51	1.48

Table 1: Comparison of lattice and analytical results

The lattice results are, at least in the range of the parameter N_c covered, larger by a factor of 3/2. The outcome of this calculation suggests that the perturbative part of the potential may constitute a sizeable part of the full potential.

To summarize, we have demonstrated gauge invariance and especially infrared finiteness of the perturbative static potential at one–loop level in arbitrary dimensions. While our D–dimensional formula (5) reproduces the long known terms in $D = 4$, setting $D = 3$ gives a new analytic result. In coordinate space, it consists of the leading logarithmic (coulombic) plus a linear term, the latter of which is compared with lattice simulations. It would be interesting to have information on the next (quadratic) term in the expansion, which bears information on the range of validity of the perturbative potential. An investigation is currently in progress.

Acknowledgments

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