Dimensionally reduced QCD at high temperature

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Abstract

Finite-temperature QCD at high temperature T exhibits three different momentum scales T, gT and g^2T. Naive perturbation theory in a small gauge coupling g does not work beyond leading order. In the framework of effective theories, a separation of contributions originating from the different momentum scales can be systematically performed, allowing for higher-order weak-coupling expansions. In this talk, we review finite-temperature QCD in the EQCD/MQCD framework, and outline a specific high-precision calculation of matching coefficients therein.

1 Introduction

The bulk equilibrium properties of matter at high temperatures can in principle be calculated within a quantum field theory at finite temperature T. The equilibrium properties of such a system are described by its free energy F or, equivalently, its thermodynamic pressure P, as functions of T. In nature, such high-temperature conditions are present in the early universe, while a large ongoing experimental program is reproducing similar conditions in heavy-ion collisions at RHIC or the LHC. On the theoretical side, as pointed out by Linde in 1979 [1], a naive perturbative expansion of the free energy in the small gauge coupling g breaks down at order g^6. The effects which are responsible for the breakdown are qualitatively new and of nonperturbative nature and therefore only accessible through lattice simulations.

The first correction of order $\mathcal{O}(g^2)$ to $P(T)$ was calculated more than 30 years ago [2]. Beyond that order, it was not just simply determining one more term in the perturbative expansion. Naively, one would expect the next correction to be of $\mathcal{O}(g^4)$, but in fact it is a screening effect leading to odd powers in the coupling expansion and starting to contribute at $\mathcal{O}(g^3)$. At high temperatures, the typical distance of particles in the plasma is of order $1/T$. On the other hand, interactions mediated by exchange of particles are long-range effects and get screened over distances larger than $1/gT$. Screening of the chromoelectric force can be taken into account by resumming an infinite number of ring diagrams and its contribution was first calculated in [3] to $\mathcal{O}(g^3)$. Higher corrections were calculated [4, 5, 6] using the same or similar resummation techniques to $\mathcal{O}(g^5)$. Beyond that order the expansion is afflicted with infrared divergencies due to the lack of a screening mass for the chromomagnetic force.

A solution based on the old idea of dimensional reduction was given in [7] by constructing a sequence of two effective field theories. The so-called electrostatic QCD (EQCD) is constructed by integrating

*Talk presented at INTERNATIONAL SCHOOL OF NUCLEAR PHYSICS 33rd Course: From Quarks and Gluons to Hadrons and Nuclei, Erice, Italy, September 16-24, 2011 out the hard scale T whereas the second one called magnetostatic QCD (MQCD) can be obtained by further integrating out the soft scale gT. The free energy or the pressure of hot QCD is then decomposed into its different contributions from the momentum scales T, gT and g^2T starting at $\mathcal{O}(g^0), \mathcal{O}(g^3)$ and $\mathcal{O}(g^6)$, respectively. In contrast to the momentum scales T and gT, the low momentum scale g^2T is only accessible nonperturbatively via lattice simulations. In this framework, the already known $\mathcal{O}(g^5)$ contribution was recalculated [8], and the unknown $\mathcal{O}(g^6 \ln(1/g))$ correction was determined [9]. The weak coupling expansion of the pressure normalized to the Stefan-Boltzmann pressure $P_{\rm SB}$ reads

$$\frac{P(T)}{P_{\rm SB}} = 1 + c_2 g^2 + c_3 g^3 + (c_4' \ln g + c_4)\, g^4 + c_5 g^5 + (c_6' \ln g + c_6)\, g^6 + \mathcal{O}(g^7)\,. \tag{1}$$

The coefficient c_6 is still unknown due to the fact that a 4-loop calculation within thermal QCD is involved, requiring the evaluation of a large number of yet-unknown sum-integrals, at present is a major challenge. The $\mathcal{O}(g^6)$ contribution is of great interest because at this order, all three above-mentioned physical scales contribute to $P(T)$ for the first time.

2 The basic setting

The underlying theory is finite-temperature QCD with N_f flavours of massless quarks and gauge group $SU(N_c)$. Before gauge fixing the corresponding bare Euclidean Lagrangian is given by

$$\begin{aligned} S_{\rm QCD} &= \int_0^\beta d\tau \int d^d x \mathcal{L}_{\rm QCD}\,, \\ \mathcal{L}_{\rm QCD} &= \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar{\psi}\gamma_\mu D_\mu \psi\,, \end{aligned} \tag{2}$$

with $\beta = 1/T, d = 3-2\epsilon$, $\mu, \nu = 0, \ldots, d$, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, $D_\mu = \partial_\mu - igA_\mu$, $A_\mu = A^a_\mu T^a$, with ${\rm Tr}[T^a T^b] = \delta^{ab}/2$, $\gamma^\dagger_\mu = \gamma_\mu$, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, g is the bare gauge coupling, and ψ carries Dirac, colour, and flavour indices. At high temperatures particle masses can be neglected and the quantities we are interested in are solely functions of temperature T and the gauge coupling g. The fundamental quantity is the partition function $\mathcal{Z}(T)$. Taking the logarithm and the limit $V \to \infty$ we get the pressure or minus the free energy defined by

$$P_{\rm QCD}(T) \equiv \lim_{V\to\infty} \frac{T}{V} \ln \mathcal{Z}(T) = \lim_{V\to\infty} \frac{T}{V} \ln \int \mathcal{D}A^a_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-S_{\rm QCD}\right), \tag{3}$$

where V denotes the d-dimensional volume. In the imaginary time formalism the bosonic fields are periodic in imaginary time τ with period $1/T$. Thus the fields A^a_μ can be expanded into their Fourier modes $A^a_{\mu,n} \exp[i2\pi nT\tau]$ with the corresponding propagators $[\mathbf{p}^2 + (2\pi nT)^2]^{-1}$. For nonstatic modes, $2\pi nT$ acts like a mass and at sufficiently high temperature T these modes decouple like infinitely heavy particles from the theory. In contrast to zero-temperature field theories with heavy particles, the decoupling is not ‘complete’ [10]. At sufficiently high temperatures, and for modes with momenta $p \sim gT$, the dynamics of Eq. (2) reduce to a simpler 3-dimensional $SU(N_c)$ coupled to an adjoint scalar A^a_0

$$\begin{aligned} P_{\rm QCD} &= P_{\rm E}(T) + \lim_{V\to\infty} \frac{T}{V} \ln \int \mathcal{D}A_k \mathcal{D}A_0 \exp\left(-S_{\rm E}\right), \\ S_{\rm E} &= \int d^d x \mathcal{L}_{\rm E}\,, \\ \mathcal{L}_{\rm E} &= \frac{1}{2}{\rm Tr} F^2_{kl} + {\rm Tr}\,[D_k, A_0]^2 + m^2_{\rm E} {\rm Tr} A^2_0 + \lambda^{(1)}_{\rm E} \left({\rm Tr} A^2_0\right)^2 + \lambda^{(2)}_{\rm E} {\rm Tr} A^4_0 + \ldots\,, \end{aligned} \tag{4}$$

where $k = 1, \ldots, d, F_{kl} = i/g_{\rm E}\left[D_k, D_l\right], D_k = \partial_k - ig_{\rm E}A_k$. The electrostatic gauge field A_0^a and magnetostatic gauge field A_i^a appearing in Eq. (4) can be related (up to normalisation) to the zero modes of A_μ^a in thermal QCD, Eqs. (2). The effective field theory shown above is known as electrostatic QCD (EQCD) and contains four effective couplings $m_{\rm E}^2, g_{\rm E}^2, \lambda_{\rm E}^{(1)}, \lambda_{\rm E}^{(2)}$ as well as the matching parameter $P_{\rm E}(T)$. These coefficients can be obtained by performing a matching computation, requiring the same result on the QCD and EQCD side within the domain of validity. For the moment let us only note the leading behaviour

$$P_{\rm E} \sim T^4\,,\ m_{\rm E}^2 \sim g^2T^2\,,\ g_{\rm E}^2 \sim g^2T\,,\ \lambda_{\rm E}^{(1)} \sim g^4T\,,\ \lambda_{\rm E}^{(2)} \sim g^4T\,. \tag{5}$$

As soon as these coefficients are determined up to sufficiently high accuracy, it remains to evaluate the pressure using lattice simulations in EQCD, or to perturbatively expand it in a small coupling expansion in $g_{\rm E}$. Unfortunately, the latter approach is afflicted with infrared divergencies and causes a breakdown of the expansion. However, EQCD still contains two dynamical scales gT and g^2T. Integrating out the color electric field A_0 (having a mass $m_{\rm E} \sim gT$) results in a pure 3-dimensional $SU(N_c)$ effective field theory called magnetostatic QCD (MQCD) only containing the magnetostatic gauge field A_k

$$\begin{aligned}\lim_{V\to\infty}\frac{T}{V}\ln\int\mathcal{D}A_k\mathcal{D}A_0\exp\left(-S_{\rm E}\right) &\equiv P_{\rm M}(T) + \lim_{V\to\infty}\frac{T}{V}\ln\int\mathcal{D}A_k\exp\left(-S_{\rm M}\right)\,,\\ S_{\rm M} &= \int d^dx\mathcal{L}_{\rm M}\,,\\ \mathcal{L}_{\rm M} &= \frac{1}{2}{\rm Tr}F_{kl}^2 + \ldots\,,\end{aligned} \tag{6}$$

where $F_{kl} = i/g_{\rm M}\left[D_k, D_l\right], D_k = \partial_k - ig_{\rm M}A_k$. One again finds a new effective coupling $g_{\rm M}$ as well as a new matching parameter $P_{\rm M}$, which are can be related to the parameters of EQCD, Eq. (5) by suitable matching computations and behave as

$$g_{\rm M}^2 \sim g_{\rm E}^2\,,\ P_{\rm M}(T) \sim m_{\rm E}^3T\,. \tag{7}$$

Higher-order operators were neglected in Eqs. (4,6) simply due to fact that they do not contribute at the level we are currently working. The schematic structure of higher-order operators in Eqs. (4) and (6) reads

$$\delta\mathcal{L}_{\rm E} = g^2\frac{D_kD_l}{(2\pi T)^2}\mathcal{L}_{\rm E}\,,\quad \delta\mathcal{L}_{\rm M} = g_{\rm E}^2\frac{D_kD_l}{m_{\rm E}^3}\mathcal{L}_{\rm M}\,, \tag{8}$$

and lead to a contribution in the most conservative case at

$$\frac{\delta P_{\rm QCD}(T)}{T} \sim \delta\mathcal{L}_{\rm E} \sim g^7T^3\,,\quad \frac{\delta P_{\rm QCD}(T)}{T} \sim \delta\mathcal{L}_{\rm M} \sim g^9T^3\,. \tag{9}$$

The contribution of the ultrasoft scale g^2T is completely nonperturbative and its contribution can be obtained via a lattice measurement of the plaquette operator [11] giving

$$P_{\rm G}(T) \equiv \lim_{V\to\infty}\frac{T}{V}\ln\int\mathcal{D}A_k\exp\left(-S_{\rm M}\right) \tag{10}$$

and subsequently translated from lattice to continuum regularisation scheme with the help of lattice perturbation theory [12]. The Lagrangian given in Eq. (6) depends only on one parameter and because of dimensional reasons $P_{\rm G}(T)$ behaves as $P_{\rm G}(T) \sim Tg_{\rm M}^6$. Putting all contributions together we finally obtain the desired decomposition

$$P_{\rm QCD}(T) = P_{\rm E}(T) + P_{\rm M}(T) + P_{\rm G}(T)\,. \tag{11}$$

A more detailed summary can be found in Refs. [9] or [13].

3 Matching computation

So far we have reviewed the effective field theory framework we are working in. However, in order to actually perform computations we still need to determine the emerging coefficients up to sufficiently high accuracy. Let us start recalling what is already known about the matching coefficients. As we have seen, there are five different effective couplings to be determined, the EQCD coefficients $m_{\rm E}^2$, $g_{\rm E}^2$, $\lambda_{\rm E}^{(1)}$, $\lambda_{\rm E}^{(2)}$ and the MQCD effective coupling $g_{\rm M}^2$. Up to 2-loop accuracy, they read

$$\begin{aligned} m_{\rm E}^2 &= T^2\left[g^2\left(\alpha_{E4}+\alpha_{E5}\epsilon+\mathcal{O}(\epsilon^2)\right)+\frac{g^4}{(4\pi)^2}\left(\alpha_{E6}+\beta_{E2}\epsilon+\mathcal{O}(\epsilon^2)\right)+\mathcal{O}(g^6)\right], \\ g_{\rm E}^2 &= T\left[g^2+\frac{g^4}{(4\pi)^2}\left(\alpha_{E7}+\beta_{E3}\epsilon+\mathcal{O}(\epsilon^2)\right)+\frac{g^6}{(4\pi)^4}\left(\gamma_{E1}+\mathcal{O}(\epsilon)\right)+\mathcal{O}(g^8)\right], \\ \lambda_{\rm E}^{(1,2)} &= T\left[\frac{g^4}{(4\pi)^2}\left(\beta_{E4,5}+\mathcal{O}(\epsilon)\right)+\mathcal{O}(g^6)\right], \\ g_{\rm M}^2 &= g_{\rm E}^2\left[1-\frac{1}{48}\frac{g_{\rm E}^2N_c}{\pi m_{\rm E}}-\frac{17}{4608}\left(\frac{g_{\rm E}^2N_c}{\pi m_{\rm E}}\right)^2+\mathcal{O}\left(\left(\frac{g_{\rm E}^2N_c}{\pi m_{\rm E}}\right)^3\right)\right], \end{aligned} \tag{12}$$

where we have used the notation of [9, 14] and [15].

In the following we want to focus on the $m_{\rm E}^2$ and $g_{\rm E}^2$ and push the perturbative expansion one step further to the 3-loop level. Going back to the Lagrangian in Eq. (4), we observe that the effective screening mass $m_{\rm E}^2$ and effective gauge coupling g_E^2 can be computed from suitable 2-, 3-, and 4-point functions. We perform the matching computation in a strict perturbation in g^2 and remove all infrared divergencies with an appropriate infrared cutoff.

The mass parameter $m_{\rm E}^2$ can be understood as the large momentum contribution to the electric screening mass $m_{\rm el}^2$ in the full theory. The screening mass $m_{\rm el}^2$ is defined by the pole of the propagator for $A_0^a(\tau,\mathbf{x})$ at $\mathbf{p}^2=-m_{\rm el}^2$ and $p_0=0$

$$p^2+\Pi(p^2)=0\,, \tag{13}$$

with $\Pi(p^2)\equiv\Pi_{00}(p^2)$. On the EQCD side, the mass parameter $m_{\rm E}^2$ is defined in a similar way by

$$p^2+m_{\rm E}^2+\Pi_{\rm EQCD}(p^2)=0\,, \tag{14}$$

evaluated at $p^2=-m_{\rm el}^2$, and $\Pi_{\rm EQCD}$ denotes the self-energy on the EQCD side. Taylor expanding the self energy $\Pi(\mathbf{p}^2)$ in Eq. (13) we obtain the next-to-next to leading order in terms of Taylor coefficients

$$\begin{aligned} m_{\rm el}^2 &= g^2\Pi_1(0)+g^4\left[\Pi_2(0)-\Pi_1'(0)\Pi_1(0)\right]+g^6\big[\Pi_3(0)-\Pi_1'(0)\Pi_2(0)- \\ &\quad -\Pi_2'(0)\Pi_1(0)+\Pi_1''(0)\Pi_1^2(0)+\Pi_1(0)\Pi_1'^2(0)\big]+\mathcal{O}(g^8)\,. \end{aligned} \tag{15}$$

The only scale in $\Pi_{\rm EQCD}(p^2)$ is p^2 and therefore the Taylor expansion simplifies the computation considerably. All dimensionally regularized integrals are vanishing and with Eq. (14) we immediately get $m_{\rm E}^2=m_{\rm el}^2$.

To simplify the calculation of $g_{\rm E}^2$, we choose background field gauge [16] and the Lagrangian in Eqs. (4) reads symbolically

$$\mathcal{L}_{\rm eff}\sim c_2A^2+c_3gA^3+c_4g^2A^4+\dots\,, \tag{16}$$

where A denotes the background field potential and $c_i=1+\mathcal{O}(g^2)$. Defining $A_{\rm eff}^2\equiv c_2A^2$ we get

$$\mathcal{L}_{\rm eff}\sim A_{\rm eff}^2+c_3c_2^{-3/2}gA_{\rm eff}^3+c_4c_2^{-2}g^2A_{\rm eff}^4+\dots\,, \tag{17}$$

and under consideration of the gauge invariant structure in terms of the effective background potential we obtain $g_{\rm E}=c_3c_2^{-3/2}g=c_4^{1/2}c_2^{-1}g$. In addition, gauge invariance in the original potential A and Eq. (16) yields $c_3=c_2=c_4$ and

$$g_{\rm E}=c_2^{-1/2}g\,. \tag{18}$$

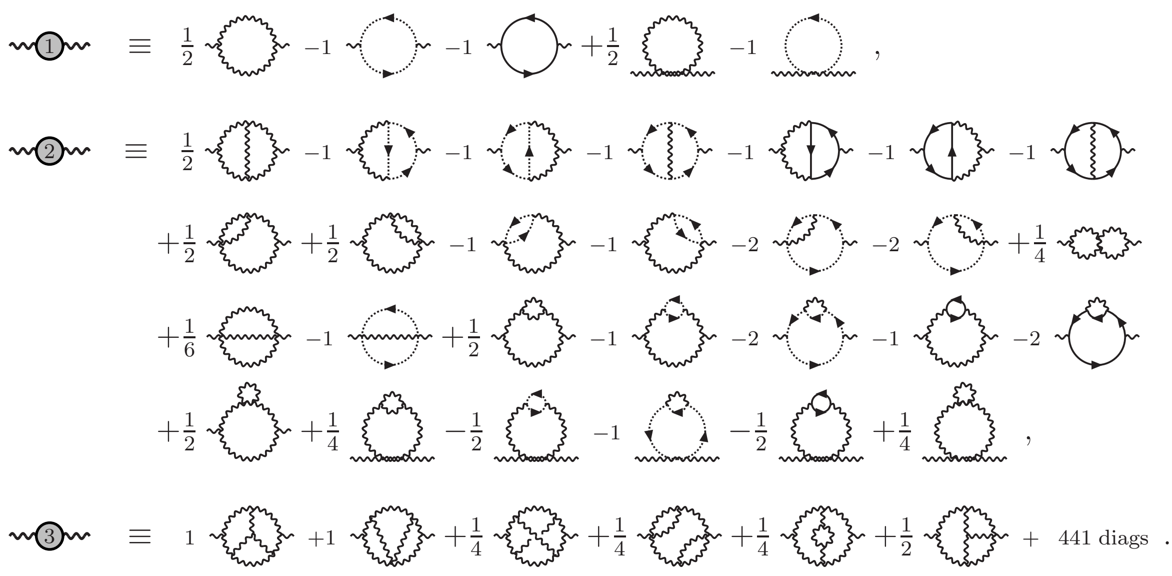

Figure 1: *The 1-loop, 2-loop and some 3-loop self-energy diagrams in the background field gauge. Wavy lines represent gauge fields, dotted lines ghosts, and solid lines fermions.*

From here, we proceed in the same way as for the effective mass $m_{\rm E}^2$ and obtain

$$g_{\rm E}^2 = T\left\{g^2 - g^4\Pi'_{\rm T1}(0) + g^6\left[(\Pi'_{\rm T1}(0))^2 - \Pi'_{\rm T2}(0)\right] + \right.$$
$$\left. +g^8\left[2\,\Pi'_{\rm T1}(0)\Pi'_{\rm T2}(0) - (\Pi'_{\rm T1}(0))^3 - \Pi'_{\rm T3}(0)\right] + \mathcal{O}(g^{10})\right\}\,, \quad (19)$$

where

$$\Pi_{00}(\mathbf{p}) \equiv \Pi_{\rm E}(\mathbf{p}^2)\,,\quad \Pi_{ij}(\mathbf{p}) \equiv \left(\delta_{ij} - \frac{p_i p_j}{\mathbf{p}^2}\right)\Pi_{\rm T}(\mathbf{p}^2) + \frac{p_i p_j}{\mathbf{p}^2}\Pi_{\rm L}(\mathbf{p}^2)\,. \quad (20)$$

The external momentum p is taken purely spatial, $p = (0, \mathbf{p})$, while the heat bath is timelike, with Euclidean four-velocity $u = (1, 0)$, so that $u \cdot u = 1, u \cdot p = 0$. Under these circumstances, $\Pi_{\mu\nu}$ has three independent components (Π_{0i}, Π_{i0} vanish identically). The loop corrections to the spatially longitudinal part $\Pi_{\rm L}$ also vanish, so that only the two non-trivial functions $\Pi_{\rm E}$ and $\Pi_{\rm T}$ remain. A more detailed derivation can be found in [15, 19].

Both coefficients are now expressed in terms of Taylor coefficients cf. Eqs. (15) and (19) evaluated at zero external momentum $p^2 = 0$. In other words, we are left with vacuum tadpoles at finite temperature up to 3-loop. A systematic reduction with the help of integration-by-parts relations and symmetry properties of the different topologies, both implemented using Laporta's algorithm [17], enabled us to reduce the calculation of the 3-loop Taylor coefficients that we need to rather compact expressions of the form [18, 19]

$$\Pi_3 = \sum_i a_i\, A_i + \sum_j b_j\, B_j\,, \quad (21)$$

where $A \in \{I_b I_b I_b, I_f I_b I_b, I_f I_f I_b, I_f I_f I_f\}$ and $B \in \{J, K, L\}$ according to the topologies shown in Fig. 2, with various powers of propagators as well as numerator factors of the form $u \cdot k$, where k is a loop momentum. The coefficients a_i, b_j are rational functions of the space-time dimension d. Important cross-checks have been performed to confirm the validity of our result e.g. $\Pi_{\rm L3} = 0, \Pi'_{\rm L3} = 0$ as well as gauge parameter independence of $m_{\rm E}^2$ and $g_{\rm E}^2$.

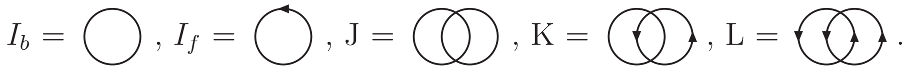

Figure 2: *A shorthand for the bosonic and fermionic master sum-integrals encountered after 3-loop reduction. Each arrow-line corresponds to a fermionic propagator.*

4 Outlook

The reduction step for all sum-integrals required for the diagrams of Fig. 1 has been completed successfully, and the number of master integrals in Eq. (21) is small enough (in the case of $m_{\rm E}^2$ of the order of 30) to tackle their analytic evaluation. However, we are still facing some problems related to the evaluation of such 3-loop basketball-like sum-integrals [20] beyond the constant terms in the ϵ-expansion; for a number of solved 3-loop examples, see e.g. [21].

Expanding beyond constant terms is necessary because many of the prefactors in (21) are singular when expanded around $d = 3 - 2\epsilon$. As a possible strategy for improving this situation, it might be advantageous to perform a change of basis [22] to get rid of or at least reduce the number of divergent prefactors. Work along these lines is under way.

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