4.2 kanonische Transformationen

wollen Hamiltonsche Bewgln (s. S. 55) $\frac{dq_a}{dt} = +\frac{\partial H}{\partial p_a}$, $\frac{dp_a}{dt} = -\frac{\partial H}{\partial q_a}$

noch eleganter / symmetrischer schreiben

betrachte eine beliebige im Phasenraum def. Funktion $f(q,p,t)$

$$\Rightarrow \frac{df}{dt} = \sum_{a=1}^{s} \frac{\partial f}{\partial q_a} \boxed{\frac{dq_a}{dt}} + \sum_{a=1}^{s} \frac{\partial f}{\partial p_a} \boxed{\frac{dp_a}{dt}} + \frac{\partial f}{\partial t}$$

$$= \sum_{a=1}^{s} \left(\frac{\partial f}{\partial q_a} \frac{\partial H}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial H}{\partial q_a} \right) + \frac{\partial f}{\partial t}$$

def. Poisson-Klammer $\{f,g\} \equiv \sum_{a=1}^{s} \left(\frac{\partial f}{\partial q_a} \frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q_a} \right)$

wobei $f(q,p,t)$, $g(q,p,t)$ beliebige reellwertige Fkt sind

$$= \{f, H\} + \frac{\partial f}{\partial t}$$

damit ist auch $\boxed{\frac{dq_a}{dt} = \{q_a, H\} \quad , \quad \frac{dp_a}{dt} = \{p_a, H\}}$

denn: $\{q_a, H\} = \sum_b \underbrace{\frac{\partial q_a}{\partial q_b}}_{\overset{!}{=}\delta_{ab}} \frac{\partial H}{\partial p_b} - \sum_b \underbrace{\frac{\partial q_a}{\partial p_b}}_{\overset{!}{=}0} \frac{\partial H}{\partial q_b} = \frac{\partial H}{\partial p_a}$ ✓

$\{p_a, H\} = \sum_b \underbrace{\frac{\partial p_a}{\partial q_b}}_{\overset{!}{=}0} \frac{\partial H}{\partial p_b} - \sum_b \underbrace{\frac{\partial p_a}{\partial p_b}}_{\overset{!}{=}\delta_{ab}} \frac{\partial H}{\partial q_b} = -\frac{\partial H}{\partial q_a}$ ✓

$\rightsquigarrow$ Eigenschaften der Poisson-Klammer:

- $\{g, f\} = \sum_a \left(\frac{\partial g}{\partial q_a} \frac{\partial f}{\partial p_a} - \frac{\partial g}{\partial p_a} \frac{\partial f}{\partial q_a} \right) = -\{f, g\}$ ist antisymmetrisch
- für eine Konstante c gilt: $\{f, c\} = 0$
- $\{c_1 f_1 + c_2 f_2, g\} = c_1 \{f_1, g\} + c_2 \{f_2, g\}$ ist bilinear
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ _ Jacobi-Identität

Beweis durch explizites Ausschreiben ...

$$\{h, \{f, g\}\} = \sum_{a,b} \frac{\partial h}{\partial q_b} \left(\frac{\partial^2 f}{\partial p_b \partial q_a} \frac{\partial g}{\partial p_a} + \frac{\partial f}{\partial q_a} \frac{\partial^2 g}{\partial p_b \partial p_a} - \frac{\partial^2 f}{\partial p_b \partial p_a} \frac{\partial g}{\partial q_a} - \frac{\partial f}{\partial p_a} \frac{\partial^2 g}{\partial p_b \partial q_a} \right)$$
$$- \sum_{a,b} \frac{\partial h}{\partial p_b} \left(\frac{\partial^2 f}{\partial q_b \partial q_a} \frac{\partial g}{\partial p_a} + \frac{\partial f}{\partial q_a} \frac{\partial^2 g}{\partial q_b \partial p_a} - \frac{\partial^2 f}{\partial q_b \partial p_a} \frac{\partial g}{\partial q_a} - \frac{\partial f}{\partial p_a} \frac{\partial^2 g}{\partial q_b \partial q_a} \right)$$

andere Terme addieren, Indizes umbenennen ... $= 0$ qed

- $\{q_a, q_b\} = 0 = \{p_a, p_b\}$

denn z.B. $\{q_a, q_b\} = \sum_c \left(\frac{\partial q_a}{\partial q_c} \underbrace{\boxed{\frac{\partial q_b}{\partial p_c}}}_{=0} - \underbrace{\boxed{\frac{\partial q_a}{\partial p_c}}}_{=0} \frac{\partial q_b}{\partial q_c} \right) = 0$ ✓

- $\{q_a, p_b\} = \sum_c \left(\underbrace{\frac{\partial q_a}{\partial q_c}}_{=\delta_{ac}} \underbrace{\frac{\partial p_b}{\partial p_c}}_{=\delta_{bc}} - \underbrace{\frac{\partial q_a}{\partial p_c}}_{=0} \underbrace{\frac{\partial p_b}{\partial q_c}}_{=0} \right) = \sum_c \delta_{ac}\delta_{bc} = \delta_{ab}$

⇝ sehr wichtiger Spezialfall; hat direktes Analogon in Quantenmechanik

→ betrachte nun Transformationen der Koordinaten

$Q_a = Q_a(q,t)$ ist Punkttrafo im Koord.-Raum ((z.B. $(x,y) \to (r,\varphi)$))

$\left\{ \begin{array}{l} Q_a = Q_a(q,p,t) \\ P_a = P_a(q,p,t) \end{array} \right\}$ ist Punkttrafo im Phasenraum ((allgemeiner als ↑))

man spricht von einer kanonischen Transformation, falls die neuen Koord. die Werte von Poisson-Klammern nicht ändern;

d.h. $\{f,g\}_{qp} = \{f,g\}_{QP}$, insbesondere $\{Q_a, Q_b\} = 0 = \{P_a, P_b\}$

$\{Q_a, P_b\} = \delta_{ab}$

Bsp $(s=1)$ $\{f,g\}_{qp} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$, $\{f,g\}_{QP} = \frac{\partial f}{\partial Q}\frac{\partial g}{\partial P} - \frac{\partial f}{\partial P}\frac{\partial g}{\partial Q}$

$f(Q,P)$
$g(Q,P)$

$$= \left(\frac{\partial f}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial f}{\partial P}\frac{\partial P}{\partial q}\right)\left(\frac{\partial g}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial g}{\partial P}\frac{\partial P}{\partial p}\right) - \left(\frac{\partial f}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial f}{\partial P}\frac{\partial P}{\partial p}\right)\left(\frac{\partial g}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial g}{\partial P}\frac{\partial P}{\partial q}\right)$$

$$= \left(\frac{\partial f}{\partial Q}\frac{\partial g}{\partial P} - \frac{\partial f}{\partial P}\frac{\partial g}{\partial Q}\right)\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right)$$

$$= \{f,g\}_{QP}\,\{Q,P\}_{qp}$$

⇒ für kanon. Trafo $1 \overset{!}{=} \{Q,P\}_{qp} = \det\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \equiv \frac{\partial(Q,P)}{\partial(q,p)}$

allg. lineare Trafo: $Q = a_{11}q + a_{12}p + b_1$, $P = a_{21}q + a_{22}p + b_2$

⇒ $\frac{\partial(Q,P)}{\partial(q,p)} = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \equiv \det A \overset{!}{=} 1$

⇒ die Matrix A muss unimodular sein;

z.B. $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ Drehung

$A = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ Skalentransformation

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $Q = -p$, $P = q$

$\rightsquigarrow$ allg. Fall ($s \geq 1$):

$$\{f,g\} = \sum_a \left(\frac{\partial f}{\partial q_a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q_a} \right) = \left(\left\{\frac{\partial f}{\partial q_a}\right\}, \left\{\frac{\partial f}{\partial p_a}\right\} \right) \underbrace{\begin{pmatrix} 0 & 1\!\!1_{s\times s} \\ -1\!\!1_{s\times s} & 0 \end{pmatrix}}_{\equiv J} \begin{pmatrix} \left\{\frac{\partial g}{\partial q_a}\right\} \\ \left\{\frac{\partial g}{\partial p_a}\right\} \end{pmatrix}$$

es gilt auch:

$$\begin{pmatrix} \left\{\frac{\partial g}{\partial q_a}\right\} \\ \left\{\frac{\partial g}{\partial p_a}\right\} \end{pmatrix} = \begin{pmatrix} \left\{ \sum_b \left(\frac{\partial Q_b}{\partial q_a}\frac{\partial g}{\partial Q_b} + \frac{\partial P_b}{\partial q_a}\frac{\partial g}{\partial P_b} \right) \right\} \\ \left\{ \sum_b \left(\frac{\partial Q_b}{\partial p_a}\frac{\partial g}{\partial Q_b} + \frac{\partial P_b}{\partial p_a}\frac{\partial g}{\partial P_b} \right) \right\} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial Q_b}{\partial q_a} & \frac{\partial P_b}{\partial q_a} \\ \frac{\partial Q_b}{\partial p_a} & \frac{\partial P_b}{\partial p_a} \end{pmatrix}}_{\equiv M} \begin{pmatrix} \left\{\frac{\partial g}{\partial Q_b}\right\} \\ \left\{\frac{\partial g}{\partial P_b}\right\} \end{pmatrix}$$

also ist $\{f,g\}_{qp} = \{f,g\}_{QP} \iff M^T J M = J$

solche Matrizen M bilden eine "symplektische" Gruppe $Sp(2s)$

((denn: $A^T J A = J$ und $B^T J B = J$

$\Rightarrow (AB)^T J AB = B^T A^T J AB = B^T J B = J$))

die $Sp(2s)$ ist Grundlage der "Mathematik der klassischen Mechanik"

((s. z.B. [V. Arnold, Mathem. Meth. der klass. Mech.]))

Bem.: • Zeitabhängigkeit kann auch als kanon. Trafo gesehen werden:

$Q_a = Q_a(q_0, p_0, t)$, $P_a = P_a(q_0, p_0, t)$ mit Anfangsbed. q_0, p_0

denn: $Q_b = q_b + \Delta t \frac{dq_b}{dt} = q_b + \Delta t \frac{\partial H}{\partial p_b}$

$P_b = p_b + \Delta t \frac{dp_b}{dt} = p_b - \Delta t \frac{\partial H}{\partial q_b}$

$$\Rightarrow M = \begin{pmatrix} 1\!\!1 + \Delta t \frac{\partial^2 H}{\partial q_a \partial p_b} & -\Delta t \frac{\partial^2 H}{\partial q_a \partial q_b} \\ \Delta t \frac{\partial^2 H}{\partial p_a \partial p_b} & 1\!\!1 - \Delta t \frac{\partial^2 H}{\partial p_a \partial q_b} \end{pmatrix}, \quad M^T J M = \ldots = J + \mathcal{O}(\Delta t^2)$$

qed.

• Ziel: finde kanon. Trafo, so dass die Q_a zyklisch sind

dann: $H = H(P_a, t)$, $\frac{dP_a}{dt} = -\frac{\partial H}{\partial Q_a} = 0$, $\frac{dQ_a}{dt} = \frac{\partial H}{\partial P_a}$

falls $H(P_a, \cancel{t}) \Rightarrow Q_a(t) = \frac{\partial H}{\partial P_a} \cdot t + Q_a(0)$ System ist integrierbar

• wegen $M^T J M = J \Rightarrow |\det M| = 1$ gilt

$$\int dQ_1 \ldots dQ_s \, dP_1 \ldots dP_s = \int dq_1 \ldots dq_s \, dp_1 \ldots dp_s \underbrace{\left| \det \frac{\partial(Q,P)}{\partial(q,p)} \right|}_{= |\det M^T| = 1}$$

$\Rightarrow$ Volumenelement im Phasenraum ist eine kanonische Invariante!

((wichtig in statistischer Mechanik; s. auch §4.3))