QCD **Table of Contents** 09.04.2013 – 18.07.2013

Lecture homepage is `http://www.physik.uni-bielefeld.de/~yorks/qcd13`

Quantum Chromodynamics (QCD)

Tue 8.30 - 10
Thu 12.30 - 14

4S, E6-118, 6221, yorks @ physik...

www.physik.uni-bielefeld.de/~yorks/qcd13

→ language? Ger/En

prerequisites : QFT (or Laermann's QFT 5513)
Elementary Particle Physics (for context)

literature : → see webpage, Semesterapparat

topics : → ✓.

Credits : 5 = 3 + 2
↑ oral exam
attending lecture ⇝ signup sheet

exercises : mini-reviews @ start of each lecture

1. Introduction

Nature is extremely strange – but also very beautiful.
We have built a system of understanding (here):

- QM + Special Rel $\Rightarrow$ QFT
- objects: space-filling fields;
  excitations: particles
- "Standard Model" $\hat{=}$ 3 basic conceptual structures
  - $\rightarrow$ gauge system: $SU(3)_c \times SU(2)_L \times U(1)_Y$
    $\sim$ 3 parameters g_i
  - $\rightarrow$ gravity system: Einstein-Hilbert action
    + minimal matter coupling
    $\rightarrow$ 2 parameters G_N, Λ
  - $\rightarrow$ Higgs system: no deep principle
    $\sim$ many parameters
    provisional concept?
- (extremely) accurately tested/confirmed by many experiments

1.1. QCD Appetizer

"zoom" into part of gauge system: QCD ← the $SU(3)_c$ above

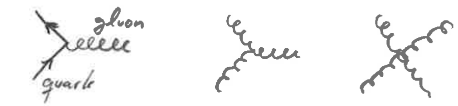
theory of strong interactions

$\hat{=}$ models you know from particle physics (quarks, color; partons)

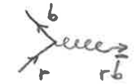
+ mathematical structure you know from QFT
  (non-Abelian gauge theory, [Yang, Mills 1954])

in analogy to QED: specify QCD via Feynman rules

Vertices:



Some qualitative remarks:

- gluon couples to color charge
  color of quark typically changes at qqg-vertex
  eg. 

  , gluon carries the difference
  ((the fact that quarks carry 3 (eg. red/green/blue) color charges has been determined experimentally; more later))
- gluons therefore interact also among themselves
  (in contrast to the electrically neutral photon)
- QCD has very few parameters
  "gauge invariance" requires $\succ \sim g_s$, $\times \sim g_s^2$
- define $\alpha_s \equiv \frac{g_s^2}{4\pi}$ (cf. $\alpha_{EM} = \frac{e^2}{4\pi} \approx \frac{1}{137}$)

  now, $\alpha_s \gtrsim \frac{1}{10}$ is "large"
  - $\Rightarrow$ perturbation theory is not "as perfect" as in QED
  - $\rightarrow$ QCD is "more interesting", has a very rich structure, features surprising effects (more later; material of this semester)
  - $\rightarrow$ theoretical calculations typically have errors $\gtrsim 1\%$
  - $\rightsquigarrow$ one important method to "solve" QCD is (numerical) Lattice-QCD

Some highlights:

- central feature: asymptotic freedom

  QCD shows different faces at long and short distances

long distance		short distance
low energy		high energy ($Q \sim 1/r$)
confinement	⟵ interplay ⟶	asymptotic freedom
non-perturbative hadronic structure		hard scattering cross sections
eg. Lattice QCD		perturbative methods

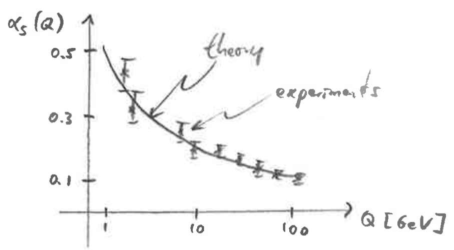
- rough qualitative picture of asymptotic freedom:
  (more later → "renormalization")

  value of α_s depends on distance (i.e. energy)

screening of the charge like in QED — anti-screening non-abelian

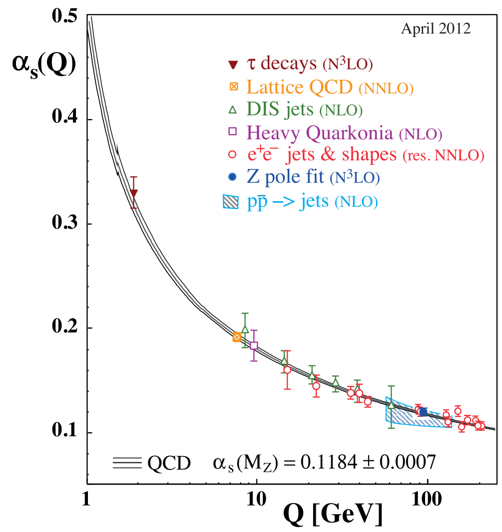
$\alpha_s(r) \nearrow$ if $r \searrow$ — $\alpha_s(r) \searrow$ if $r \searrow$

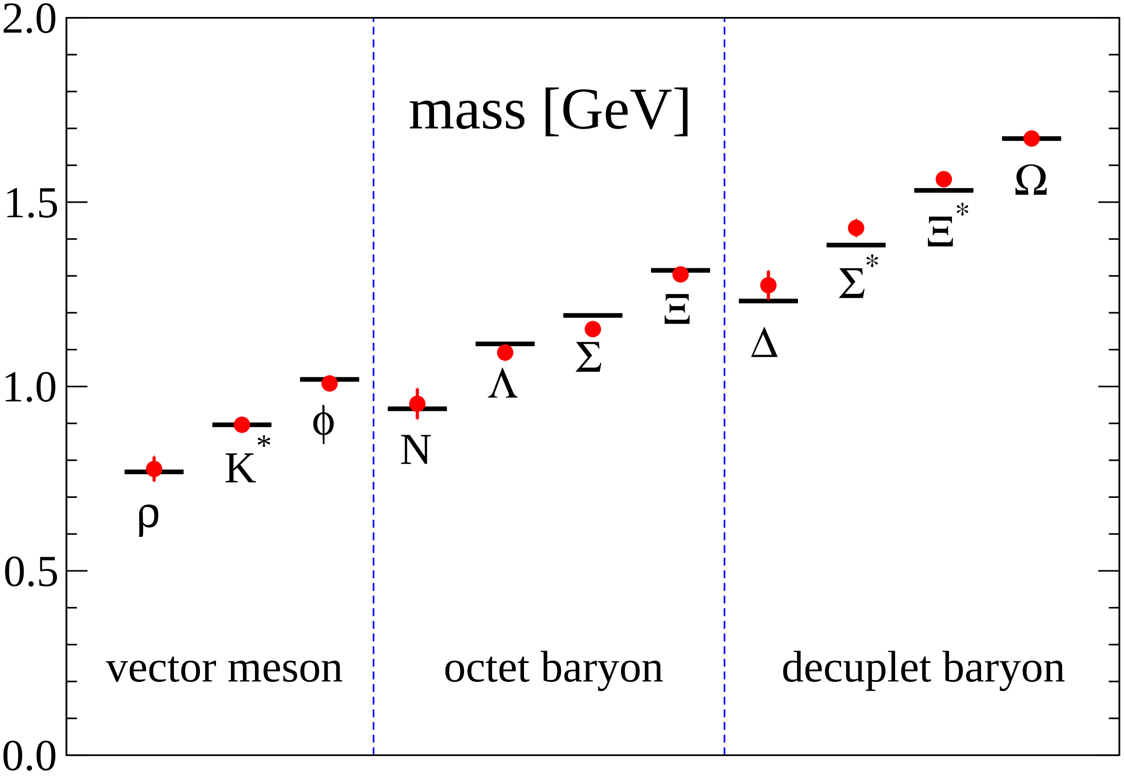
who wins?
$$\alpha_s(Q^2) \approx \frac{4\pi}{\left(-\frac{2}{3}N_f + 11\right)\ln\left(Q^2/\Lambda^2\right)}$$



Nobel 2004: Gross / Politzer / Wilczek

← plot online [PDG; LEP EWWG]





1.2 reality checks

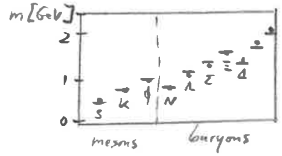
- hadron spectrum
  ↳ (bound states of quarks; eg $K = s\bar{d}$, $p = uud$, $\Lambda = uds$, ...)

  in "our" world, at long distances, observe not quarks + gluons,
  but hadrons (mesons $q\bar{q}$, baryons qqq)

  → "solve" QCD eqs by computer : Lattice QCD ← powerful: presently O(10) TFlops

  ⇒ what one gets are just the observed particles + masses
  (no gluons; no fractional charges)

  upshot : QCD postdicts the low-lying hadron masses



← plot online
[Aoki et al, PACS-CS 2008]
((discretize eg. $Vol(3fm)^4$ → $48^3 \times 64$ points ;
Euclidean time $t \to -it_E$; physics may differ))

- experimental checks of QCD
  collider physics

  eg. LEP (@ CERN, 1989-2000) : $e^+e^- \to X$ ← stuff hitting the detectors

  → asymptotic freedom enables us to compute
  interactions of quarks + gluons at short distances ;
  detectors are a long distance away, see hadrons (not free partons)

  ⇒ for comparison of theory ⇔ experiment, need also
  infrared safety : classes of quantities which are
  independent of long-distance physics, hence pQCD calculable

  factorization : even wider class of processes, can
  be factorized into universal long-distance piece
  and process-dependent (but pQCD calculable) short-distance pc.

  (more later!)

get (QM!) (1) $X = e^+e^-$ or $\tau^+\tau^-$ or ... ⇒ detailed QED check
(2) $X >$ 10 particles : $\pi, s, p, \bar{p}, \ldots$ ⇒ QCD "Jets"

case (1) : no color charge → mainly QED interactions

simple final states: coupling $\alpha_{em} \approx \frac{1}{137}$ small

→ most of the time (99%) nothing happens

→ $e^+e^-\gamma$ ~ 1% ⇒ check QED details

→ $e^+e^-\gamma\gamma$ ~ 0.01% ⇒ ...

case (2) : $X \in$ {"greek & latin soup" constructed from quarks + gluons

observed pattern:

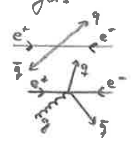
or ...

flow of energy + momentum in "jets"

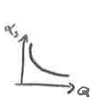
$\alpha_s \gtrsim \frac{1}{10}$ → 2 jets 90%

→ 3 jets 9%

→ 4 jets 0.9%



- perturbative QCD and hard physics

we have seen (pg 4) ((and will calculate later)) that $\alpha_s(Q^2) =$ 

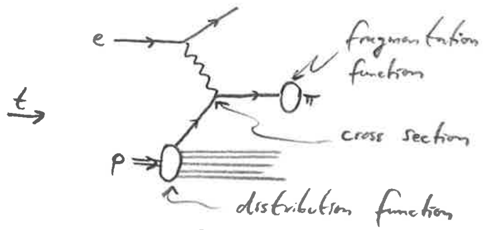
⇒ for large enough 4-momentum squared Q^2 coupling should be small enough for perturbation theory to converge

((more precisely: one gets asymptotic expansions which converge only when "higher twist" and genuine non-perturbative contributions such as "instantons" are also accounted for))

→ perturbative QCD is the basis for interpreting most experiments.

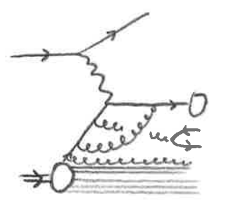
⇒ so pQCD is the most important topic to learn here.

e.g. deep inelastic scattering:



(LO)

vs.



(real life)

- QCD and search for "New Physics"

Specific example: anomalous magnetic moment of muon a_μ

→ determined experimentally and theoretically (within the SM) with such high precision that it became a very sensitive test for many ideas for "physics beyond the SM"

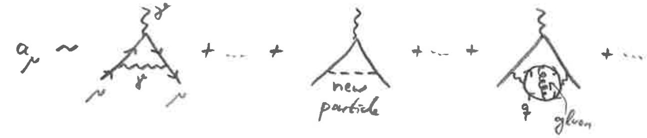
$a_\mu(\text{exp}) = 11659208\,(\pm 6) \cdot 10^{-10}$

$a_\mu(\text{theor}) = 11659186\,(\pm 8) \cdot 10^{-10}$

deviation: 2-3 σ, not "significant" yet

↑ dominated by uncertainty of QCD contributions

→ strategy for "New Physics" search:

$a_\mu \sim$ 

typically get rather stringent limits on e.g. the minimal allowed mass of hypothetical new particles; obviously, any deviation between $a_\mu(\text{exp})$ and $a_\mu(\text{theor})$ could be a signal for new physics.

⇒ does the lack of precision in our QCD calculations keep us from clearly "seeing" signals of exciting new physics?

1.3 Color charge in QCD

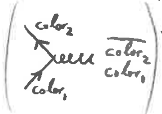
- in addition to its electric charge $\left(\left(\begin{matrix} u\ c\ t \leftarrow +\frac{2}{3} \\ d\ s\ b \leftarrow -\frac{1}{3}\end{matrix}\ ;\ \bar{q}\ \text{neg.}\right)\right)$

each quark carries a <u>color</u> charge.

3 possible values, e.g. r = red, g = green, b = blue

((experimentally determined, more later; often we generalize $3 \to N_c$))

- if a quark emits a gluon, its color may or may not change



→ 9 ways of coupling a gluon between initial + final quark

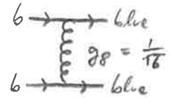
e.g. $g_1 = r\bar{g}$, $g_2 = r\bar{b}$, $g_3 = g\bar{r}$, $g_4 = g\bar{b}$, $g_5 = b\bar{r}$,
$g_6 = b\bar{g}$, $g_7 = \frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$, $g_8 = \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$, } SU(3) color "octet"
$g_0 = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$ } SU(3) "singlet"

- experimental evidence: from scattering expt's we know that matter (mesons = $q\bar{q}$, baryons = qqq) is composed of quarks, yet those hadrons must be neutral to the strong force.
  ⇒ stable particles (hadrons) are "colorless";
  more precisely: they are in "color singlet state"
  ⇒ color singlet gluon state g_0 is not needed / observed.

- strength of coupling between 2 quarks ~ color factors:

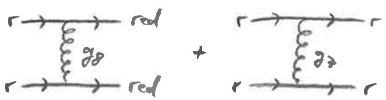
((QED:  $\sim e_1 e_2 \alpha_{em}$ where e.g. $e_{u,c,t} = +\frac{2}{3}$, $e_{d,s,b} = -\frac{1}{3}$ etc.))

QCD:  $\sim \frac{c_1}{\sqrt{2}} \frac{c_2}{\sqrt{2}} \alpha_s$ where $\frac{1}{\sqrt{2}}$ are historical; c_i from g_i above

example: 

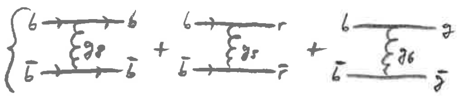
$g_8 = \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$ $\sim \frac{1}{2}\left(-\frac{2}{\sqrt{6}}\right)\left(-\frac{2}{\sqrt{6}}\right) = \frac{1}{3}$ $(\times \alpha_s)$

same result due to color symmetry

vs 

$\sim \frac{1}{2}\frac{1}{\sqrt{6}}\frac{1}{\sqrt{6}} + \frac{1}{2}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$

example: single gluon exchange between q and $\bar{q}$ in color singlet state
$(q\bar{q})_{singlet} = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$ ⇒ consider e.g. $b\bar{b}$, mult. ×3



$\sim 3\,\frac{1}{2}\,\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\left\{-\frac{2}{\sqrt{6}}\frac{2}{\sqrt{6}} - 1\cdot 1 - 1\cdot 1\right\} = -\frac{4}{3}$

(($\bar{q}$ opposite charge to q ⇒ − sign for $\bar{q}$ vertex))

⇒ color force can be both repulsive and attractive.

1.4 Elements of group theory

- the color charge introduced above can be treated much more rigorously.
  → symmetry at work.
  → before (re-) learning the connection symmetry ⇔ charge from QED (cf. § 2.1), let us review some basic facts about the theory of continuous symmetry groups
  → (much) more detail e.g. in [H. Georgi: Lie Algebras in Particle Physics] or at http://www.physik.uni-bielefeld.de/~laine/symmetrien/cover.html

- our provisional color assignment to gluons (cf. § 1.3) $g_1 \ldots g_8$ can be rewritten in a different basis (just different linear combinations) of 3×3 - Matrices (label e.g. rows by colors, columns by anticolors $\begin{matrix} & \bar{r}\ \bar{g}\ \bar{b} \\ \begin{matrix} r \\ g \\ b \end{matrix} & \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \end{matrix}$)

$T^1 = \frac{1}{2}\begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&0 \end{pmatrix}$ $T^2 = \frac{1}{2}\begin{pmatrix} 0&-i&0 \\ i&0&0 \\ 0&0&0 \end{pmatrix}$ $T^3 = \frac{1}{2}\begin{pmatrix} 1&0&0 \\ 0&-1&0 \\ 0&0&0 \end{pmatrix}$ $T^4 = \frac{1}{2}\begin{pmatrix} 0&0&1 \\ 0&0&0 \\ 1&0&0 \end{pmatrix}$

$T^5 = \frac{1}{2}\begin{pmatrix} 0&0&-i \\ 0&0&0 \\ i&0&0 \end{pmatrix}$ $T^6 = \frac{1}{2}\begin{pmatrix} 0&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}$ $T^7 = \frac{1}{2}\begin{pmatrix} 0&0&0 \\ 0&0&-i \\ 0&i&0 \end{pmatrix}$ $T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&-2 \end{pmatrix}$

("generalization of Pauli matrices")

→ actually, $T^a = \frac{1}{2}\lambda^a$, where λ^a are "Gell-Mann matrices", $a = 1 \ldots 3^2-1$ $= 1 \ldots 8$
→ they form a possible representation of the infinitesimal generators of the "special unitary group" SU(3), the fundamental representation
→ some important properties (check ?!):

$[T^a, T^b] = i f^{abc} T^c$ — Lie algebra
↳ antisymmetric structure constants

$\{T^a, T^b\} = \frac{1}{3}\delta^{ab}\mathbb{1}_{3\times3} + d^{abc} T^c$
↳ symmetric structure constants

⇒ $T^a T^b = \frac{1}{2}\left(\frac{1}{3}\delta^{ab}\mathbb{1}_{3\times3} + (d^{abc} + i f^{abc}) T^c\right)$

$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl}\right)$ — Fierz identity

→ often we will need traces
$Tr(T^a) = 0$
$Tr(T^a T^b) = \frac{1}{2}\delta^{ab}$ — normalization
etc.

→ could calculate f^{abc} by multiplying Lie algebra with T^d, then taking trace:

$$f^{abc} = \frac{2}{i}\left(\mathrm{Tr}(T^aT^bT^c) - \mathrm{Tr}(T^bT^aT^c)\right)$$

result (check ?!) : $f^{123} = 1\,,\ f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}$,
$f^{458} = f^{678} = \frac{\sqrt{3}}{2}$; rest by antisymmetry

- from a more general viewpoint, we have just seen one example of a broader mathematical concept: representations of Lie Groups

  math: group contains abstract entities that obey certain algebraic rules

  QM: interested in groups of unitary operators acting on vector space of states

  here: interested in continuously generated groups

  contain elements arbitrarily close to identity.

  can reach general group element by repeated action of infinitesimal ones

  $$g(\alpha) = 1 + i\alpha^a T^a + \mathcal{O}(\alpha^2)$$

  ↑ group parameters; ↑ Hermitean op's; "generators" of symm. group

  a group with this structure is called a "Lie group"

- the set T^a spans space of infinitesimal group transformations

  ⇒ Commutator is a linear combination of generators

  $$[T^a, T^b] = i f^{abc} T^c$$

  vector space spanned by generators + commutator ≡ Lie Algebra

  → $[T^a,[T^b,T^c]] + [T^b,[T^c,T^a]] + [T^c,[T^a,T^b]] = 0$ Jacobi identity

  ⇒ $f^{abe}f^{ecd} + f^{ace}f^{edb} + f^{ade}f^{ebc} = 0$

- for us, symmetry ≙ unitary transformation of a set of fields

  → interested in Lie groups with finite # of generators : "compact"

- classification of Lie Algebras

  → if one T^a commutes with all others : Abelian subgroup, $\psi \to e^{i\alpha}\psi$, $U(1)$ (group of phase rotations)

  → if set of T^a's cannot be divided into two mutually commuting sets: "simple"

  → general Lie algebra ≡ direct sum of non-Abelian simple components + additional Abelian generators

  → $SU(N)$ $((UU^\dagger = 1, \det U = 1))$, $SO(N)$ $((RR^T = 1, \det R = 1))$, $Sp(N)$; $G_2, F_4, E_{6,7,8}$ is complete set of compact simple Lie groups !

  (Killing, Cartan ca. 1894)

1.5 Notation and conventions

- natural units $\hbar = c = k_B = 1$

  $$\Rightarrow [\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1} = \mathrm{GeV}^{-1}$$

- vectors + tensors

  indices $\mu = 0,1,2,3$ or t, x, y, z

  metric tensor $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

  four-vectors $x^\mu = (x^0, \vec{x})$; $\partial_\mu = \partial_{x^\mu} = (\partial_0, \vec{\nabla})$

  totally antisymmetric tensor $\varepsilon^{0123} \equiv 1$ $((\Rightarrow \varepsilon_{0123} = -1,\ \varepsilon^{1230} = -1$ etc$))$

- matrices

  Pauli $\sigma^i\sigma^j = \delta^{ij}\mathbb{1}_{2\times2} + i\varepsilon^{ijk}\sigma^k$

  $$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  Dirac $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

  standard basis: $\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}$, $\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$

  $$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

- Einstein summation convention

  e.g. $p_\mu x^\mu \equiv \sum_{\mu=0}^{3} p_\mu x^\mu = p_0 x^0 + (-\vec{p})\cdot\vec{x} = p^0x^0 - \vec{p}\cdot\vec{x}$

  e.g. $A^a T^a \equiv \sum_a A^a T^a$

2. Basics

2.1 Reminder: QED and gauge invariance

- gauge symmetry is a fundamental principle that determines the form of the Lagrangian
- consider $\psi(x)$ (complex-valued Dirac-field)

  we now demand the theory to be invariant under local phase transformations: $\boxed{\psi(x) \to e^{i\alpha(x)}\psi(x)}$

  Q: which Lagrangian terms can we construct that are invariant?

  A1: terms that are also invariant under global trafos

  e.g. $\bar{\psi}(x)\psi(x)$ ((recall Dirac-adjoint $\bar{\psi} \equiv \psi^\dagger\gamma^0$))

  A2: for terms with derivatives, we need some preparation:

  ((recall: derivative in e.g. n^μ-direction def'd as differential quotient))

  $$\frac{\psi(x+\epsilon n) - \psi(x)}{\epsilon} \xrightarrow{\epsilon\to 0} n^\mu \partial_\mu \psi(x)$$

  feel completely different phase transformation! ↯

  ⇒ for meaningful comparison, introduce a compensating (scalar) phase factor, transforming as $U(y,x) \to e^{i\alpha(y)} U(y,x) e^{-i\alpha(x)}$

  ($U(x,x) = 1$; $\psi(y)$, $U(y,x)\psi(x)$ transform the same)

⇒ def. covariant derivative $\frac{\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)}{\epsilon} \xrightarrow{\epsilon\to 0} n^\mu D_\mu \psi(x)$

for infinitesimal separation of y,x, expand:

$$U(x+\epsilon n, x) \approx 1 - ie\, \epsilon n^\mu A_\mu(x) + O(\epsilon^2)$$

definition ("$-ie$"); ↑ new vector field! "connection"

⇒ cov. deriv.: $D_\mu \psi(x) = \partial_\mu \psi(x) + ie A_\mu(x)\psi(x)$

where $A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$ (consistent w/ U-transformation)

now: $\boxed{D_\mu \psi(x) \xrightarrow{\text{(check!)}} = e^{i\alpha(x)} D_\mu \psi(x)}$

transforms the same way as the field $\psi(x)$ ✓✓

⇒ $\bar{\psi}(x) D_\mu \psi(x)$ also invariant.

- summary 1: local phase rotation symmetry

  ⇒ def. of covariant derivative

  and existence of vector field A_μ (connection)

  and transformation properties of A_μ

  → all terms that are globally ($\psi \to e^{i\alpha}\psi$, α const) invariant are also locally invariant if we replace all $\partial_\mu \to D_\mu$.

- how about (locally invariant) kinetic terms for A_μ?

  (a) construction using $U(y,x)$

  ($u(x,x)=0$ from $U(x,x)=1$)

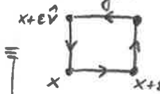
  U is pure phase: $U(y,x) = e^{iu(y,x)}$, $u(y,x)\in\mathbb{R}$

  assume $U(y,x) = [U(x,y)]^\dagger$ ((⇒ $u(y,x) = -u(x,y)$))

  ⇒ $u(y,x) = \sum_{n=0}^{\infty} (y-x)^{2n+1} f_n(y+x)$ is odd under $y \leftrightarrow x$

  ⇝ can write $U(x+\epsilon n, x) = e^{-ie\,\epsilon n^\mu A_\mu(x+\frac{\epsilon n}{2})} + O(\epsilon^3)$

  now, use this for comparing phase products around a small square

  $U(x) \equiv$ 

  ($\hat\mu$: unit vector in μ-direction, e.g. $\hat{1}$)

  $$= U(x, x+\epsilon\hat\nu)\, U(x+\epsilon\hat\nu, x+\epsilon\hat\mu+\epsilon\hat\nu)\, U(x+\epsilon\hat\mu+\epsilon\hat\nu, x+\epsilon\hat\mu)\, U(x+\epsilon\hat\mu, x)$$

  $$= e^{-ie\epsilon\{-A_\nu(x+\frac{\epsilon\hat\nu}{2}) - A_\mu(x+\frac{\epsilon\hat\mu}{2}+\epsilon\hat\nu) + A_\nu(x+\epsilon\hat\mu+\frac{\epsilon\hat\nu}{2}) + A_\mu(x+\frac{\epsilon\hat\mu}{2})\} + O(\epsilon^3)}$$

  $$\approx 1 - ie\epsilon\frac{\epsilon}{2}\{-\partial_\nu A_\nu(x) - \partial_\mu A_\mu(x) - 2\partial_\nu A_\mu + 2\partial_\mu A_\nu(x) + \partial_\nu A_\nu(x) + \partial_\mu A_\mu(x)\} + O(\epsilon^3)$$

  $$= 1 - ie\epsilon^2\big(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\big) + O(\epsilon^3)$$

  (ϵ^2: area of square); $\equiv F_{\mu\nu}(x)$ ≙ electromagnetic field strength tensor

  but $U(x)$ is locally invariant by construction!

  ⇒ $F_{\mu\nu}(x)$ is a locally invariant function of A_μ!

(6) construction using D_μ

since (see above) $\psi \to e^{i\alpha(x)}\psi$, $D_\mu\psi \to e^{i\alpha(x)} D_\mu \psi$

it also follows that $D_\mu D_\nu \psi \to e^{i\alpha(x)} D_\mu D_\nu \psi$

or $[D_\mu, D_\nu]\psi \to e^{i\alpha(x)}[D_\mu, D_\nu]\psi$ (*)

now, note that

$$[D_\mu, D_\nu]\psi = [\partial_\mu,\partial_\nu]\psi + ie\left([\partial_\mu, A_\nu] + [A_\mu, \partial_\nu]\right)\psi - e^2[A_\mu, A_\nu]\psi$$

$$= ie\left(\partial_\mu A_\nu \psi + \partial_\mu A_\nu \psi - A_\nu \partial_\mu \psi + A_\mu \partial_\nu \psi - \partial_\nu A_\mu \psi - \partial_\nu A_\mu \psi\right)$$

$$= ie\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\cdot\psi \quad \text{has no derivative acting outside ()!}$$

$\Rightarrow [D_\mu, D_\nu] = ie F_{\mu\nu}$

$\Rightarrow$ in (*), $F_{\mu\nu}$ is just a multiplicative factor, must be invariant.

- can now write the most general locally invariant Lagrangian (for the electron field ψ and its associated vector field A_μ)

$$\mathcal{L}_{QED} = \bar\psi\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - c\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

remarks:
- ▶ used operators of dimension ≤ 4 here
  in general, there are many additional gauge-invariant op's, e.g.:
  $\mathcal{L}_5 \sim \bar\psi[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\psi$
  $\mathcal{L}_6 \sim (\bar\psi\psi)^2$, $(\bar\psi\gamma^5\psi)^2$, ...
  → all these are non-renormalizable interactions (see later)
  → irrelevant for physics, in Wilsonian sense
  (see e.g. § 12.1 in Peskin/Schroeder)
- ▶ the coefficient $c \equiv 0$ if we postulate invariance under (discrete) P, T symmetries
  → then only 2 free parameters in $\mathcal{L}$: m, e (hidden in D_μ)

- summary 2: local phase rotation symmetry of electron field ψ
  $\Rightarrow$ existence + transformation properties of em. vector potential A_μ
  $\Rightarrow$ most general (4d, renormalizable, T or P invariant) Lagrangian is unique: Maxwell-Dirac-Lagrangian!

2.2 Generalization: Yang-Mills Lagrangian

geometric construction can be generalized:
invariance under local phase rotations
→ invariance under any (continuous) symmetry group

here, use 3d rotation group (O(3) or SU(2)) for brevity
→ in the end, simple generalization to arbitrary local symmetry.

- consider $\psi(x) = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}$ (doublet of Dirac fields)

demand invariance under local 3d rotations: $\boxed{\psi(x) \to e^{i\frac{\sigma^i}{2}\alpha^i(x)}\psi(x)}$, $e^{i\frac{\sigma^i}{2}\alpha^i(x)} \equiv V(x)$

((where σ^i = Pauli matrices $= \left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\}$; $\sum_{i=1}^{3}$ suppressed))

($e^{\text{matrix}} = \sum_{n=0}^{\infty}\frac{(\text{Matrix})^n}{n!}$)

Q: construct invariant Lagrangian?

→ need again a covariant derivative!
→ now, compensating phase factor has to be a matrix, with transformation $U(y,x) \to V(y)U(y,x)V^\dagger(x)$
→ again, $U(x,x) = 1$ and $U^\dagger U = UU^\dagger = \mathbb{1}$ unitary
  $\Rightarrow$ can expand in terms of (hermitean: $\sigma^\dagger = \sigma$) SU(2)-generators:

$$U(x+\varepsilon n, x) \approx \mathbb{1} + ig\,\varepsilon n^\mu A_\mu^i\frac{\sigma^i}{2} + O(\varepsilon^2)$$

convention; ↑ new (matrix-valued) vector field

(-i: different from QED, since we take $e<0$, $g>0$)

$\Rightarrow$ covariant derivative: $\left(\left(n^\mu D_\mu\psi = \lim_{\varepsilon\to 0}\frac{\psi(x+\varepsilon n) - U(x+\varepsilon n, x)\psi(x)}{\varepsilon}\right)\right)$

$$D_\mu = \partial_\mu - ig A_\mu^i\frac{\sigma^i}{2}$$

where $A_\mu^i(x)\frac{\sigma^i}{2} \longrightarrow V(x)\left(A_\mu^i(x)\frac{\sigma^i}{2} + \frac{i}{g}\partial_\mu\right)V^\dagger(x)$ (consistency w/ U-trafo)

now, infinitesimally, $\psi \to \left(1 + i\alpha^i\frac{\sigma^i}{2} + \ldots\right)\psi$

$$D_\mu\psi \to \text{(check!)} = \left(1 + i\alpha^i\frac{\sigma^i}{2}\right)D_\mu\psi$$

again, transforms the same way as field $\psi(x)$ ✓
((also valid for finite transformations (check!)))

$\Rightarrow$ again, $\bar\psi(x)\left(i\gamma^\mu D_\mu - m\right)\psi(x)$ is locally invariant.

- gauge-invariant terms containing A_μ^i only?
  - → here, use construction via D_μ:
    from above, we have $[D_\mu, D_\nu]\psi(x) \to V(x)[D_\mu, D_\nu]\psi(x)$ (**)

  now, note that

$$[D_\mu, D_\nu]\psi = \underbrace{[\partial_\mu, \partial_\nu]}_{0}\psi - ig\left([\partial_\mu, A_\nu^i \tfrac{\sigma^i}{2}] + [A_\mu^i \tfrac{\sigma^i}{2}, \partial_\nu]\right)\psi - g^2 \left[A_\mu^i \tfrac{\sigma^i}{2}, A_\nu^j \tfrac{\sigma^j}{2}\right]\psi$$

does not vanish, as in QED → $= A_\mu^i A_\nu^j \left[\tfrac{\sigma^i}{2}, \tfrac{\sigma^j}{2}\right] = A_\mu^i A_\nu^j i\varepsilon^{ijk}\tfrac{\sigma^k}{2}$

($\sigma^i\sigma^j = \delta^{ij}\mathbb{1} + i\varepsilon^{ijk}\sigma^k$)

$$= -ig\underbrace{\left(\partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g\varepsilon^{ijk}A_\mu^j A_\nu^k\right)}_{\equiv F_{\mu\nu}^i(x)}\frac{\sigma^i}{2}\psi$$

$\equiv F_{\mu\nu}^i(x)$ = (non-Abelian) field strength tensor

  - → as before, $[D_\mu, D_\nu]$ is not a derivative, but a constant (matrix)!
  - ⇒ from (**), the field strength is not invariant now,
    but transforms as $F_{\mu\nu}^i \frac{\sigma^i}{2} \to V(x)\, F_{\mu\nu}^i \frac{\sigma^i}{2} V^\dagger(x)$
  - ⇒ can construct the locally invariant terms from traces (using cyclicity and $V^\dagger V = \mathbb{1}$)
    e.g. $\mathrm{Tr}\left(F_{\mu\nu}^i \frac{\sigma^i}{2} F^{\mu\nu j}\frac{\sigma^j}{2}\right) = \frac{1}{2}F_{\mu\nu}^i F^{\mu\nu i} \equiv \frac{1}{2}(F_{\mu\nu}^i)^2$

- adding up: $\boxed{\mathcal{L} = \bar\psi(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}(F_{\mu\nu}^i)^2}$ Yang-Mills Lagrangian
  - two parameters: m, g
  - variations → equations of motion: Dirac eqn + eqn for vector field

- generalize to other continuous symmetry groups:
  $V \to n\times n$ unitary matrices ; $\psi(x)$ is n-plet ; $\psi(x) \to V(x)\psi(x)$
  expand $V(x) \approx \mathbb{1} + iT^a\alpha^a(x) + \mathcal{O}(\alpha^2)$
  ↑ $(T^a)^\dagger = T^a$ set of generators of symmetry group
  all as above, with $\frac{\sigma^i}{2} \to T^a$
  for def. of $F_{\mu\nu}^a$ use $[T^a, T^b] = if^{abc}T^c$
  ↑ completely antisy. structure const.

- summary:
  invariance of n-plet ψ under local "gauge" transformations $\psi(x) \to V(x)\psi(x)$
  with $V(x) \equiv n\times n$ unitary matrices $= e^{iT^a\alpha^a(x)}$
  where $(T^a)^\dagger = T^a$ are hermitean generators
  with structure constants f^{abc} given by $[T^a, T^b] = if^{abc}T^c$
  - ⇒ covariant derivative $D_\mu = \partial_\mu - igA_\mu^a T^a$
    contains one vector field for each independent generator of local symmetry
    $A_\mu^a T^a \to V(x)\left(A_\mu^a T^a + \frac{i}{g}\partial_\mu\right)V^\dagger(x)$ guarantees $D_\mu\psi(x) \to V(x)D_\mu\psi(x)$
  - ⇒ field strength tensor $[D_\mu, D_\nu] = -igF_{\mu\nu}^a T^a$
    ((or $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$))
    transforms as $F_{\mu\nu}^a T^a \to V(x)F_{\mu\nu}^a T^a V^\dagger(x)$
  - ⇝ for later reference: infinitesimal transformations
    $\psi \to \psi + iT^a\alpha^a(x)\psi + \mathcal{O}(\alpha^2)$
    $A_\mu^a \to A_\mu^a + \left(f^{abc}A_\mu^b + \frac{1}{g}\delta^{ac}\partial_\mu\right)\alpha^c(x) + \mathcal{O}(\alpha^2)$
    $F_{\mu\nu}^a \to F_{\mu\nu}^a + f^{abc}F_{\mu\nu}^b\alpha^c(x) + \mathcal{O}(\alpha^2)$
  - ⇒ most general gauge-invariant renormalizable Lagrangian
    (conserving P, T): $\mathcal{L} = \bar\psi(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}(F_{\mu\nu}^a)^2$

- jargon: Abelian symmetry group of QED
  vs non-Abelian symmetry group of the more general theories above.
  → non-Abelian gauge theory ≡ QFT associated with a non-commuting local symmetry

2.3 QCD and its symmetries

- Quantum Chromodynamics (QCD) is a Yang-Mills theory with gauge group $SU(3)$.
  - matter fields (the q above) are **quarks**;
    they are in the fundamental representation of $SU(3)$, have spin $\frac{1}{2}$
    there are six types ("**flavors**") of quarks: u, d, c, s, t, b
    index of gauge group is called **color** index
    $\Rightarrow$ write as $q^{\alpha,A}$; color index $\alpha = 1,2,3$
    flavor index $A = u,d,c,s,t,b$
  - the $3^2-1=8$ vector fields (or gauge bosons) A_μ^a, $a=1,\dots,8$ are called **gluons**

- $$\mathcal{L}_{QCD} = \bar{q}^{\alpha,A}\left(i\gamma^\mu\left(\partial_\mu \delta_{\alpha\beta} - ig A_\mu^a T^a_{\alpha\beta}\right) - m_A \delta_{\alpha\beta}\right) q^{\beta,A} - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$
  ((sum over color indices α,β; sum over flavor index A))
  $T^a_{\alpha\beta}$: generators of $SU(3)$ in fundamental rep.
  m_A: each quark flavor can have a different mass

- sometimes, it is useful to consider the generalizations
  $SU(3) \to SU(N_c)$ $\Rightarrow$ colors: $\alpha,\beta = 1,\dots,N_c$;
  gluons: $a = 1,\dots,N_c^2-1$
  6 quark flavors $\to N_f$ quark flavors $\Rightarrow A = 1,\dots,N_f$

- QCD possesses not only the exact local $SU(N_c)$ color symmetry, but has also important approximate global symmetries:
  $\to$ consider (x-independent) rotations in flavor space
  ((note: global phase-redefinition for each flavor ($A=u,d,\dots$) separately $\Rightarrow$ $\mathcal{L}_{QCD}$ invariant $\Rightarrow$ quark number conserved))
  $\to$ rotations between different flavors make sense if some masses are (approximately) degenerate
  ((note: in nature, $m_u \sim 3\,MeV$, $m_d \sim 5\,MeV$
  $\Rightarrow m_d - m_u \ll m_{Hadron} \sim 150\,MeV \Rightarrow \mathcal{L}$ has increased symmetry))

$\to$ assume e.g. $m_u \approx m_d$ $\Rightarrow$ $M \equiv \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \approx m_u \mathbb{1}_{2\times2}$

write $q = \begin{pmatrix} q^u \\ q^d \end{pmatrix}$, then $\mathcal{L}_{QCD} \ni \bar{q}(i\not{D} - M)q$

is invariant under $q \to e^{i\sum_{j=0}^{3}\alpha_j\sigma^j} q$ (($\sigma^{1,2,3}$ = Pauli, $\sigma^0 \equiv \mathbb{1}_{2\times2}$))

$\in U(2) = U(1) \otimes SU(2)$

$U(1)$: quark number symmetry (see above); $SU(2)$: "isospin symmetry", exact only if $m_u = m_d$

((note: these symmetries are, via Noether's theorem, associated with vector currents $j_\mu^i = \bar{q}\gamma_\mu\sigma^i q$, hence often $SU(2)_V$))

((note: if e.g. $m_u \approx m_d \approx m_s$ is a useful approximation, then symmetry is enhanced, $SU(3)_V$; etc))

$\to$ for massless flavors, the symmetry becomes even larger: $M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

use left- and right-handed projectors
$\gamma_{L/R} = \frac{1 \mp \gamma^5}{2}$ (($\Rightarrow \gamma_L^2 = \gamma_L$, $\gamma_R^2 = \gamma_R$, $\gamma_L\gamma_R = 0$))

decompose $q^u = (\gamma_L + \gamma_R) q^u \equiv q^u_L + q^u_R$ etc

now $q_L \equiv \begin{pmatrix} q^u_L \\ q^d_L \end{pmatrix}$, and $\mathcal{L}_{QCD} \ni \bar{q}_L i\not{D} q_L + \bar{q}_R i\not{D} q_R$

((note: $M \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ would have coupled L,R: $\mathcal{L} \ni -q_L^\dagger M q_R$ + h.c.))

$\Rightarrow$ independent transformations $q_L \to U_L q_L$, $q_R \to U_R q_R$ permitted!

$\rightsquigarrow U(2)_L \otimes U(2)_R$ symmetry

$= U(1)_L \otimes U(1)_R \otimes SU(2)_L \otimes SU(2)_R$, called **chiral symmetry** ((since acting separately on L,R))

((note: the symmetry $SU(N_f)_L \otimes SU(N_f)_R$ is sometimes rewritten as the product $SU(N_f)_V \otimes SU(N_f)_A$ "axial",

using $Q \equiv \begin{pmatrix} q_L \\ q_R \end{pmatrix}$, $\mathcal{L}_{QCD} \ni \bar{Q} i\not{D} Q$,

invariant under $Q \to e^{i\alpha^a T^a} Q$ and $Q \to e^{i\beta^a T^a \gamma^5} Q$))

T^a: generators of $SU(N_f)$ flavor symmetry in fundamental rep.; γ^5: to check this invariance, see (left)

($\{\gamma^5, \gamma^\mu\} = 0$, $(\gamma^5)^\dagger = \gamma^5$
$\Rightarrow \gamma^\mu e^{i\beta\gamma^5} = e^{-i\beta\gamma^5}\gamma^\mu$ via $e^u = \sum \frac{u^n}{n!}$
$\Rightarrow \gamma^0\gamma^\mu e^{i\beta\gamma^5} = e^{i\beta\gamma^5}\gamma^0\gamma^\mu$
$\Rightarrow \bar{q}\not{\partial}q = q^\dagger\gamma^0\gamma^\mu\partial_\mu q \to q^\dagger e^{-i\beta\gamma^5}\gamma^0\gamma^\mu e^{i\beta\gamma^5}\partial_\mu q = \bar{q}\not{\partial}q$)

- similar to the above approximate global symmetries for light quarks (neglecting effects of order m_q), can also consider <u>heavy quark symmetries</u> (neglecting effects of order $1/m_Q$).
  → systematics, "heavy quark effective theories", see e.g. [M. Neubert, Phys. Rept. 245 (1994) 259]

- other important exact symmetries of QCD are the <u>discrete global symmetries</u>: C, P, T
  ((these agree with the <u>observed properties</u> of the strong interactions; for tests and limits, see Particle Data Group, pdg.lbl.gov))
  → analysis of $\mathcal{L}_{QCD}$ under C, P, T is complicated (at quantum level) due to the possible dim-4 operator we had discovered (see pg. 14)

$$\mathcal{L}_\Theta = \frac{\Theta g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\,\mu\nu} \quad , \text{ where } \quad \tilde{F}^{a\,\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma}$$

  ↑ conventional normalization; on pg 14: "c"

  → $\mathcal{L}_\Theta$ would violate both P and T, in contradiction to observations
  ⇒ set $\Theta = 0$, or at least $\Theta \ll 1$?!
  ↑ ↯ (could be regenerated by known CP effects in weak int.)

  → actually, $F^a_{\mu\nu}\tilde{F}^{a\,\mu\nu} = \partial_\alpha \left\{ 2\varepsilon^{\alpha\beta\gamma\delta} A^a_\beta \left(\partial_\gamma A^a_\delta - \frac{2}{3} g f^{abc} A^b_\gamma A^c_\delta \right) \right\}$
  is a total derivative
  contributes only surface term to action $S = i\int d^4x\, \mathcal{L}$
  therefore plays no role in perturbative QCD

  → however, $\mathcal{L}_\Theta$ can have real physical effects due to non-perturbative effects (QCD vacuum can have non-trivial topology ⇒ surface terms contribute; the {...} is <u>not</u> gauge-invariant)
  [see e.g. Erice lectures by S. Coleman (1977), F. Wilczek (1983)]

  → problem: observations tell $\Theta < 10^{-9}$ (neutron dipole moment)
  "naturally", Θ should be large (coming from <u>strong</u> interactions)
  ⇒ "<u>strong CP problem</u>"
  ⇒ several <u>proposed</u> solutions; e.g. Peccei-Quinn-symmetry
  ⇒ new particles: <u>Axions</u>

<u>2.4 Quantization, path integral (remarks only)</u>

- so far, have seen non-Abelian gauge symmetry at work ⇒ $\mathcal{L}_{QCD}$.
  now, work out consequences for particle physics interactions
  → need rules for computing Feynman diagrams
  → apply rules to compute amplitudes, cross sections

- local gauge symmetry ⇒ some Lagrangian dof's are unphysical
  ($\hat{=}$ can be adjusted arbitrarily by gauge trafos)
  (see QFT lecture)
  cf. QED: in functional integral $\int DA\, e^{iS[A]}$ the photon part was

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu}\right) = \frac{1}{2}\int d^4x\, A_\mu(x) \left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right) A_\nu(x) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k) \left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right) \tilde{A}_\nu(-k)$$

  (FT: $A(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{A}(k)$, $\int d^4x\, e^{-ikx} = (2\pi)^4 \delta^{(4)}(k)$)

  ⇒ for $\tilde{A}_\mu(k) = k_\mu a(k)$, $S = 0$ ⇒ $\int DA\, e^{i0}$ diverges ! ↯
  ↑ arbitrary scalar fct

  ((⇔ (...) has no inverse: cannot solve $(-k^2 g_{\mu\nu} + k_\mu k_\nu)\tilde{D}_F^{\nu\sigma}(k) = i g_\mu{}^\sigma$ for Feynman propag. $\tilde{D}_F$))

  recall Abelian gauge invariance $A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$
  ⇒ field configurations that are gauge-equivalent to $A_\mu(x) = 0$ did ↯

  → the way out was <u>Faddeev-Popov</u> gauge fixing [Phys. Lett. 25B (1967) 29]
  result: $S \to S + \int d^4x \left(-\frac{1}{2\xi}(\partial_\mu A^\mu)^2\right)$
  ⇔ can solve $\left(-k^2 g_{\mu\nu} + (1-\frac{1}{\xi}) k_\mu k_\nu\right)\tilde{D}_F^{\nu\sigma}(k) = i g_\mu{}^\sigma$:

$$\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2 + i\varepsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right) \quad \text{photon propagator}$$

  → propagator depends on arbitrary parameter ξ ? ↯
  physics does <u>not</u>: QED vertex γ_μ is such that ξ drops out of S-matrix elements
  (due to the <u>Ward-Takahashi</u> identities)

  ⇝ similar structure in QCD; ξ-cancellations more complicated.

- we will make use of functional methods
  - → most useful for interacting QFT's: path integral method, relying on functional integration
  - → for (many) more details: [QFT lecture] [Peskin/Schroeder, §9]

- reminder of a functional derivative:

  def. $\delta_{J(x)} J(y) = \delta^{(4)}(x-y)$ or $\delta_{J(x)} \int d^4y\, J(y)\phi(y) = \phi(x)$

  ⇒ can take functional derivatives as usual,

  e.g. $\delta_{J(x)}\, e^{i\int d^4y\, J(y)\phi(y)} = i\phi(x)\, e^{i\int d^4y\, J(y)\phi(y)}$

  e.g. $\delta_{J(x)} \int d^4y\, (\partial_\mu J(y)) A^\mu(y) = -\partial_\mu A^\mu(x)$ ((after partial integration))

- reminder of the generating functional of correlation functions

  $Z[J] = \int \mathcal{D}\phi\, e^{i\int d^4x\,[\mathcal{L} + J(x)\phi(x)]}$ (↑ source term)

  such that (time ordering) $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \dfrac{\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\, e^{i\int d^4x\, \mathcal{L}}}{\int \mathcal{D}\phi\, e^{i\int d^4x\, \mathcal{L}}}$

  $= \dfrac{1}{Z[0]} (-i\delta_{J(x_1)})(-i\delta_{J(x_2)})\, Z[J]\Big|_{J=0}$ very elegant!

- to see the elegance of the $Z[J]$ formulation, consider a free scalar theory, $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$

  ⇒ $\int d^4x\, [\mathcal{L}_0 + J\phi] \overset{PI}{=} \int d^4x\, [\frac{1}{2}\phi(-\partial^2 - m^2 + i\varepsilon)\phi + J\phi]$

  (↑ convergence factor for functional integral, $\varepsilon > 0$)

  complete the square: $\phi \to \phi + i\int d^4y\, D_F(x-y) J(y)$

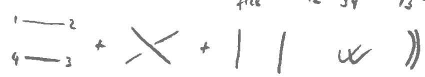
  where $(-\partial^2 - m^2 + i\varepsilon) D_F(x-y) = -i\delta^{(4)}(x-y)$, D_F is Greens fct

  $= \int d^4x\, [\mathcal{L}_0 + \frac{i}{2} J(x) \int d^4y\, D_F(x-y) J(y)]$

  ⇒ $Z_{free}[J] = Z_{free}[0]\, e^{-\frac{1}{2}\int d^4x\, d^4y\, J(x) D_F(x-y) J(y)}$

  ⇒ two-point function $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle_{free} \overset{(check?!)}{=} \cdots = D_F(x_1 - x_2)$

  ⇒ four-point function $\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle_{free} = D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}$

  ((where $D_{ij} \equiv D_F(x_i - x_j)$; ))

  ⇒ etc

(($D_{iz} \equiv D_F(x_i - z)$))

- now, consider an interacting theory, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ (e.g. $-\frac{\lambda}{4!}\phi^4$)

  look at generating functional again

  $Z[J] = \int \mathcal{D}\phi\, e^{i\int d^4x\,[\mathcal{L}_0 + \mathcal{L}_I + J\phi]}$

  $= \int \mathcal{D}\phi\, e^{i\int d^4x\, \mathcal{L}_I(\phi \to -i\delta_J)}\, e^{i\int d^4x\,[\mathcal{L}_0 + J\phi]}$

  (↑ ϕ-independent! ; as in free theory ⇒ shift as above)

  $= e^{i\int d^4x\, \mathcal{L}_I(\phi\to -i\delta_J)} \cdot e^{-\frac{1}{2}\int d^4x \int d^4y\, J(x) D_F(x-y) J(y)} \cdot \int \mathcal{D}\phi\, e^{i\int d^4x\, \mathcal{L}_0}$ (↑ $Z_{free}[0]$)

  such that the correlation functions follow from

  $\langle 0|T\,\mathcal{O}(\phi)|0\rangle = \dfrac{1}{Z[0]}\, \mathcal{O}(\phi \to -i\delta_J)\, Z[J]\Big|_{J=0}$

  $= \dfrac{\mathcal{O}(\phi\to -i\delta_J)\, e^{i\int d^4x\, \mathcal{L}_I(\phi\to -i\delta_J)}\, e^{-\frac{1}{2}\int d^4x \int d^4y\, J(x) D_F(x-y) J(y)}\Big|_{J=0} \cdot Z_{free}[0]}{e^{i\int d^4x\, \mathcal{L}_I(\phi\to -i\delta_J)}\, e^{-\frac{1}{2}\int d^4x \int d^4y\, J(x) D_F(x-y) J(y)}\Big|_{J=0} \cdot Z_{free}[0]}$

  ((note: in denominator, sum of "vacuum diagrams"))

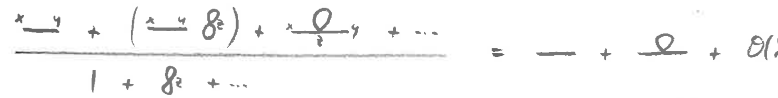
  → perturbative expansion (Feynman diagrams) follow from expanding $e^{i\int\mathcal{L}_I}$ in terms of (small) coupling constants (here: λ)

  → all combinatorics for evaluating correlation fcts is just in exponentials!

  ⇒ two-point function $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle =$

  $= \dfrac{\delta_{iJ(x_1)}\,\delta_{iJ(x_2)}\,\{1 + \int d^4z\, (-\frac{i\lambda}{4!})\,\delta^4_{iJ(z)} + \mathcal{O}(\lambda^2)\}\, e^{-\frac{1}{2}\int d^4x \int d^4y\, J(x) D_F(x-y) J(y)}\Big|_{J=0}}{\{\ \cdots\ \}\, e^{-\frac{1}{2}\cdots}\Big|_{J=0}}$

  ((check!)) $= \dfrac{D_{12} + (-\frac{i\lambda}{4!})\int d^4z\,(3 D_{12} D_{zz} D_{zz} + 12\, D_{1z} D_{2z} D_{zz}) + \mathcal{O}(\lambda^2)}{1 + (-\frac{i\lambda}{4!})\int d^4z\; 3\, D_{zz} D_{zz} + \mathcal{O}(\lambda^2)}$

  $+\ \mathcal{O}(\lambda^2)$

  → this cancellation is actually generic: $\dfrac{(\text{connected pieces}) \cdot e^{(\text{disconnected pieces})}}{e^{(\text{disconnected pieces})}}$

  works for all higher correlation fcts as well.

2.5 QCD Feynman rules

→ recall from above remarks that for a perturbative treatment, we need to read off propagators (cf pg 21) and vertices (cf pg 23) from the Lagrangian $\mathcal{L}$.

→ $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$

- $\mathcal{L}_I$: interactions ⇒ vertices
- $\mathcal{L}_0$: bilinear in fields ⇒ propagators

→ recall: $\mathcal{L}_{QCD} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(i\not{D} - m)\psi$ (pg 18)

$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ (pg 17)

$D_\mu = \partial_\mu - ig A^a_\mu T^a$

$= \mathcal{L}_0 + \bar{\psi} A^a_\mu g \gamma^\mu T^a \psi - g f^{abc} g^{\nu\sigma}(\partial^\mu A^a_\nu) A^b_\mu A^c_\sigma - \frac{1}{4} g^2 f^{eab} f^{ecd} g^{\mu\sigma} g^{\nu\tau} A^a_\mu A^b_\nu A^c_\sigma A^d_\tau$

- quark-gluon vertex: from $e^{i\int \mathcal{L}_I}$

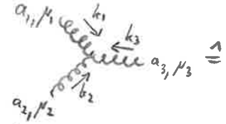
$i \cdot g \gamma^\mu T^a \;\hat{=}$ 

$\left(A(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{A}(k) \right)$

- 3-gluon-vertex: need to fix conventions

in Fourier space, $\partial_\mu \to -ik_\mu$ ⇒ $i \cdot (-g f^{abc} g^{\nu\sigma})(-ik^\mu)$

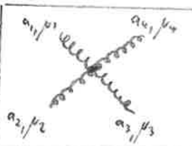
symmetrize this wrt A: 3! possible permutations



$\hat{=}\; g f^{123} \left\{ (k_1 - k_2)_3\, g_{12} + (k_2 - k_3)_1\, g_{23} + (k_3 - k_1)_2\, g_{31} \right\}$

(f^{123}: color indices $a_1, \ldots$; $g_{12}, \ldots$: Lorentz indices $\mu_1, \ldots$)

- 4-gluon-vertex: $i \cdot (-\frac{1}{4} g^2 f^{e12} f^{e34} g^{13} g^{24})$, 4! possible permutations (sets of 4 are equal)



$\hat{=} -ig^2 \left\{ f^{12e} f^{e34} (g_{13} g_{24} - g_{14} g_{23}) + (1324) + (1423) \right\}$

→ for the propagators, need to look at

$S_0 = \int d^4x\, \mathcal{L}_0 = \int d^4x \left\{ \frac{1}{2} A^a_\mu \delta^{ab} (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A^b_\nu + \sum_{\text{flavor}} \bar{\psi}_f (i\not{\partial} - m_f) \psi_f \right\}$

- mini-review: anticommuting (Grassmann) numbers

$\{\theta, \eta\} = 0$ ⇒ $\theta^2 = 0$, Taylor $f(\theta) = a + b\theta$ terminates!

integrals: $\int d\theta = 0$, $\int d\theta\, \theta \equiv 1$

complex Grassmann #s: $\theta = \theta_1 + i\theta_2$, $\theta^* = \theta_1 - i\theta_2$, $(\theta\eta)^* = \eta^*\theta^* = -\theta^*\eta^*$

complex Gauss int: $\int d\theta^* d\theta\, e^{-\theta^* b \theta} = \int d\theta^* d\theta (1 - \theta^* b \theta) = \int d\theta^* d\theta (1 + \theta\theta^* b) = b$

another one: $\int d\theta^* d\theta\, \theta\theta^* e^{-\theta^* b \theta} = \int d\theta^* d\theta\, \theta\theta^* = 1$

higher dim Gauss int: $\left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\theta_i^* B_{ij} \theta_j} = (\;) e^{-\sum \theta_i^* b_i \theta_i} = \prod_i b_i = \det B$

(B hermitean; diagonalize by unitary trafo.)

derivatives: $\partial_\theta \theta = 1$; e.g. $\partial_\theta \eta\theta = -\partial_\theta \theta\eta = -\eta$ etc.

- quark propagator: consider one quark flavor, $\mathcal{L}_0 \ni \bar{\psi}(i\not{\partial} - m)\psi$

($\bar{\eta}, \eta$: Grassmann-valued source fields)

$Z^\psi_{free}[\bar{\eta}, \eta] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi\, e^{i\int d^4x [\bar{\psi}(i\not{\partial} - m + i\varepsilon)\psi + \bar{\eta}\psi + \bar{\psi}\eta]}$

shift ψ to complete square (see pg. 22), $\psi \to \psi + S\eta$ symbolically

$= Z^\psi_{free}[0,0]\, e^{-\int d^4x \int d^4y\, \bar{\eta}(x) S_F(x-y) \eta(y)}$

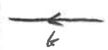
where $S_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{\not{k} - m + i\varepsilon}$

((is again Greens fct: $(i\not{\partial} - m + i\varepsilon) S_F(x-y) = i\delta^{(4)}(x-y)$))

now e.g. $\langle 0 | T \psi(x_1) \bar{\psi}(x_2) | 0 \rangle = \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \psi(x_1)\bar{\psi}(x_2)\, e^{i\int d^4x \mathcal{L}}}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{i\int d^4x \mathcal{L}}}$

$= \frac{1}{Z^\psi_{free}[0,0]} \left(-i\delta_{\bar{\eta}(x_1)}\right)\left(+i\delta_{\eta(x_2)}\right) Z^\psi_{free}[\bar{\eta}, \eta] \Big|_{\bar{\eta} = \eta = 0}$

$= S_F(x_1 - x_2)$ Feynman propagator ✓



$\hat{=} \frac{i}{\not{k} - m + i\varepsilon} = \frac{i(\not{k} + m)}{k^2 - m^2 + i\varepsilon}$

(($(\not{k} - m)(\not{k} + m) = \not{k}\not{k} - m^2$, $\not{k}\not{k} = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} k_\mu k_\nu = k^2$))

→ for gluon propagator, will have same problem as in QED:
$(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu)$ has no inverse (see pg 21)
need Faddeev-Popov gauge fixing

- mini-review: defining the functional integral of a gauge theory

$Z = \int \mathcal{D}\phi\, G(\phi)$; ϕ some gauge-fields (A^a_μ) (($G(A) = e^{i\int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}\right)}$))

gauge invariance: $G(\phi) = G(\phi_\Lambda)$, $\int \mathcal{D}\phi = \int \mathcal{D}\phi_\Lambda$ (($(A_\mu)_\Lambda = V(A_\mu + \frac{i}{g}\partial_\mu)V^\dagger$, $V = e^{iT^a \alpha^a}$)) see §2.3

$= \int \mathcal{D}\phi\, G(\phi)\, \underbrace{\Delta(\phi,b) \int \mathcal{D}\Lambda\, \delta(f(\phi_\Lambda) - b)}_{\equiv 1 \text{ (defines } \Delta)}$ ← arbitrary field (b)

"gauge condition" (($f^a(A) = \partial^\mu A^a_\mu$)) ← covariant gauge

note: $\Delta(\phi,b) = \Delta(\phi_\Lambda,b)$ owing to $\int \mathcal{D}\Lambda$ in its definition

$= \int \mathcal{D}\Lambda \int \mathcal{D}\phi_\Lambda\, G(\phi_\Lambda)\, \Delta(\phi_\Lambda,b)\, \delta(f(\phi_\Lambda) - b)$ used gauge invariance of $\int \mathcal{D}\phi, G, \Delta$ (at each Λ)

$= \left(\int \mathcal{D}\Lambda\right) \int \mathcal{D}\phi\, G(\phi)\, \Delta(\phi,b)\, \delta(f(\phi) - b)$ renamed int variable $\phi_\Lambda \to \phi$

↳ "volume" of gauge orbit factors out; cancels in expectation values!

now average over b, with weight $B(b)$ (($B(b) = e^{-\frac{i}{2\xi}\int d^4x\, b^a b^a}$))

$= \frac{\left(\int \mathcal{D}\Lambda\right)}{\left(\int \mathcal{D}b\, B(b)\right)} \int \mathcal{D}\phi\, G(\phi) \int \mathcal{D}b\, B(b)\, \Delta(\phi,b)\, \delta(f(\phi) - b)$

$= \frac{\left(\int \mathcal{D}\Lambda\right)}{\left(\int \mathcal{D}b\, B(b)\right)} \int \mathcal{D}\phi\, G(\phi)\, B(f(\phi))\, \Delta(\phi, f(\phi))$ used δ-fct

now compute the "Faddeev-Popov determinant" Δ from its definition:

$\Delta(\phi, f(\phi)) = \left[\int \mathcal{D}\Lambda\, \delta(f(\phi_\Lambda) - f(\phi))\right]^{-1}$ cross δ-peak with infinit. gauge trafo

$= \left[\int \mathcal{D}\Lambda\, \delta(\Lambda \mathcal{F}(\phi) + \mathcal{O}(\Lambda^2))\right]^{-1}$ $f(\phi_\Lambda) \approx f(\phi) + \Lambda \mathcal{F}(\phi) + \mathcal{O}(\Lambda^2)$

$= \left[\frac{1}{\det \mathcal{F}(\phi)} \int \mathcal{D}\Lambda\, \delta(\Lambda)\right]^{-1} = \det \mathcal{F}(\phi)$

$= \frac{\left(\int \mathcal{D}\Lambda\right)}{\left(\int \mathcal{D}b\, B(b)\right)} \int \mathcal{D}\phi\, G(\phi)\, B(f(\phi))\, \det \mathcal{F}(\phi)$, where $\mathcal{F}(\phi) = (\partial_\phi f(\phi))(\partial_\Lambda \phi_\Lambda)_{\Lambda=0}$

(($\mathcal{F}(A) = \partial^\mu \left(f^{abc} A^b_\mu + \frac{1}{g}\delta^{ac}\partial_\mu\right)$, pg 17))

note: in QED, $(A_\mu)_\Lambda = A_\mu - \frac{1}{e}\partial_\mu \alpha$ (pg 12),
so $\mathcal{F}$ does not depend on A, hence $\det \mathcal{F}$ cancels in correlators.
⇒ in QCD, $\det \mathcal{F}(A)$ remains inside the functional integral.

- gluon propagator

collection from above, $G(\phi) B(f(\phi)) = e^{i\int d^4x\, \mathcal{L}_{QCD}}\, e^{i\int d^4x \left(-\frac{1}{2\xi}\right)(\partial^\mu A^a_\mu)^2}$

such that $S_0 \ni \int d^4x\, \frac{1}{2} A^a_\mu \delta^{ab} \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu + \frac{1}{\xi}\partial^\mu \partial^\nu\right) A^b_\nu$

$\overset{!}{=} \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{2} \tilde{A}^a_\mu(k)\, \delta^{ab} \left(-k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu\right) \tilde{A}^b_\nu(-k)$

a,μ  b,ν $\hat{=}\ \frac{-i}{k^2 + i\varepsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}$ ((since $(\ldots)^{ab}_{\mu\nu} \cdot (\ldots)^{bc}_{\nu\rho} \overset{\checkmark}{=} i\delta^{ac} g_\mu^{\ \rho}$)) ((check?!))

↳ gauge parameter; often, use $\xi = 1$ Feynman gauge
as in QED, physics is ξ-independent

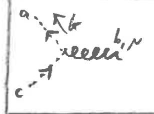
- Faddeev-Popov ghost fields

have to take care of $\det \mathcal{F}(\phi)$ factor (on bottom of pg 26)
using Grassmann numbers again (see Gauss integral on pg 25), rewrite

$\det \mathcal{F}(A) = \det\left(\frac{1}{g}\partial^\mu [g f^{abc} A^b_\mu + \delta^{ac}\partial_\mu]\right)$

$\overset{!}{=} \int \mathcal{D}c\, \mathcal{D}\bar{c}\, e^{i\int d^4x\, \bar{c}^a \left(-\partial^\mu [\partial_\mu \delta^{ac} + g f^{abc} A^b_\mu]\right) c^c}$, $[\ldots] \equiv D^{ac}_\mu$

where the "FP ghosts" are anticommuting fields
(but there are no γ-matrices ⇒ spin 0!)

⇒ ghost propagator	a  b $\hat{=}\ \frac{i}{k^2 + i\varepsilon}\delta^{ab}$
⇒ ghost-gluon-vertex	a, k, c, b,μ  $\hat{=}\ -g f^{abc} k_\mu$

((for physical interpretation of ghosts, see e.g. Peskin/Schroeder § 16.3))

→ now, know all propagators and vertices of QCD,
so we can again (as in ϕ^4 theory, see § 2.4) do
perturbative expansions via generating functional.

- short summary:

  from pg. 18, 26, 27 we have now

$$\mathcal{L}_{QCD} = \sum_f \bar\psi_f \,(i\not{D} - m_f)\,\psi_f - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar c^a(-\partial^\mu D^{ac}_\mu)c^c$$

where $D^{ac}_\mu = \partial_\mu \delta^{ac} + g f^{abc} A^b_\mu$ is the "covariant derivative in the adjoint representation"

→ this expression is still invariant, but not under a local gauge transformation as in § 2.2; the relevant transformation now includes the ghost fields $\bar c, c$ in an essential way and is called "BRST" transformation

[Becchi/Rouet/Stora, Ann. Phys. 98 (1976) 287; Tyutin, Theor. Math. Phys. 27 (1976) 316]

(a symmetry with continuous but anticommuting parameters)
((more: QFT lecture; e.g. Peskin/Schroeder §))

→ the formal treatment of pg. 26 had implicitly assumed that the gauge condition $f(\phi)$ (($f^a(A^a_\mu)$)) selects (via the delta-fct) one unique representative ϕ ((A^a_μ)) for each "gauge orbit" ϕ_Ω.
However, Gribov [Gribov, Nucl. Phys. B 139 (1978) 1] has demonstrated that for non-Abelian theories, this cannot always be guaranteed.
In practice, this fact has little relevance.

- set of "Feynman rules" as usual

  → see QFT lecture; Particle physics lecture; ...

  draw diagrams – fix symmetry factors – insert Feynman rules for propagators + vertices – perform traces and Lorentz algebra – regularize divergent integrals – Wick rotation – evaluate loop integrals – ...

3. Fundamentals

→ our $\mathcal{L}_{QCD}$ contains operators of dimension ≤ 4

⇒ theory is renormalizable: all divergences can be removed by a finite number of counterterms

here: illustrate some divergences in QCD
important physical consequence: asymptotic freedom!

- mini-review: renormalization [see e.g. Peskin/Schroeder §10]

  loops → $\int d^4k$ → ultraviolet (UV) divergences (from large k ⟺ small x)
  common and natural in QFT.

  → counting of UV div's (in 1PI diags): superficial degree of divergence

  → idea: "hide" div's in $\mathcal{L}$ by rescaling fields + couplings

  $\phi_B \to Z_\phi\, \phi_R$ etc, $Z = 1 + O(g_R^2)$ (B: bare, R: renormalized)

  → need intermediate regularization of divergent loop integrals
  many possibilities (discretize space-time; cutoff large momenta; Pauli-Villars; ...)
  most elegant for us: dimensional regularization $\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^dk}{(2\pi)^d}$,

  ($d = 4 - 2\varepsilon$, $\varepsilon \to 0$)

  $d \in \mathbb{C}$; analytic cont. $d \to 4$; div's will be poles $\sim \frac{1}{d-4}$

  Introduce artificial mass-scale μ

  e.g. $(m^2)^{-\varepsilon} = \mu^{-2\varepsilon}\left(\frac{\mu^2}{m^2}\right)^\varepsilon \stackrel{\uparrow}{=} \mu^{-2\varepsilon}\left(1 + \varepsilon \ln \frac{\mu^2}{m^2} + O(\varepsilon^2)\right)$ [$x^\varepsilon = e^{\varepsilon \ln x} \approx$ Taylor ...]

  → the renormalization group (RG) equations describe the μ-dependence of parameters / Greens functions

- mini-review: dimensional regularization

$$\int d^dk\, f(k+q) = \int d^dk\, f(k)$$

$$\int d^dk\, f(\lambda k) = |\lambda|^{-d} \int d^dk\, f(k) \quad \Rightarrow \quad \int d^dk = 0 = \int \frac{d^dk}{(k^2)^{\alpha \neq d/2}} \quad (!)$$

$$\int d^dk\, e^{-\alpha k^2} = \left[\int_{-\infty}^{\infty} dk_1\, e^{-\alpha k_1^2}\right]^d = \left(\frac{\pi}{\alpha}\right)^{d/2}$$

$$\int d^dk\, f(|k|) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dk\, k^{d-1} f(k) \qquad \text{d-dim. spherical coordinates}$$

$$\int d^dk\, k^\mu f(|k|) = 0 \qquad ((\text{odd in } k))$$

$$\int d^dk\, k^\mu k^\nu f(|k|) \stackrel{!}{=} g^{\mu\nu} I = \int d^dk\, \frac{g^{\mu\nu} k^2}{d} f(|k|) \qquad ((\text{since } \int d^dk\, k^2 f = g^\mu_\mu I = d\, I))$$

- important integrals

$$\Gamma(n) = (n-1)! = \int_0^\infty dt\, t^{n-1} e^{-t}$$

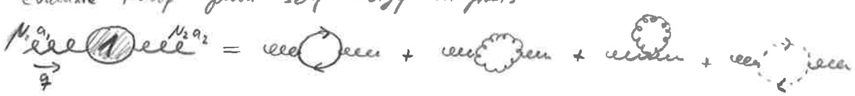
$$\int_0^\infty dx \frac{x^{2a-1}}{(x^2+1)^b} = \frac{\Gamma(a)\Gamma(b-a)}{2\Gamma(b)} \quad (\text{if } \mathrm{Re}[a]>0,\ \mathrm{Re}[b-a]>0)$$

$$\frac{1}{A_1^{a_1} A_2^{a_2} \cdots A_n^{a_n}} = \frac{\Gamma(a_1+\dots+a_n)}{\Gamma(a_1)\cdots\Gamma(a_n)} \int_0^1 dx_1 \dots dx_n \frac{x_1^{a_1-1}\cdots x_n^{a_n-1}\, \delta(1-\sum_{i=1}^n x_i)}{[x_1 A_1 + \dots + x_n A_n]^{a_1+\dots+a_n}}$$ Feynman parametrization

$(A_i = \hat{A}_i + i\varepsilon)$

3.1 one-loop divergences in QCD

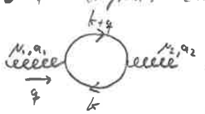
→ goal: evaluate 1-loop gluon self-energy diagrams



use dimensional regularization, $d^4k \to d^dk$, $d = 4-2\varepsilon$

use Feynman gauge $\xi = 1$ for gluon propagator (for simplicity)

- 1st diagram: consider one quark flavor, do $\sum_f$ in the end



$$= -\mathrm{Tr}\int \frac{d^dk}{(2\pi)^d} (ig\gamma^{\mu_1}T^{a_1}) \frac{i(\not k+m)}{k^2-m^2+i\varepsilon} (ig\gamma^{\mu_2}T^{a_2}) \frac{i(\not k+\not q+m)}{(k+q)^2-m^2+i\varepsilon}$$

↑ trace over γ's and T's; one closed fermion loop

[reminder: Dirac matrices

def $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$; $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$

$\mathrm{tr}(\mathbb{1}) \equiv N$ (we take $N=4$; also $N=2^{d/2}$, $N=d$ seen)

$\mathrm{tr}(\gamma^\mu) = 0$; $\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0$; ...

$\mathrm{tr}(\gamma^\mu\gamma^\nu) = N g^{\mu\nu}$; $\mathrm{tr}(\gamma^{1234}) = N(g^{12}g^{34} - g^{13}g^{24} + g^{14}g^{23})$; ...]

$$= -g^2 (\mathrm{tr}\mathbb{1}) \underbrace{(\mathrm{tr}\, T^{a_1}T^{a_2})}_{\frac{1}{2}\delta^{12},\ \text{see p. 9}} \int \frac{d^dk}{(2\pi)^d} \frac{N^{\mu_1\mu_2}(k, k+q)}{(k^2-m^2+i\varepsilon)((k+q)^2-m^2+i\varepsilon)}$$

$$N^{13}(k,p) \overset{\mathrm{tr}(\gamma's)}{=} m^2 g^{13} + m\cdot 0 + (g^{12}g^{34} - g^{13}g^{24} + g^{14}g^{23}) k_2 p_4$$
$$= (m^2 - k\cdot p) g^{13} + k^1 p^3 + p^1 k^3$$

for denominator, use Feynman parameters, see p. 30

$$\frac{1}{(k^2-m^2+i\varepsilon)((k+q)^2-m^2+i\varepsilon)} = \int_0^1 dx \frac{1}{[(1-x)(k^2-m^2+i\varepsilon) + x(k^2+2kq+q^2-m^2+i\varepsilon)]^2}$$
$$= \int_0^1 dx \frac{1}{[(k+xq)^2 + \underbrace{x(1-x)q^2 - m^2}_{\equiv -\Delta} + i\varepsilon]^2}$$

now, shift $k \to k - xq$ under integral $\int d^dk$

$$= -g^2 \cdot 4 \cdot \tfrac{1}{2}\delta^{a_1a_2} \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{N^{\mu_1\mu_2}(k-xq, k+(1-x)q)}{[k^2-\Delta+i\varepsilon]^2}$$

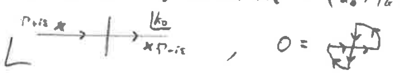
numerator $= (m^2 - (k^2 - x(1-x)q^2 + \mathcal{O}(k)))g^{\mu_1\mu_2} + 2k^{\mu_1}k^{\mu_2} - 2x(1-x)q^{\mu_1}q^{\mu_2} + \mathcal{O}(k)$

linear (in k) terms integrate to 0; $\quad k^{\mu_1}k^{\mu_2} \to \frac{g^{\mu_1\mu_2}k^2}{d}$, see p. 29 bottom

$$= \left(m^2 + \left(\tfrac{2}{d}-1\right)k^2 + x(1-x)q^2\right) g^{12} - 2x(1-x) q^1 q^2$$

to evaluate the $\int d^dk$ integral, perform a Wick rotation $k^0 = ik_E^0$

[$k^2 - m^2 + i\varepsilon = k_0^2 - \vec{k}^2 - m^2 + i\varepsilon = (k_0 + \sqrt{\vec{k}^2+m^2} - i\varepsilon)(k_0 - \Gamma + i\varepsilon)$



$k^0 = ik_E^0$]

$$= -g^2 \cdot 4 \cdot \tfrac{1}{2}\delta^{a_1a_2}\int_0^1 dx\ i\left\{\left[(m^2 + x(1-x)q^2)g^{12} - 2x(1-x)q^1q^2\right] I_2^0(\Delta) + \left(1-\tfrac{2}{d}\right) g^{12} I_2^1(\Delta)\right\}$$

where $I_n^a(\Delta) \equiv \int \frac{d^dk_E}{(2\pi)^d} \frac{(k_E^2)^a}{[k_E^2+\Delta]^n}$ (basic 1-loop "tadpole" integral)

$$= \frac{1}{(2\pi)^d} \frac{2\pi^{d/2}}{\Gamma(d/2)} \underbrace{\int_0^\infty dk_E \frac{k_E^{d-1+2a}}{(k_E^2+1)^n}}_{= \frac{\Gamma(a+d/2)\Gamma(n-a-d/2)}{2\Gamma(n)}} |\Delta|^{d/2+a-n}$$ see p. 30

$$= \frac{\Gamma(a+\frac{d}{2})\Gamma(n-a-\frac{d}{2})}{(4\pi)^{d/2}\Gamma(\frac{d}{2})\Gamma(n)|\Delta|^{n-a-d/2}}$$

note that $I_2^1(\Delta) = I_2^0(\Delta)\cdot|\Delta|\cdot\frac{d}{2}\frac{1}{1-\frac{d}{2}} = I_2^0(\Delta)\frac{|\Delta|}{\frac{2}{d}-1}$ $\quad (x\Gamma(x) = \Gamma(x+1))$

$$= -g^2\cdot 4\cdot\tfrac{1}{2}\delta^{12}\int_0^1 dx\ i\, I_2^0(\Delta)\left\{\left(\underbrace{m^2 + x(1-x)q^2 - |\Delta|}_{\Delta>0:\ = 2x(1-x)q^2}\right)g^{12} - 2x(1-x)q^1q^2\right\}$$

$$= -4ig^2\delta^{a_1a_2}\left(g^{\mu_1\mu_2}q^2 - q^{\mu_1}q^{\mu_2}\right)\int_0^1 dx\, I_2^0(m^2 - x(1-x)q^2)\, x(1-x)$$

$$= \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 dx \frac{x(1-x)}{[m^2 - x(1-x)q^2]^{2-d/2}}$$

• 2^{nd} diagram : calculation parallels the one above !

(Annotations over the first line: symmetry factor; Feynman gauge; color indices a_i.)

$$= \frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\left(\frac{-i}{k^2}\right)\left(\frac{-i}{(k+q)^2}\right) g^2 f^{134} f^{243} \times$$

(Annotation: Lorentz indices μ_i.)

$$\times\left[(q-k)^4 g^{13} + (2k+q)^1 g^{34} + (-k-2q)^3 g^{41}\right]\times$$

$$\times\left[(-k-2q)_3 g^2_4 + (2k+q)^2 g_{34} + (-k+q)_4 g_3{}^2\right]$$

Feynman parameters ; shift $k \to k - xq$; $\Delta \equiv -x(1-x)q^2$;
in numerator, linear $k \to 0$, $k^\mu k^\nu \to g^{\mu\nu}\frac{k^2}{d}$

$$= \frac{1}{2} g^2 f^{134} f^{234} \int_0^1 dx \int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-\Delta)^2} \times$$

$$\times\left\{ g^{12}\left[kk\, 6\,\frac{d-1}{d} + qq\left((2-x)^2+(1+x)^2\right)\right]\right.$$

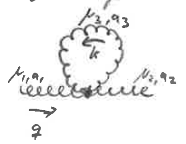
$$\left. - q^1q^2\left[(2-d)(1-2x)^2 + 2(1+x)(2-x)\right]\right\}$$

Wick rotate, use basic 1-loop tadpole integrals, $I_2^1 = I_2^0 \frac{d}{d-2}\Delta$

$$= i g^2 f^{a_1a_3a_4} f^{a_2a_3a_4}\int_0^1 dx\, I_2^0(-x(1-x)q^2)\ \frac{1}{2}\left\{ g^{\mu_1\mu_2}q^2\left[6\,\frac{d-1}{2-d}\,x(1-x) + 5 - 2x + 2x^2\right]\right.$$

$$\left. - q^{\mu_1}q^{\mu_2}\left[(2-d)(1-2x)^2 + 2(1+x)(2-x)\right]\right\}$$

• 3^{rd} diagram : template for $\overset{4\quad 3}{\underset{1\quad 2}{\times}}$: $f^{12a}f^{a34}(g_{13}g_{24} - g_{14}g_{23}) + 1324 + 1423$
(under the terms: 12 33 ; 1323 1323 ← here)



$$= \frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\left(\frac{-i}{k^2}\right)(-ig^2)\left\{ f^{12a}\cancel{f^{a33}}^{\,0}(..) + 2 f^{13a}f^{a23}\underbrace{\left(g^{12}g^3{}_3 - g^{13}g_3{}^2\right)}_{=\, g^{12}(d-1)}\right\}$$

↑ symm. factor

$$= -g^2 f^{134} f^{234} g^{12}(d-1)\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2} \times \frac{k^2+2kq+q^2}{(k+q)^2}$$

(Under $\frac{1}{k^2}$: would be zero ; under the fraction: try to make it look like 2^{nd} diag)

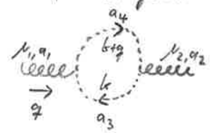
Fy par, shift $k \to k - xq$, $\Delta \equiv -x(1-x)q^2$, numerator $k = 0$

$$= -g^2 f^{134} f^{234} g^{12}(d-1)\int_0^1 dx \int\frac{d^dk}{(2\pi)^d}\frac{k^2 + (1-x)^2q^2}{(k^2-\Delta)^2}$$

Wick rotate , use basic 1-loop tadpole ints

$$= i g^2 f^{a_1a_3a_4} f^{a_2a_3a_4}\int_0^1 dx\, I_2^0(-x(1-x)q^2)\ g^{\mu_1\mu_2} q^2 (1-d)\left[\frac{d}{2-d}\,x(1-x) + (1-x)^2\right]$$

• 4^{th} diagram



(Annotations: closed loop of anticomm. fields; color; Lorentz.)

$$= -\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2}\frac{i}{(k+q)^2}(-g)^2 f^{413}(k+q)^1 f^{324} k^2$$

$$= -g^2 f^{134} f^{234}\int\frac{d^dk}{(2\pi)^d}\frac{(k+q)^1 k^2}{k^2(k+q)^2}$$

Fy par, shift $k \to k - xq$, $\Delta \equiv -x(1-x)q^2$, numerator $k = 0$, $k^\mu k^\nu \to g^{\mu\nu}\frac{k^2}{d}$

$$= -g^2 f^{134} f^{234}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-\Delta)^2}\left\{\frac{g^{12}k^2}{d} - x(1-x)q^1q^2\right\}$$

Wick rotate, use 1-loop tadpole integrals

$$= i g^2 f^{a_1a_3a_4} f^{a_2a_3a_4}\int_0^1 dx\, I_2^0(-x(1-x)q^2)\ x(1-x)\left\{-g^{\mu_1\mu_2}q^2\frac{1}{2-d} + q^{\mu_1}q^{\mu_2}\right\}$$

• sum diagrams 2+3+4

$$2+3+4 = i g^2 f^{134} f^{234}\int_0^1 dx\, I_2^0(-x(1-x)q^2)\ *$$

$$*\left\{ g^{\mu_1\mu_2}q^2\left[3\,\frac{d-1}{2-d}\,x(1-x) + \frac{1}{2}(5-2x+2x^2) + (1-d)\frac{d}{2-d}\,x(1-x) + (1-d)(1-x)^2 - \frac{1}{2-d}\,x(1-x)\right]\right.$$

$$\left. - q^{\mu_1}q^{\mu_2}\left[\frac{2-d}{2}(1-2x)^2 + (1+x)(2-x) \qquad - x(1-x)\right]\right\}$$

(Under the terms: from [2nd diagram] ; [3rd diagram] ; [4th diagram])

note that first line is invariant under $x \to 1-x$
$\{\dots\}$ is polynomial $\sim 1 + x + x^2$
re-express this in terms of $a \equiv 1-2x$ which is odd under $x \to 1-x$
((i.e. $x^2 = \frac{1}{4}(a^2-2a+1)$, $x = \frac{1}{2}(1-a)$))

$$= i g^2 f^{134} f^{234}\int_0^1 dx\, I_2^0(-x(1-x)q^2)\ \left\{ g^{12}q^2\left[\left(1-\tfrac{d}{2}\right)a^2 + \cancel{\tfrac{1-d}{2}a} + 2\right] - q^1q^2\left[\left(1-\tfrac{d}{2}\right)a^2 + 2\right]\right\}$$

(integrates to 0 (odd under $x \to 1-x$))

$$= i g^2 \underbrace{f^{a_1a_3a_4} f^{a_2a_3a_4}}_{=\, N_c\, \delta^{a_1a_2}}\left(g^{\mu_1\mu_2}q^2 - q^{\mu_1}q^{\mu_2}\right)\int_0^1 dx\, I_2^0(-x(1-x)q^2)\left[\left(1-\tfrac{d}{2}\right)(1-2x)^2 + 2\right]$$

((wow : same Lorentz-structure as [fermion loop] !))

$= N_c\, \delta^{a_1a_2}$ for $SU(N_c)$

derivation : from Lie algebra (pg. 9) $[T^a, T^b] = i f^{abc} T^c$

$$\Rightarrow f^{abc} = -2i\, tr\left([T^a,T^b]T^c\right)$$
$$= -2i\left\{(T^a)_{ij}(T^b)_{jk}(T^c)_{ki} - (T^b)_{ij}(T^a)_{jk}(T^c)_{ki}\right\}$$

then use Fierz identity (pg. 9) $(T^a)_{ij}(T^a)_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$,
normalization (pg. 9) $tr(T^aT^b) = \frac{1}{2}\delta^{ab}$, and $\delta_{ii} = N_c$

- summing up all four diagrams (see pg. 31, 33)

$$= ig^2\delta^{a_1a_2}\left(g^{\mu_1\mu_2}q^2 - q^{\mu_1}q^{\mu_2}\right)\int_0^1 dx \; *$$

$$* \left\{-4x(1-x)\sum_f I_2^0\left(m_f^2 - x(1-x)q^2\right) + N_c\left[\left(1-\tfrac{d}{2}\right)(1-2x)^2 + 2\right] I_2^0\left(-x(1-x)q^2\right)\right\}$$

now, use (pg. 31) $I_2^0(\Delta) = \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\frac{1}{\Delta^{2-d/2}} = \frac{1}{2-\frac{d}{2}}\frac{\Gamma(3-\frac{d}{2})}{(4\pi)^{d/2}}\frac{1}{\Delta^{2-d/2}}$

$\overset{d=4-2\varepsilon}{\approx} \frac{1}{\varepsilon}\frac{1}{(4\pi)^2} + O(\varepsilon^0)$

$\Rightarrow \{\dots\} \approx \frac{1}{\varepsilon}\frac{1}{(4\pi)^2}\left\{-4x(1-x)N_f + N_c\left[-(1-2x)^2+2\right]\right\} + O(\varepsilon^0)$

$\Rightarrow \int_0^1 dx\{\dots\} \approx \frac{1}{\varepsilon}\frac{1}{(4\pi)^2}\left\{-\frac{2}{3}N_f + \frac{5}{3}N_c\right\} + O(\varepsilon^0)$

$$\approx i\frac{g^2}{16\pi^2}\delta^{a_1a_2}\left(g^{\mu_1\mu_2}q^2 - q^{\mu_1}q^{\mu_2}\right)\frac{1}{\varepsilon}\left\{\boxed{\tfrac{5}{3}}N_c - \frac{2}{3}N_f + O(\varepsilon)\right\}$$

$\boxed{\tfrac{5}{3}} \to \left(\frac{13}{6} - \frac{\xi}{2}\right)$ for general value of gauge parameter ξ

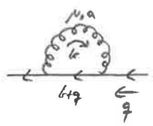
⤳ cannot (yet) take limit $\varepsilon \to 0$

need to remove the $\frac{1}{\varepsilon}$ divergence by a counterterm (see pg 29; §3.3 below)

⤳ want to first compute other 1-loop divergences

3.2 more 1-loop divergences in QCD

→ goal: evaluate 1-loop diagrams (again, in dim. reg. ($d=4-2\varepsilon$) and Feynman gauge ($\xi=1$))

- 

$$= \int\frac{d^dk}{(2\pi)^d}\left(ig\gamma^\mu T^a\right)\frac{i(\not k+\not q+m)}{(k+q)^2-m^2+i\varepsilon}\left(ig\gamma_\mu T^a\right)\left(\frac{-i}{k^2}\right)$$

$\gamma^\mu\gamma_\mu = \gamma^\mu\gamma^\nu g_{\mu\nu} = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}g_{\mu\nu} = \frac{1}{2}2g^{\mu\nu}\mathbb{1}g_{\mu\nu} = d\,\mathbb{1}$

$\gamma^\mu\gamma^\nu\gamma_\mu = \{\gamma^\mu,\gamma^\nu\}\gamma_\mu - \gamma^\nu\gamma^\mu\gamma_\mu = 2\gamma^\nu - \gamma^\nu d = (2-d)\gamma^\nu$

$T^a_{ij}T^a_{jk} = \frac{1}{2}\left(\delta_{ik}\delta_{jj} - \frac{1}{N_c}\delta_{ij}\delta_{jk}\right) = \frac{N_c^2-1}{2N_c}\delta_{ik}$

$$= -g^2\frac{N_c^2-1}{2N_c}\mathbb{1}_{\text{color}}\int\frac{d^dk}{(2\pi)^d}\frac{(2-d)(\not k+\not q)+md\,\mathbb{1}_{\text{Dirac}}}{(k^2)\left((k+q)^2-m^2\right)}$$

Fy par, $\frac{1}{()()} = \int_0^1 dx\frac{1}{[(1-x)k^2+x(k^2+2kq+q^2-m^2)]^2} = \int_0^1 dx\frac{1}{[(k+xq)^2+\underbrace{x(1-x)q^2-xm^2}_{=-\Delta}]^2}$

shift $k \to k - xq$

$$= -g^2\frac{N^2-1}{2N}\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx\frac{(2-d)(\not k + (1-x)\not q)+md}{(k^2-\Delta)^2}$$

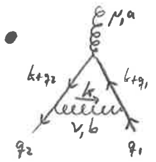
($\not k \to 0$ (odd))

Wick rotate, use basic tadpole from pg 31

$$= -ig^2\frac{N^2-1}{2N}\int_0^1 dx\, \underbrace{I_2^0\left(xm^2 - x(1-x)q^2\right)}_{\approx\frac{1}{\varepsilon}\frac{1}{(4\pi)^2}\text{, see pg. 34}}\left\{(2-d)(1-x)\not q + md\right\}$$

$$\approx i\frac{g^2}{16\pi^2}\frac{N_c^2-1}{2N_c}\frac{1}{\varepsilon}\left\{\boxed{1}\not q - \boxed{4}m + O(\varepsilon)\right\}$$

$\boxed{1} \to \xi$, $\boxed{4} \to (3+\xi)$ for general value of ξ

- 

$$= \int\frac{d^dk}{(2\pi)^d}(ig)^3 T^bT^aT^b\frac{\gamma^\nu i(\not k+\not q_2+m)\gamma^\mu i(\not k+\not q_1+m)\gamma_\nu}{((k+q_2)^2-m^2)\quad((k+q_1)^2-m^2)}\frac{(-i)}{k^2}$$

$\gamma^{\mu\nu s}{}_\mu = (2g^{\mu\nu}-\gamma^\nu\gamma^\mu)\gamma^s{}_\mu = 2\underbrace{\gamma^{s\nu}}_{=2g^{s\nu}-\gamma^{\nu s}} - \gamma^\nu\underbrace{\gamma^{\mu s}{}_\mu}_{=(2-d)\gamma^s} = 4g^{\nu s} - (4-d)\gamma^{\nu s}$

$\gamma^{\mu\nu s\sigma}{}_\mu = (2g^{\mu\nu}-\gamma^\nu\gamma^\mu)\gamma^{s\sigma}{}_\mu = 2\gamma^{s\sigma}\gamma^\nu - \gamma^\nu\left(4g^{s\sigma} - (4-d)\gamma^{s\sigma}\right) = -2\gamma^{\sigma s\nu} + (4-d)\gamma^{\nu s\sigma}$

$T^b_{ij}T^a_{jk}T^b_{kl} = T^a_{jk}\frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) = \frac{1}{2}\delta_{il}T^a_{jj} - \frac{1}{2N}T^a_{il} = -\frac{1}{2N}T^a$ (with $T^a_{jj} = 0$)

$$= g^3\left(-\frac{1}{2N}T^a\right)\int\frac{d^dk}{(2\pi)^d}\frac{N^\mu}{((k+q_2)^2-m^2)((k+q_1)^2-m^2)k^2}$$

$N^\mu = m^2(2-d)\gamma^\mu + m(k+q_2)_\nu\left(4g^{\nu\mu} - (4-d)\gamma^{\nu\mu}\right) + m(k+q_1)_\nu\left(4g^{\mu\nu} - (4-d)\gamma^{\mu\nu}\right)$
$+ (k+q_2)_\nu(k+q_1)_\sigma\left[-2\gamma^{\sigma\mu\nu} + (4-d)\gamma^{\nu\mu\sigma}\right]$

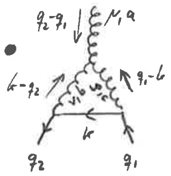
Superficial degree of divergence of this integral: $\frac{k^4}{k^6} \to$ log!

$\Rightarrow$ look at $k \gg \{q_1, q_2, m\}$ only, to extract this leading log div

$$\sim g^3\left(-\frac{1}{2N}T^a\right)\int\frac{d^dk}{(2\pi)^d}\frac{\left[-2\gamma^{\sigma\mu\nu} + (4-d)\gamma^{\nu\mu\sigma}\right]k_\nu k_\sigma}{(k^2)^3} + \text{const.}$$

numerator $\to [\dots]\frac{g_{\nu\sigma}k^2}{d} = \frac{k^2}{d}\left[-2\gamma^{\sigma\mu}{}_\sigma + (4-d)\gamma^{\nu\mu}{}_\nu\right] \overset{\text{(pg. 34)}}{=} \frac{k^2}{d}(2-d)^2\gamma^\mu$

$$= g^3\left(-\frac{1}{2N}T^a\right)\frac{(2-d)^2}{d}\gamma^\mu\underbrace{\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^2}}_{= iI_2^0(0)\ \text{(after Wick rot.)}}$$

- 

$$= \int \frac{d^dk}{(2\pi)^d} (ig)^2 T^b T^c \frac{\gamma_\nu \, i(\not k + m)\, \gamma_\rho}{(k^2-m^2)} \frac{(-i)}{(k-q_2)^2} \frac{(-i)}{(k-q_1)^2} \, g f^{abc} \,*$$

$$* \left[(2q_2 - q_1 - k)^\rho g^{\mu\nu} + (2k - q_1 - q_2)^\mu g^{\nu\rho} + (2q_1 - q_2 - k)^\nu g^{\rho\mu} \right]$$

$$\Big\lceil\; T^b T^c f^{abc} = \tfrac{1}{2}[T^b, T^c] f^{abc} = \tfrac{i}{2} f^{bcd} T^d f^{abc} = \tfrac{i}{2} N_c T^a$$
$$\Big\lfloor \qquad f^{bcd} f^{abc} = N_c \delta^{ad} \text{ (see pg. 33)}$$

again, extract only leading log div, $k \gg \{q_1, q_2, m\}$

$$\sim -g^3 \frac{N_c}{2} T^a \int \frac{d^dk}{(2\pi)^d} \frac{\gamma_\nu \not k \gamma_\rho \left[-k^\rho g^{\mu\nu} + 2k^\mu g^{\nu\rho} - k^\nu g^{\rho\mu}\right]}{(k^2)^3} + \text{const.}$$

replace $k^\mu k^\nu \to \frac{g^{\mu\nu} k^2}{d}$

numerator $\to \gamma_\nu \gamma_\sigma \gamma_\rho \left[-g^{\sigma\rho} g^{\mu\nu} + 2 g^{\sigma\mu} g^{\nu\rho} - g^{\sigma\nu} g^{\rho\mu}\right]$

$$= -\gamma^\mu \gamma^\rho{}_\rho + 2\gamma^\rho \gamma^\mu{}_\rho - \gamma_\nu \gamma^\nu \gamma^\mu \overset{(pg. 34)}{=} -d\gamma^\mu + 2(2-d)\gamma^\mu - d\gamma^\mu = 4(1-d)\gamma^\mu$$

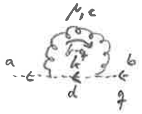
$$= -g^3 \, 2N_c T^a \frac{1-d}{d} \gamma^\mu \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^2}$$
$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^2} = i I_2^0(0) \quad \text{(after Wick rot.)}$$

- sum of last two diagrams

$$= -ig^3 T^a \gamma^\mu I_2^0(0) \left\{ \frac{1}{2N} \frac{(2-d)^2}{d} + 2N \frac{1-d}{d} \right\} + \text{const.}$$

$$\overset{d=4-2\varepsilon}{\approx} -ig T^a \gamma^\mu \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \left\{ \boxed{1} \frac{1}{2N} - \frac{3}{4} \boxed{2} N + \mathcal{O}(\varepsilon) \right\}$$

$\boxed{1} \to \xi$, $\boxed{2} \to (\xi + 1)$ for general value of ξ

- 

$$= \int \frac{d^dk}{(2\pi)^d} \left(-g f^{acd} q^\mu\right)\left(-g f^{dcb} k_\mu\right) \frac{i}{k^2} \frac{-i}{(k-q)^2}$$

$$= -g^2 f^{acd} f^{cdb} \int \frac{d^dk}{(2\pi)^d} \frac{q \cdot k}{k^2 (k-q)^2}$$

$f^{acd} f^{cdb} = N_c \delta^{ab}$ (pg. 33)

denominator invariant under $k \to q - k$

write numerator $k = \frac{1}{2}k + \frac{1}{2}k \to \frac{1}{2}k + \frac{1}{2}(q-k) = \frac{1}{2}q$

$$= -g^2 N_c \delta^{ab} \frac{q^2}{2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 (k-q)^2}$$

extract leading log div, $k \gg q$; Wick rot; use tadpole

$$\sim -ig^2 N_c \delta^{ab} \frac{q^2}{2} I_2^0(0) + \text{const.}$$

$$\overset{d=4-2\varepsilon}{\approx} -i \frac{g^2}{16\pi^2} \delta^{ab} q^2 \frac{1}{\varepsilon} \left\{ \boxed{2} \frac{N_c}{4} + \mathcal{O}(\varepsilon) \right\}$$

$\boxed{2} \to (3 - \xi)$ for general value of ξ

3.3 one-loop counterterms in QCD

→ now, renormalize the theory.

use freedom of redefining fields, parameters/couplings

schematically: $\phi_B = \sqrt{Z_\phi}\, \phi_R$, $\phi \in \{\psi, A, c\}$ — $B \equiv$ "bare"

$\lambda_B = Z_\lambda \lambda_R$, $\lambda \in \{m, g, \xi\}$ — $R \equiv$ "renormalized"

where the multiplicative renormalization factors Z_i depend on the renormalized parameters (and the dimension d), and are taken to be dimension<u>less</u>, $Z_i = 1 + \delta Z_i$, $\delta Z_i \sim g^2$ (see below)

- recall (pg. 28)

$$\mathcal{L}_B = \bar\psi_B (i\gamma^\mu D_\mu - m_B)\psi_B - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar c^a (-\partial^\mu D^{ac}_\mu) c^c$$

with $D_\mu = \partial_\mu - ig_B A^a_\mu T^a$, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, $D^{ac}_\mu = \partial_\mu \delta^{ac} + g f^{abc} A^b_\mu$

(in this line, all ψ, A, c, m, g, ξ should have an index $\underline{B}$)

$$= \overbrace{\bar\psi i\not\partial\psi}^{Z_\psi} - \overbrace{m\bar\psi\psi}^{Z_m Z_\psi} + \overbrace{g\bar\psi\gamma^\mu A^a_\mu T^a \psi}^{Z_g Z_\psi Z_A^{1/2}}$$

$$- \overbrace{\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2}^{Z_A} - \overbrace{\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2}^{Z_A Z_\xi^{-1}} - \overbrace{\frac{1}{2} g f^{abc} A^b_\mu A^c_\nu (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})}^{Z_g Z_A^{3/2}}$$

$$- \overbrace{\frac{1}{4} g^2 f^{abc} A^b_\mu A^c_\nu f^{ade} A^{d\mu} A^{e\nu}}^{Z_g^2 Z_A^2} - \overbrace{\bar c^a \partial^\mu \partial_\mu c^a}^{Z_c} - \overbrace{g f^{abc} \bar c^a \partial^\mu A^b_\mu c^c}^{Z_g Z_c Z_A^{1/2}}$$

(here, without Z's: index B; <u>with</u> Z's: index R for all ψ, A, c, m, g, ξ)

$$= \mathcal{L}_R + \mathcal{L}^{c.t.} \quad \leftarrow \text{counterterms}$$

$$\mathcal{L}^{c.t.} = (Z_\psi - 1)\bar\psi\left(i\not\partial - \frac{Z_m Z_\psi - 1}{Z_\psi - 1} m\right)\psi + \left(Z_g Z_\psi Z_A^{1/2} - 1\right) [\text{vertex}]$$

$$- (Z_A - 1)\left[\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \frac{Z_A Z_\xi^{-1} - 1}{Z_A - 1} \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2\right] + \left(Z_g Z_A^{3/2} - 1\right) [\text{3-gluon vertex}]$$

$$+ \left(Z_g^2 Z_A^2 - 1\right) [\text{4-gluon vertex}] - (Z_c - 1)\bar c \partial^\mu \partial_\mu c + \left(Z_g Z_c Z_A^{1/2} - 1\right) [\text{ghost vertex}]$$

(now, all indices are $\underline{R}$, and omitted)

3.3 one-loop counterterms in QCD

→ now, renormalize the theory.

use freedom of redefining fields, parameters/couplings

schematically: $\phi_B = \sqrt{Z_\phi}\,\phi_R \;,\quad \phi \in \{\psi, A, c\}$ B = "bare"

$\lambda_B = Z_\lambda \lambda_R \;,\quad \lambda \in \{m, g, \xi\}$ R = "renormalized"

where the multiplicative renormalization factors Z_i depend on the renormalized parameters (and the dimension d), and are taken to be dimension**less**, $Z_i = 1 + \delta Z_i$, $\delta Z_i \sim g^2$ (see below)

• recall (pg. 28)

$$\mathcal{L}_B = \bar\psi_B (i\gamma^\mu D_\mu - m_B)\psi_B - \frac14 F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar c^a(-\partial^\mu D^{ac}_\mu)c^c$$

with $D_\mu = \partial_\mu - i g A^a_\mu T^a$, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, $D^{ac}_\mu = \partial_\mu \delta^{ac} + g f^{abc} A^b_\mu$

(in this line, all ψ, A, c, m, g, ξ should have an index $\underline{B}$)

$$= \overbrace{\bar\psi i\partial\!\!\!/ \psi}^{Z_\psi} - \overbrace{m\bar\psi\psi}^{Z_m Z_\psi} + \overbrace{g\bar\psi\gamma^\mu A^a_\mu T^a\psi}^{Z_g Z_\psi Z_A^{1/2}}$$

$$- \overbrace{\frac14(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2}^{Z_A} - \overbrace{\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2}^{Z_A Z_\xi^{-1}} - \overbrace{\frac12 g f^{abc} A^b_\mu A^c_\nu (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})}^{Z_g Z_A^{3/2}}$$

$$- \overbrace{\frac14 g^2 f^{abc} A^b_\mu A^c_\nu f^{ade} A^{d\mu} A^{e\nu}}^{Z_g^2 Z_A^2} - \overbrace{\bar c^a \partial^\mu \partial_\mu c^a}^{Z_c} - \overbrace{g f^{abc}\bar c^a \partial^\mu A^b_\mu c^c}^{Z_g Z_c Z_A^{1/2}}$$

(here, without Z's: index B; **with** Z's: index R for all ψ, A, c, m, g, ξ)

$= \mathcal{L}_R + \mathcal{L}^{c.t.}$ ← counterterms

$$\mathcal{L}^{c.t.} = (Z_\psi - 1)\bar\psi\Big(i\partial\!\!\!/ - \frac{Z_m Z_\psi - 1}{Z_\psi - 1} m\Big)\psi + (Z_g Z_\psi Z_A^{1/2} - 1)$$

$$-(Z_A - 1)\Big[\frac14(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \frac{Z_A Z_\xi^{-1} - 1}{Z_A - 1}\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2\Big] + (Z_g Z_A^{3/2} - 1)$$

$$+ (Z_g^2 Z_A^2 - 1) - (Z_c - 1)\bar c\,\partial^\mu\partial_\mu c + (Z_g Z_c Z_A^{1/2} - 1)$$

(now, all indices are $\underline{R}$, and omitted)

→ treat $\mathcal{L}^{c.t.}$ as additional interactions.

→ get additional Feynman rules

vertices are easy (have the same form as before),

$= (Z_g Z_\psi Z_A^{1/2} - 1)$, $= (Z_g Z_A^{3/2} - 1)$

$= (Z_g^2 Z_A^2 - 1)$, $= (Z_g Z_c Z_A^{1/2} - 1)$

and there are also "two-point-vertices" now:

$$= i\big[(Z_\psi - 1)q\!\!\!/ - (Z_m Z_\psi - 1)m\big]$$

$$= -i\delta^{ab}\Big[(Z_A - 1)(g^{\mu\nu}q^2 - q^\mu q^\nu) + \underbrace{(Z_A Z_\xi^{-1} - 1)}_{\doteq 0,\ \text{see pg. 39}}\frac{1}{\xi}q^\mu q^\nu\Big]$$

$$= i\delta^{ab}(Z_c - 1)q^2$$

• from our explicit results for 1-loop divergences in §3.1, §3.2, we can now fix the yet-unknown constants Z!

• finite $\overset{!}{=}$ + + $\mathcal{O}(g^4)$

(pg. 35) $\overset{!}{=}\; i\frac{g^2}{16\pi^2}\frac{1}{\varepsilon}\frac{N_c^2-1}{2N_c}\big(q\!\!\!/\,\xi - (\xi+3)m\big) + \mathcal{O}(\varepsilon^0) + i\big[(Z_\psi - 1)q\!\!\!/ - (Z_m Z_\psi - 1)m\big] + \mathcal{O}(g^4)$

$$\Rightarrow\; \underline{\underline{Z_\psi \overset{!}{=} 1 - \frac{g^2}{16\pi^2}\Big(\frac{1}{\varepsilon}\frac{N_c^2-1}{2N_c}\xi + \mathcal{O}(\varepsilon^0)\Big)}} + \mathcal{O}(g^4)$$

what one puts here is a matter of choice.

often used: "minimal subtraction" (MS) scheme: put 0 here

many other schemes possible, e.g. modified MS ($\hat{=}\ \overline{MS}$), see below

$$\Rightarrow\; Z_m Z_\psi \overset{!}{=} 1 - \frac{g^2}{16\pi^2}\frac{1}{\varepsilon}\frac{N_c^2-1}{2N_c}(3+\xi) + \mathcal{O}(g^4)$$

$$\Leftrightarrow\; \underline{\underline{Z_m = 1 - \frac{g^2}{16\pi^2}\frac{1}{\varepsilon}\frac{N_c^2-1}{2N_c}\,3 + \mathcal{O}(g^4)}} \quad \text{in MS scheme}$$

note: ξ-independent

- finite $\overset{!}{=}$  $+ \mathcal{O}(g^4)$

(pg. 34) $= i \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \delta^{ab} (g^{\mu\nu} q^2 - q^\mu q^\nu)\left(\left(\frac{13}{6} - \frac{\xi}{2}\right)N_c - \frac{2}{3}N_f\right) + \mathcal{O}(\varepsilon^0)$

$- i \delta^{ab} \left[(Z_A - 1)(g^{\mu\nu} q^2 - q^\mu q^\nu) + (Z_A Z_\xi^{-1} - 1) \frac{1}{\xi} q^\mu q^\nu \right] + \mathcal{O}(g^4)$

$\Rightarrow \underline{Z_A \overset{!}{=} 1 + \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \left(\left(\frac{13}{6} - \frac{\xi}{2}\right)N_c - \frac{2}{3}N_f\right)} + \mathcal{O}(g^4)$

$\Rightarrow Z_A Z_\xi^{-1} \overset{!}{=} 1 + 0 \cdot g^2 + \mathcal{O}(g^4)$

$\Leftrightarrow \underline{Z_\xi = Z_A} + \mathcal{O}(g^4)$

→ note actually, one can show that $Z_\xi = Z_A$ exactly, to all orders of g^2, due to gauge invariance: the BRST symmetry gives rise to the so-called Ward/Takahashi/Slavnov/Taylor-identities, one of which guarantees that the longitudinal ($q^\mu q^\nu$ piece) part of the gluon propagator does not get radiative corrections, $q^\mu \Pi^{ab}_{\mu\nu}(q) = 0$

- finite $\overset{!}{=}$  $+ \mathcal{O}(g^5)$

(pg. 36) $= -ig T^a \gamma^\mu \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \frac{\xi - 3N_c^2 \frac{\xi+1}{2}}{2N_c} + \mathcal{O}(\varepsilon^0) + (Z_g Z_\psi Z_A^{\frac{1}{2}} - 1)\, ig T^a \gamma^\mu + \mathcal{O}(g^5)$

$\Rightarrow Z_g Z_\psi Z_A^{\frac{1}{2}} \overset{!}{=} 1 + \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \frac{\xi - 3N_c^2 \frac{\xi+1}{2}}{2N_c} + \mathcal{O}(g^4)$

$\Leftrightarrow \underline{Z_g = 1 - \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \left(\frac{11}{6}N_c - \frac{1}{3}N_f\right)} + \mathcal{O}(g^4)$

note: ξ-independent

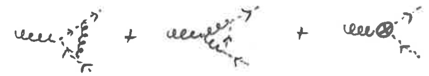
- finite $\overset{!}{=}$ 

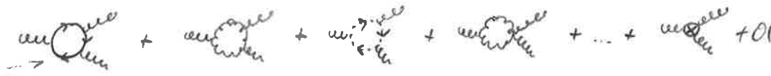
(pg. 36) $= -i \delta^{ab} q^2 \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} (3-\xi) \frac{N_c}{4} + \mathcal{O}(\varepsilon^0) + i \delta^{ab} q^2 (Z_c - 1)$

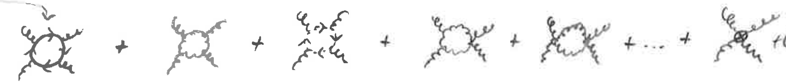
$\Rightarrow \underline{Z_c \overset{!}{=} 1 - \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \frac{N_c}{4} (\xi - 3)} + \mathcal{O}(g^4)$

→ get finite (one-loop) results after fixing Z's as above!

→ could have computed Z_g also from

finite $\overset{?!}{=}$  $+ \mathcal{O}(g^5)$

finite $\overset{?!}{=}$  $+ \mathcal{O}(g^5)$

finite $\overset{?!}{=}$  $+ \mathcal{O}(g^6)$

(check ?!)

- current status (as of May 2013) of Z's:

$Z \sim 1 + g^2 + g^4 + g^6 + g^8 + \mathcal{O}(g^{10})$

(g^2: 1-loop (see above); g^8: 4-loop ✓; $\mathcal{O}(g^{10})$: 5-loop ??)

⇒ Nobel 2004 [Gross/Wilczek, Phys. Rev. Letters 30 (1973) 1343; Politzer, Phys. Rev. Letters 30 (1973) 1346]

4-loop:

Ritbergen/Vermaseren/Larin,	Phys. Lett. B	400 (1997) 379	: Z_g
Chetyrkin,	Phys. Lett. B	404 (1997) 161	} : Z_m
Vermaseren/Larin/Ritbergen,	Phys. Lett. B	405 (1997) 327	} : Z_m
Chetyrkin/Retey,	Nucl. Phys. B	583 (2000) 3	: Z_ψ
Chetyrkin,	Nucl. Phys. B	710 (2005) 499	} : Z_c, Z_A
Czakon,	Nucl. Phys. B	710 (2005) 485	} : Z_c, Z_A

3.4 QCD Beta-function, running coupling

⤳ recall that we had regularized QCD dimensionally: $d = 4 - 2\varepsilon$.

dimensional analysis: $e^{i\int d^dx\, \mathcal{L}} \Rightarrow [\mathcal{L}] \overset{!}{=} d$ (mass-dim: $[m] \equiv 1$)

$[\partial] = 1$, $[x] = -1$

$\mathcal{L} \ni m\bar\psi\psi \Rightarrow [\psi] = \frac{d-1}{2}$

$\mathcal{L} \ni (\partial^\mu A_\mu)^2 \Rightarrow [A_\mu] = \frac{d-2}{2}$

$\mathcal{L} \ni \bar c \partial^2 c \Rightarrow [c] = \frac{d-2}{2}$

$D_\mu \sim \partial_\mu + g A_\mu \Rightarrow [g] = \frac{4-d}{2}$

→ we had defined $g_B^2 = Z_g^2 g_R^2$, $Z_g = 1 + \ldots$: $[Z_g] = 0$

dim: $(4-d) = (0) + (4-d)$

it is convenient to use integer-dimensional renormalized couplings already in d dimensions

$$g_B^2 = Z_g^2 \hat{g}_R^2 \mu^{4-d} \quad , \quad [\hat{g}_R^2] = 0$$

(μ: arbitrary mass-scale, $[\mu] = 1$)

⇒ in fact, in all our expressions above, $g^2 \to \hat{g}_R^2$ was understood

(($Z \sim 1 + g_B^2 \int \frac{d^d k}{k^4} = 1 + Z_g^2 \hat{g}_R^2 \mu^{4-d} \int \frac{d^d k}{k^4}$, Z is fct of dim-less parameters!))

(where $g_B^2 \int \frac{d^dk}{k^4}$ is dim-less; $Z_g^2 \approx$, $\hat{g}_R^2 \approx$, $\mu^{4-d}\int \frac{d^dk}{k^4}$ is dim-less)

- QCD Beta-function

immediate consequence : $\hat{g}_R^2$ is a function of μ

$\mu^2 \partial_{\mu^2}$ | $g_B^2 = Z_g^2 \hat{g}_R^2 \mu^{4-d}$ $\qquad \hat{g}_R^2 \equiv g^2$ from here on

$$\Rightarrow \quad 0 = (\mu^2 \partial_{\mu^2} Z_g^2) g^2 \mu^{4-d} + Z_g^2 (\mu^2 \partial_{\mu^2} g^2) \mu^{4-d} + Z_g^2 g^2 \frac{4-d}{2} \mu^{4-d}$$

$$(\mu^2 \partial_{\mu^2} Z_g^2) = (\mu^2 \partial_{\mu^2} g^2)(\partial_{g^2} Z_g^2) + (\mu^2 \partial_{\mu^2} \xi)(\partial_\xi Z_g^2)^0$$

$$\Leftrightarrow \quad \boxed{\beta(g^2) \equiv \mu^2 \partial_{\mu^2} g^2 = \frac{\frac{d-4}{2} g^2}{1 + g^2 (\partial_{g^2} Z_g^2) Z_g^{-2}}}$$

in $d = 4 - 2\varepsilon$ dimensions, (pg. 39),

$$Z_g = 1 - \frac{g^2}{\varepsilon} \frac{a}{2} + O(g^4)$$

where $a \equiv \frac{1}{16\pi^2} \left(\frac{11}{3} N_c - \frac{2}{3} N_f \right) \equiv \frac{\beta_0}{16\pi^2}$

$$\beta(g^2) = \frac{-\varepsilon g^2}{1 - \frac{g^2}{\varepsilon} a + O(g^4)} = -\varepsilon g^2 \left(1 + \frac{g^2}{\varepsilon} a + O(g^4) \right)$$

$$\to -a g^4 + O(g^6) \quad \text{for } \varepsilon \to 0$$

→ more convenient to use $h \equiv \frac{\hat{g}_R^2}{16\pi^2}$:

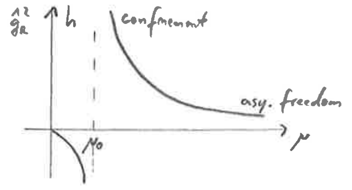
$$\beta(h) = \mu^2 \partial_{\mu^2} h = \frac{\frac{d-4}{2} h}{1 + h (\partial_h Z_g^2) Z_g^{-2}} \approx -\beta_0 h^2 - \beta_1 h^3 - \beta_2 h^4 - \beta_3 h^5 + O(h^6)$$

($\beta_0 h^2$: 1-loop; $\beta_3 h^5$: 4-loop; $O(h^6)$: unknown)

- running coupling

solve the differential equation $\mu^2 \partial_{\mu^2} h = -\beta_0 h^2$

soln: $h(\mu) = \frac{1}{\text{const.} + \beta_0 \ln(\mu^2)} = \frac{1}{\beta_0 \ln(\mu^2/\mu_0^2)}$



$\beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f > 0$ for $N_f < \frac{11}{2} N_c$

(QCD: $6 < \frac{33}{2}$ ✓)

- higher-order systematics of renormalization factors:

in general (schematically), $Z_i \sim 1 + h \frac{1}{\varepsilon} + h^2 \left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \right) + h^3 \left(\frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \right) + \ldots$

furthermore, existence of the limit $\varepsilon \to 0$ in β-fct (or, analogously, in "anomalous dimensions" $\gamma_i \equiv -\mu^2 \partial_{\mu^2} \ln Z_i$) fixes the coefficients of poles $\frac{1}{\varepsilon^{n>1}}$ in term of those of $\frac{1}{\varepsilon}$.

(in literature, often find $\gamma(h) = -\gamma_0 h - \gamma_1 h^2 - \gamma_2 h^3 \ldots$)

e.g. $Z_g = 1 + h \frac{z_{11}}{\varepsilon} + h^2 \left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon} \right) + h^3 \left(\frac{z_{33}}{\varepsilon^3} + \frac{z_{32}}{\varepsilon^2} + \frac{z_{31}}{\varepsilon} \right) + O(h^4)$

$$\Rightarrow \beta(h) = h(-\varepsilon) + h^2 (2 z_{11}) + h^3 \left(\frac{4 z_{22} - 6 z_{11}^2}{\varepsilon} + 4 z_{21} \right) + h^4 \left(\frac{2(3 z_{33} - 11 z_{22} z_{11} + 9 z_{11}^3)}{\varepsilon^2} + \frac{2(3 z_{32} - 11 z_{21} z_{11})}{\varepsilon} + 6 z_{31} \right) + O(h^5)$$

$$\stackrel{!}{=} -\beta_0 h^2 - \beta_1 h^3 - \beta_2 h^4 + \ldots$$

$\Rightarrow z_{11} = -\frac{\beta_0}{2}$, $z_{21} = -\frac{\beta_1}{4}$, $z_{31} = -\frac{\beta_2}{6}$, ...

and $z_{22} \stackrel{!}{=} \frac{3}{2} z_{11}^2 = \frac{3}{8} \beta_0^2$; $z_{32} \stackrel{!}{=} \frac{11}{3} z_{21} z_{11} = \frac{11}{24} \beta_1 \beta_0$; $z_{33} \stackrel{!}{=} -\frac{5}{16} \beta_0^3$; ...

such that $Z_g = 1 + h \left(-\frac{\beta_0}{2\varepsilon} \right) + h^2 \left(\frac{3\beta_0^2}{8\varepsilon^2} - \frac{\beta_1}{4\varepsilon} \right) + h^3 \left(-\frac{5\beta_0^3}{16\varepsilon^3} + \frac{11 \beta_1 \beta_0}{24 \varepsilon^2} - \frac{\beta_2}{6\varepsilon} \right) + \ldots$

i.e., all information is already encoded in the $\frac{1}{\varepsilon}$ poles ✓

- one often needs ε-expansions of Gamma-functions .

(recall pg 31 : $I_n^a(\Delta) \sim \frac{\Gamma(a + d/2) \Gamma(n - a - d/2)}{\Gamma(d/2)}$)

Mathematica : $\Gamma(1-\varepsilon) \sim \sum \varepsilon^{n+1}$ PolyGamma[n,1] $= \partial_n^2 \ln(\Gamma(n))$

more useful : $\Gamma(1-\varepsilon) = e^{\gamma_E \varepsilon} e^{\sum_{n=2}^{\infty} \frac{\varepsilon^n}{n} \zeta(n)}$

(Riemann Zeta $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$ $(n > 1)$)

Euler-Mascheroni $\gamma_E = -\Gamma'(1) = -\int_0^\infty dx \, e^{-x} \ln(x) = \lim_{N\to\infty} \left(\sum_{k=1}^{N} \frac{1}{k} - \ln(N) \right) \approx 0.5772\ldots$

4. QCD in e^+e^--annihilation

→ want to compare basic properties of (perturbative) QCD with experiment.

→ consider $e^+e^- \to$ hadrons

- total cross section $\sigma \sim \frac{\alpha_{em}^2}{s}$

  calculate α_s corrections, $\sigma \to \sigma \cdot (1+\alpha_s)$

  renormalization scheme dependence enters at α_s^2

  inclusive cross section: $(1+\alpha_s+\alpha_s^2+\alpha_s^3)$ known,

  high precision QCD result!

  non-perturbative corrections expected to be small

  ⇒ used as one of the most precise measurements of α_s.

- QCD predicts "jet" structure for final-state hadrons

  define jet cross sections

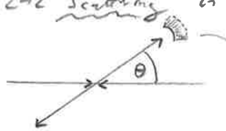
  calculate them, compare with experiment

  ⇒ can also be used to measure α_s,

  and to test/"see" triple-gluon-vertex.

4.1 $e^+e^- \to$ hadrons at leading order

remember: 2→2 scattering in CMS (center of mass system)



detector measures in

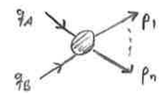
$d\Omega = \sin\theta \, d\theta \, d\varphi$

spherical coords.: $\int d\Omega = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi = \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\varphi = 4\pi$

total cross section $\sigma = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)$ ⟵ differential cross section

(see also PDG booklet §43: kinematics)

remember: Fermi's golden rule



$$\sigma_{2\to n} = \frac{1}{F} \int d\Phi_n \, |\mathcal{M}|^2$$

↖ amplitude, e.g. from Feynman diagrams

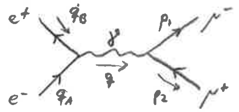
phase space integral $\left(\prod_{i=1}^{n} \int \frac{d^3 p_i}{(2\pi)^3 2E_{p_i}}\right)(2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^{n} p_i - q_A - q_B\right)$

where $F = 4\sqrt{(q_A \cdot q_B)^2 - m_A^2 m_B^2}$

$= 2\sqrt{(s-m_A^2-m_B^2)^2 - 4m_A^2 m_B^2}$, $s = (q_A+q_B)^2$ $(= (E_A+E_B)^2$ in CMS$)$

$= 4(E_A+E_B)|\vec{q}_A|$ in CMS, $q = (E, \vec{q})$

remember: amplitude $e^+e^- \to \mu^+\mu^-$ [see, e.g., Peskin/Schroeder §5.1]



$$= \bar{v}(q_B,s_B)(-ie\gamma^\mu)u(q_A,s_A)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}(p_1,s_1)(-ie\gamma^\nu)v(p_2,s_2)$$

↖ spin: +/− ↖ Dirac spinor for incoming particle

start with unpolarized beam of e^+, e^-

⇝ average over spin states s_A, s_B: $\frac{1}{4}\sum_{s_A=+,-}\sum_{s_B=+,-}$

detector does not measure spin of final state

⇝ sum over spins s_1, s_2

$$\frac{1}{4}\sum_{s_i=+/-} |\mathcal{M}|^2 = \frac{1}{4}\sum_s \left| \text{(diagram)} \right|^2 = \frac{1}{4}\sum_s \left| \frac{e^2}{q^2} \bar{v}_B \gamma u_A \, \bar{u}_1 \gamma v_2 \right|^2$$

$$= \frac{e^4}{4q^4} \sum_s \bar{v}_B \gamma u_A \bar{u}_A \gamma v_B \; \bar{u}_1 \gamma v_2 \bar{v}_2 \gamma u_1$$

do spin sums via completeness rels

$\sum_s u_A \bar{u}_A = \not{q}_A + m_e$, $\sum_s v_2 \bar{v}_2 = \not{p}_2 - m_\mu$

$$= \frac{e^4}{4q^4} \, tr\left((\not{q}_B - m_e)\gamma^\mu(\not{q}_A + m_e)\gamma^\nu\right) tr\left((\not{p}_1 + m_\mu)\gamma_\mu(\not{p}_2 - m_\mu)\gamma_\nu\right)$$

$$= \frac{e^4}{4q^4} 4\left[q_B^\mu q_A^\nu + q_A^\mu q_B^\nu - g^{\mu\nu}(q_A \cdot q_B + m_e^2)\right] 4\left[p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - g_{\mu\nu}(p_1 \cdot p_2 + m_\mu^2)\right]$$

$$= \frac{8e^4}{q^4}\left(q_A \cdot p_1 \, q_B \cdot p_2 + q_A \cdot p_2 \, q_B \cdot p_1 + m_e^2 \, p_1 \cdot p_2 + m_\mu^2 \, q_A \cdot q_B + 2m_e^2 m_\mu^2\right)$$

CMS: $\vec{q}_B = -\vec{q}_A$, $\vec{p}_2 = -\vec{p}_1$; E-cons.: $2E = E_1 + E_2 = 2E_1$

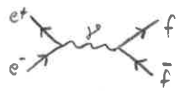
⇒ $E_A = E_B = E_1 = E_2 = E$, $\vec{q}_A^{\,2} + m_e^2 = \vec{p}_1^{\,2} + m_\mu^2$

$$= \frac{e^4}{E^4}\left(E^4 + m_e^2 E^2 + m_\mu^2 E^2 + (E^2 - m_e^2)(E^2 - m_\mu^2)\cos^2\theta\right)$$

$\theta = \sphericalangle(\vec{q}_A, \vec{p}_1)$

45

generalization: amplitude $e^+e^- \to f\bar{f}$



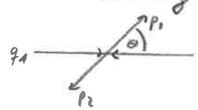
where $f \neq e$, $f \in \left\{ \begin{matrix} \nu_e, \nu_\mu, \nu_\tau, u, c, t \\ \mu, \tau, d, s, b \end{matrix} \right\}$

use charge $Q_f = \left\{ \begin{matrix} 0, 0, 0, +\frac{2}{3}, +\frac{2}{3}, +\frac{2}{3} \\ -1, -1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3} \end{matrix} \right\}$

$$\Rightarrow \langle |\mathcal{M}|^2 \rangle \equiv \sum_{\text{colors}} \sum_f \frac{1}{4} \sum_s |\mathcal{M}|^2 = \sum_{\substack{\text{colors} \\ f}} \frac{Q_f^2 e^4}{E^4} \left(E^4 + m_e^2 E^2 + m_f^2 E^2 + (E^2 - m_e^2)(E^2 - m_f^2) \cos^2\theta \right)$$

remember: phase space integration for $2 \to 2$ scattering

$A + B \to 1 + 2$ in CMS:



$$d\sigma_{2\to 2} = \frac{1}{F} d\Phi_2 |\mathcal{M}|^2 \quad , \text{ use } \delta^{(3)} \text{ for } \vec{p}_2\text{-integral}$$

$$= \frac{1}{(8\pi)^2} \frac{|\mathcal{M}|^2}{|\vec{q}_A|(E_A + E_B)} d^3\vec{p}_1 \frac{\delta\left(\sqrt{m_1^2 + \vec{p}_1^{\,2}} + \sqrt{m_2^2 + \vec{p}_2^{\,2}} - E_A - E_B\right)}{\sqrt{m_1^2 + \vec{p}_1^{\,2}}\sqrt{m_2^2 + \vec{p}_1^{\,2}}}$$

use that $|\mathcal{M}|^2 = |\mathcal{M}|^2(\vec{q}_A, \vec{q}_B, \vec{p}_1, \vec{p}_2) = |\mathcal{M}|^2(|\vec{q}_A|, |\vec{p}_1|, \cos\theta)$

spherical coords, $\rho \equiv |\vec{p}_1|$, $d^3\vec{p}_1 = \rho^2 d\rho \underbrace{\sin\theta \, d\theta \, d\varphi}_{= d\Omega}$

do $\int d\rho$ using δ fct

$$\frac{d\sigma_{2\to 2}}{d\Omega} = \frac{1}{(8\pi)^2} \frac{|\vec{p}_1|}{|\vec{q}_A|} \frac{|\mathcal{M}|^2(|\vec{q}_A|, |\vec{p}_1|, \cos\theta)}{(E_A + E_B)^2} \cdot \Theta(\sqrt{s} - m_1 - m_2)$$

total cross section

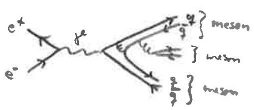
$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int_{-1}^{1} d(\cos\theta) \frac{2\pi}{(8\pi)^2} \frac{\sqrt{E^2 - m_f^2}}{\sqrt{E^2 - m_e^2}} \frac{\langle |\mathcal{M}|^2 \rangle}{(2E)^2} \Theta(E - m_f)$$

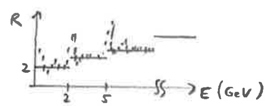
$$= \sum_{\substack{\text{colors} \\ f}} \frac{\pi}{3} \frac{Q_f^2 \alpha_{em}^2}{E^2} \frac{\sqrt{1 - m_f^2/E^2}}{\sqrt{1 - m_e^2/E^2}} \left(1 + \frac{m_e^2}{2E^2}\right)\left(1 + \frac{m_f^2}{2E^2}\right) \Theta(E - m_f)$$

$$\approx \frac{\pi}{3} \frac{\alpha_{em}^2}{E^2} \cdot \sum_{\substack{\text{colors} \\ f}} Q_f^2 \, \Theta(E - m_f), \text{ for } E \gg m_f \gg m_e$$

$$\Rightarrow R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \frac{\sum_q \sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} \approx N_c \sum_q Q_q^2 \, \Theta(E - m_q) \quad \left(\leftarrow \sum_{\text{colors}}\right)$$

$$\approx N_c \left\{ \left(\tfrac{2}{3}\right)_u^2 + \left(-\tfrac{1}{3}\right)_d^2 + \left(-\tfrac{1}{3}\right)_s^2 + \left(\tfrac{2}{3}\right)_c^2 + \left(-\tfrac{1}{3}\right)_b^2 + \left(\tfrac{2}{3}\right)_t^2 \right\} = N_c \left\{ \tfrac{4}{9}, \tfrac{5}{9}, \underset{uds}{\tfrac{2}{3}}, \underset{+c}{\tfrac{10}{9}}, \underset{+b}{\tfrac{11}{9}}, \underset{+t}{\tfrac{5}{3}} \right\}$$





$\Rightarrow N_c \stackrel{!}{=} 3$

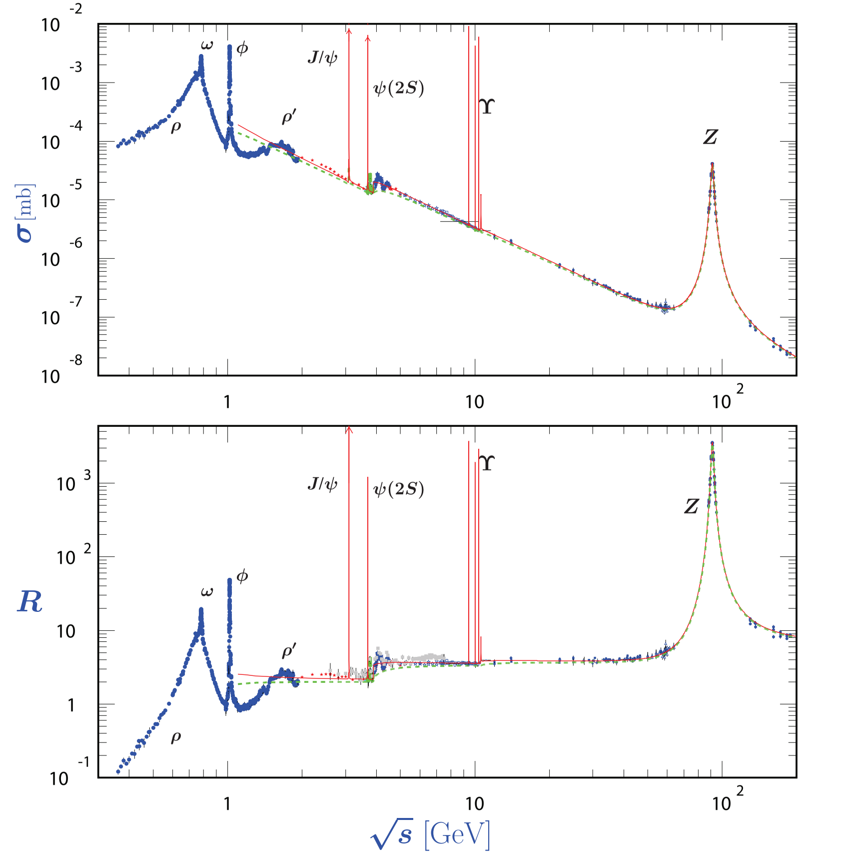


Figure 41.6: World data on the total cross section of $e^+e^- \to hadrons$ and the ratio $R(s) = \sigma(e^+e^- \to hadrons, s)/\sigma(e^+e^- \to \mu^+\mu^-, s)$. $\sigma(e^+e^- \to hadrons, s)$ is the experimental cross section corrected for initial state radiation and electron-positron vertex loops, $\sigma(e^+e^- \to \mu^+\mu^-, s) = 4\pi\alpha^2(s)/3s$. Data errors are total below 2 GeV and statistical above 2 GeV. The curves are an educative guide: the broken one (green) is a naive quark-parton model prediction, and the solid one (red) is 3-loop pQCD prediction (see "Quantum Chromodynamics" section of this *Review*, Eq. (9.7) or, for more details, K. G. Chetyrkin *et al.*, Nucl. Phys. **B586**, 56 (2000) (Erratum *ibid.* **B634**, 413 (2002)). Breit-Wigner parameterizations of J/ψ, $\psi(2S)$, and $\Upsilon(nS), n = 1, 2, 3, 4$ are also shown. The full list of references to the original data and the details of the R ratio extraction from them can be found in [arXiv:hep-ph/0312114]. Corresponding computer-readable data files are available at http://pdg.lbl.gov/current/xsect/. (Courtesy of the COMPAS (Protvino) and HEPDATA (Durham) Groups, May 2010.) See full-color version on color pages at end of book.

R in Light-Flavor, Charm, and Beauty Threshold Regions

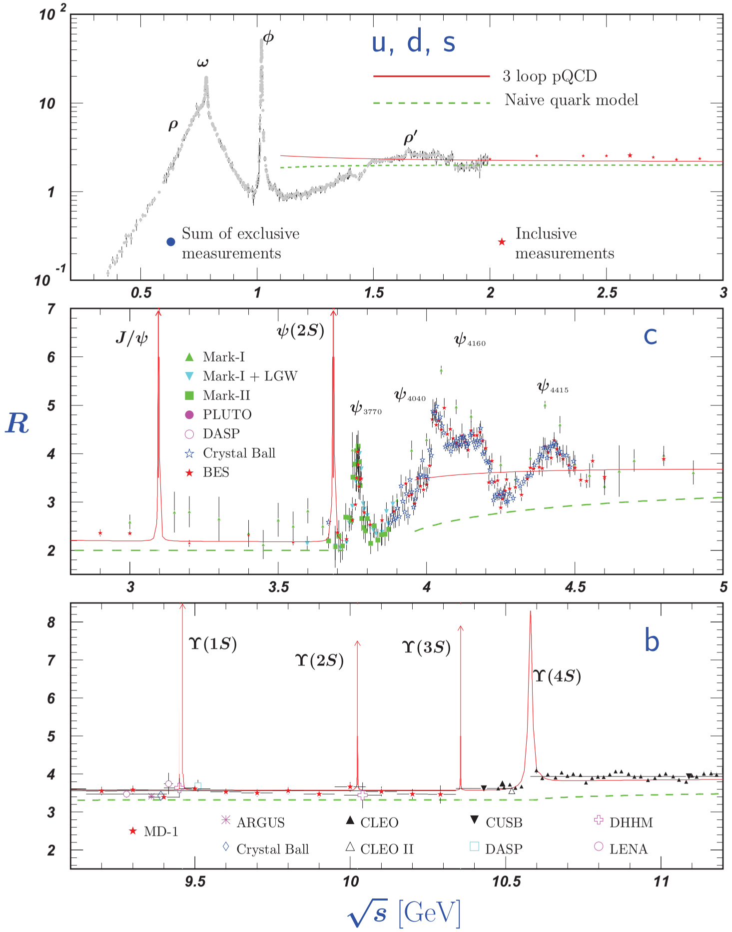


Figure 41.7: R in the light-flavor, charm, and beauty threshold regions. Data errors are total below 2 GeV and statistical above 2 GeV. The curves are the same as in Fig. 41.6. **Note:** CLEO data above $\Upsilon(4S)$ were not fully corrected for radiative effects, and we retain them on the plot only for illustrative purposes with a normalization factor of 0.8. The full list of references to the original data and the details of the R ratio extraction from them can be found in [arXiv:hep-ph/0312114]. The computer-readable data are available at http://pdg.lbl.gov/current/xsect/. (Courtesy of the COMPAS (Protvino) and HEPDATA (Durham) Groups, May 2010.) See full-color version on color pages at end of book.

4.2 The Z-peak in R(s)

- in §4.1, studied tree level  , $f \neq e$

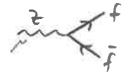
$$\frac{d\sigma_{e\bar e\to f\bar f}}{d\cos\theta} \overset{\text{pg. 45}}{=} \frac{2\pi}{(8\pi)^2}\frac{\langle|\mathcal{M}|^2\rangle}{(2E)^2}\,\theta(E-m_f) = \sum_{\substack{\text{colors}\\ f}} Q_f^2 e^4\left\{1+\frac{m_e^2+m_f^2}{E^2}+\left(1-\frac{m_e^2}{E^2}\right)\left(1-\frac{m_f^2}{E^2}\right)\cos^2\theta\right\}$$

use $\alpha = \frac{e^2}{4\pi}$, $(2E)^2 = s$ in CMS

$$= \frac{\pi\alpha^2}{2s}\sum_{c,f} Q_f^2\left\{1+4\,\frac{m_e^2+m_f^2}{s}+\left(1-\frac{4m_e^2}{s}\right)\left(1-\frac{4m_f^2}{s}\right)\cos^2\theta\right\}\theta(\sqrt{s}-2m_f)$$

- generalization in Standard Model : 

[see, e.g., Ellis/Stirling/Webber §3.1]

use SM Feynman rules :  $= -\frac{ie}{2\sin\theta_w\cos\theta_w}\gamma^\mu(V_f - A_f\gamma^5)$

where $\sin^2\theta_W \approx 0.23$ is the weak mixing angle

$A_f = \pm\frac{1}{2}$ for $f \in \left\{\begin{matrix}\nu_e, \nu_\mu, \nu_\tau, u, c, t\\ e, \mu, \tau, d, s, b\end{matrix}\right\}$ is the axial $f Z \bar f$ coupling

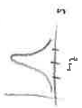
$V_f = A_f - 2Q_f\sin^2\theta_W$ vector $f Z \bar f$ coupling

now, taking $|\mathcal{M}|^2$, get interference term as well

take high E limit, $\sqrt{s} \gg m_f \gg m_e$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s}\sum_{c,f}\left\{\underbrace{(1+\cos^2\theta)}_{\to 2+\frac{2}{3}}\left(Q_f^2 - 2Q_f V_e V_f\,\kappa\,\frac{s(s-m_Z^2)}{(s-m_Z^2)^2+\Gamma_Z^2 m_Z^2} + (A_e^2+V_e^2)(A_f^2+V_f^2)\,\kappa^2\,\frac{s^2}{(s-m_Z^2)^2+\Gamma_Z^2 m_Z^2}\right) + \frac{\cos\theta}{(s-m_Z^2)^2+\Gamma_Z^2 m_Z^2}\left(-4Q_f A_e A_f\,\kappa\, s(s-m_Z^2) + 8A_e V_e A_f V_f\,\kappa^2 s^2\right)\right\}$$

(γ^2; γZ interference; Z^2; $\int_{-1}^{1} d\cos\theta \to 0$)

(s-channel Z propag. $\sim \frac{1}{s-(m_Z - i\Gamma_Z)^2}$ )

where $\kappa \equiv \frac{\sqrt{2}\,G_F\, m_Z^2}{16\pi\alpha}$, Fermi const $G_F = \frac{1}{\sqrt{2}v^2} \approx 1.166\cdot 10^{-5}\,\text{GeV}^{-2}$

$m_Z \approx 91.1876$ GeV , Z decay width $\Gamma_Z \approx 2.5$ GeV

- at small s/m_Z^2, the additional weak effects are small
  - $\Rightarrow$ neglecting them, get back result of pg. 45: $\sigma_{s \ll m_Z^2} = \frac{4\pi\alpha^2}{3s} \sum_{c,f} Q_f^2$
- on Z pole, $s = m_Z^2$, 2^{nd} line dominates

$$\Rightarrow \sigma_{s \approx m_Z^2} = \frac{4\pi\alpha^2}{3 m_Z^2} \sum_{c,f} (A_e^2 + V_e^2)(A_f^2 + V_f^2)\, \kappa^2 \frac{m_Z^2 m_Z^2}{\Gamma_Z^2 m_Z^2}$$

$$\Rightarrow R_{Z\,peak} = \frac{\sigma(e^+e^- \to Z \to hadrons)}{\sigma(e^+e^- \to Z \to \mu^+\mu^-)} = \frac{N_c \sum_q (A_q^2 + V_q^2)}{A_\mu^2 + V_\mu^2}$$

only 5 quarks lighter than Z (($m_{top} \approx 172$ GeV))

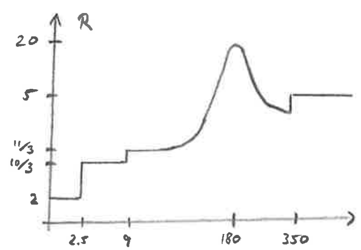
$\rightsquigarrow \sum_q$ goes over $q \in \{ u, c, d, s, b \}$

$$= \frac{3\left[2\left(\frac{1}{4} + \left(\frac{1}{2} - 2\left(\frac{2}{3}\right)\sin^2\theta_w\right)^2\right) + 3\left(\frac{1}{4} + \left(-\frac{1}{2} - 2\left(-\frac{1}{3}\right)\sin^2\theta_w\right)^2\right)\right]}{\frac{1}{4} + \left(-\frac{1}{2} - 2(-1)\sin^2\theta_w\right)^2} \approx 20.095$$

($N_c = 3$, $\sin^2\theta_w = 0.23$)

((adding the γ channel, value changes to 19.984))

- so our result would look like this: step functions from γ exchange + broad peak from Z exchange



- comparison with experiment:

((note that PDG-plot was for $\frac{\sigma(e^+e^- \to hadrons)}{\frac{4\pi\alpha^2}{3s}}$, not $\frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$))

LEP measured $R_{Z\,peak} = 20.767 \pm 0.025$

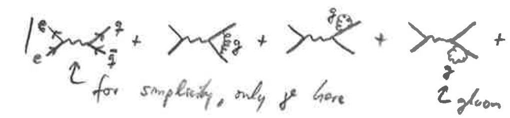
which is ~ 3.5% higher than the above lowest-order prediction

$\Rightarrow$ discrepancy is (mostly) due to higher-order QCD-corrections!

$\Rightarrow$ compute those (($\sigma \to \sigma(1 + \alpha_s + \ldots)$)), then use experiment to determine α_s.

4.3 QCD corrections to R(s)

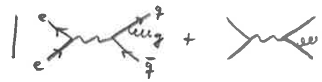
$\rightsquigarrow$ goal: compute $\sigma(e^+e^- \to hadrons) = \sigma_0 \left(1 + c\,\alpha_s + \mathcal{O}(\alpha_s^2)\right)$, $c = ?$

- $\sigma_{e^+e^- \to q\bar{q}} \sim |M|^2 = |$  $+ \ldots|^2$

  (for simplicity, only γ here; gluon)

  $\Rightarrow$ the $c_1\alpha_s$ term gets contributions from interference of tree + 1loop amplitudes. "virtual correction" c_V

- note that there is another class of diagrams, contributing to the same order:

  $\sigma_{e^+e^- \to q\bar{q}g} \sim |$  $|^2$

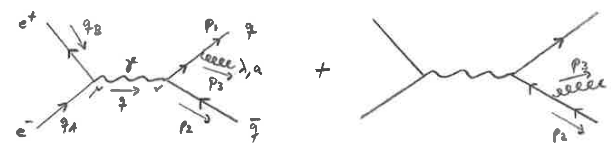
  $\Rightarrow$ "real correction" c_R

$\rightsquigarrow$ total cross section to produce (any number of) partons ($\to$ hadrons) is sum of $\sigma_{e^+e^- \to q\bar{q}} + \sigma_{e^+e^- \to q\bar{q}g} + \ldots$

$\rightsquigarrow$ in fact, both will turn out to be (infrared) divergent, only their sum is finite, hence physical.

$\rightarrow$ in practice, pick a regularization scheme (again $d = 4 - 2\varepsilon$) and remove regulator in the end ($\varepsilon \to 0$)

4.3.1 real corrections : $\sigma_{e^+e^- \to q\bar{q}g}$



(see pg. 44, plus Fy rules from pp 24,25)

(spin $= \pm\frac{1}{2}$; Dirac spinor for incoming particle; polarization vector for outgoing gluon)

$$= \bar{v}(q_B, s_B)(-ie\gamma^\mu)u(q_A, s_A)\left(\frac{-i g_{\mu\nu}}{q^2}\right)\varepsilon_\lambda^{*a} \;*$$

$$* \;\bar{u}(p_1, s_1)\left\{(i g_s \gamma^\lambda T^a)\frac{i(\not{p}_1 + \not{p}_3)}{(p_1 + p_3)^2}(-ie Q_q \gamma^\nu) + (-ie Q_q \gamma^\nu)\frac{i(-\not{p}_2 - \not{p}_3)}{(p_2 + p_3)^2}(i g_s \gamma^\lambda T^a)\right\} v(p_2, s_2)$$

note: have used Feynman gauge here, $\xi = 1$

have used massless quarks, $m_q = 0$

→ using $\bar v_B \equiv \bar v(q_B, s_B)$ etc for brevity, this reads

$$\mathcal{M} = \frac{-ie^2 Q_q g_s}{q^2}\, \varepsilon_\lambda^{*a} T^a\; \bar v_B \gamma_\mu u_A \bar u_1 \underbrace{\left\{ \gamma^\lambda \frac{\not p_1 + \not p_3}{(p_1+p_3)^2}\gamma^\mu + \gamma^\mu \frac{-\not p_2 - \not p_3}{(p_2+p_3)^2}\gamma^\lambda \right\}}_{\equiv S^{\lambda\mu}} v_2$$

→ for cross section, need $|\mathcal{M}|^2$

as in §4.1, unpolarized e^+e^- beams ⇒ average over incoming spins

spin-blind detector ⇒ sum over outgoing spins

$$\langle|\mathcal{M}|^2\rangle = \sum_{pol} \frac{1}{4}\sum_s \mathcal{M}\mathcal{M}^\dagger$$

$$= \left(\frac{e^2 Q_q g_s}{q^2}\right)^2 \underbrace{\sum_{pol}\varepsilon_\lambda^{*a}\varepsilon_\sigma^b}_{= -g_{\lambda\sigma}\delta^{ab}} \underbrace{(T^a_{ij})(T^b_{ij})^\dagger}_{= tr(T^aT^b) = \frac{1}{2}\delta^{ab}} \frac{1}{4}\sum_{s_{AB12}} \bar v_B \gamma_\mu \underbrace{u_A \bar u_A}_{\sum_{s_A} = \not q_A} \gamma_\nu^\dagger v_B\; \bar u_1 S^{\lambda\mu} \underbrace{v_2 \bar v_2}_{\sum_{s_2} = \not p_2} (S^{\sigma\nu})^\dagger u_1$$

(completeness rel., Fey gauge) (normalization) perform spin-sums via completeness rels, see pg 44 etc

(T hermitean, γ hermitean)

$$= -\frac{e^4 Q_q^2 g_s^2}{4 q^4}\frac{N_c^2-1}{2}\; \underbrace{tr(\not q_B \gamma_\mu \not q_A \gamma_\nu)}_{\equiv L_{\mu\nu}}\; \underbrace{tr(\not p_1 S^{\lambda\mu} \not p_2 S^\nu{}_\lambda)}_{\equiv G^{\mu\nu}}$$

$$\equiv L_{\mu\nu} = q_B^\sigma q_A^\rho\, 4(g_{\sigma\mu}g_{\rho\nu} - g_{\sigma\rho}g_{\mu\nu} + g_{\sigma\nu}g_{\mu\rho}) \quad \text{(see pg. 30)}$$

$$= 4(q_{B\mu}q_{A\nu} + q_{A\mu}q_{B\nu} - q_A\cdot q_B\, g_{\mu\nu})$$

→ now, total cross section is (pg 44, $m_i = 0$), using dimensional regularization,

$$\sigma_{e^+e^-\to q\bar q g} = \frac{1}{2s}\left(\prod_{i=1}^3 \int\frac{d^{d-1}p_i}{(2\pi)^{d-1}2E_{p_i}}\right)(2\pi)^d \delta^{(d)}(p_1+p_2+p_3-q_A-q_B)\,\langle|\mathcal{M}|^2\rangle$$

$$= -\frac{e^4 g_s^2 \left(\sum_q Q_q^2\right) C_F N_c}{8s (q^2)^2 (2\pi)^{2d-3}} L_{\mu\nu} \underbrace{\left(\prod_{i=1}^3\int\frac{d^{d-1}p_i}{2|\vec p_i|}\right)\delta^{(d)}(p_1+p_2+p_3-q)\, G^{\mu\nu}}_{\equiv I^{\mu\nu}(q)}$$

((used $C_F \equiv \frac{N_c^2-1}{2N_c}$))

→ note that (check ?!) $q_\mu I^{\mu\nu}(q) = 0$

$$\Rightarrow I^{\mu\nu}(q) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) I(q^2) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{g_{\sigma\rho}I^{\sigma\rho}(q)}{d-1}$$

follows from contracting the eq. with $g_{\mu\nu}$

$$\Rightarrow L_{\mu\nu}I^{\mu\nu}(q) = 4\left[q_A\cdot q_B(1-d+1) - \frac{1}{q^2}\left(2\, q\cdot q_B\, q\cdot q_A - q_A\cdot q_B\, q^2\right)\right]\frac{g_{\sigma\rho}I^{\sigma\rho}(q)}{d-1}$$

use $q_A\cdot q_B = \frac{1}{2}(q_A^2 + q_B^2) + q_A\cdot q_B = \frac{1}{2}(q_A+q_B)^2 = \frac{q^2}{2} = \frac{s}{2}$ ($q_A^2, q_B^2 = 0$ (onshell))

$$= 2q^2(2-d)\frac{g_{\mu\nu}I^{\mu\nu}(q)}{d-1}$$

→ in $\sigma_{ee\to q\bar q g}$, need now

$$g_{\mu\nu}G^{\mu\nu} = tr(\not p_1 S^{\lambda\mu}\not p_2 S_{\mu\lambda}) = \text{(check ?!)} = 4(d-2)\frac{x_1^2 + x_2^2 + \frac{d-4}{2}x_3^2}{(1-x_1)(1-x_2)}$$

where $x_i \equiv \frac{2p_i\cdot q}{q^2}$

→ plug these elements into cross section:

$$\sigma_{e^+e^-\to q\bar q g} = +\frac{e^4 g_s^2\left(\sum_q Q_q^2\right) C_F N_c}{s^2 (2\pi)^{2d-3}}\frac{(d-2)^2}{d-1}\underbrace{\left(\prod_{i=1}^3\int\frac{d^{d-1}p_i}{2|\vec p_i|}\right)\delta^{(d)}(p_{123}-q)}\frac{x_1^2+x_2^2+\frac{d-4}{2}x_3^2}{(1-x_1)(1-x_2)}$$

rewrite phase space integral in terms of x_i [see eg. Muta §5.1]

$$= \frac{\pi(\pi s)^{d-3}}{4\Gamma(d-2)}\left(\prod_{i=1}^3\int_0^1\frac{dx_i}{(1-x_i)^{\frac{4-d}{2}}}\right)\delta(2-x_1-x_2-x_3)$$

$d = 4-2\varepsilon$ ($g_s^2 = 4\pi\mu^{2\varepsilon}\alpha_s$)

$$= \alpha^2\alpha_s\left(\sum_q Q_q^2\right)C_F N_c\,\frac{2}{s}\left(\frac{4\pi\mu}{s}\right)^{2\varepsilon}\frac{(1-\varepsilon)^2}{(3-2\varepsilon)\Gamma(2-2\varepsilon)}\cdot I$$

where

$$I \equiv \left(\prod_{i=1}^3\int_0^1\frac{dx_i}{(1-x_i)^\varepsilon}\right)\delta(2-x_1-x_2-x_3)\frac{x_1^2+x_2^2-\varepsilon x_3^2}{(1-x_1)(1-x_2)}$$

$$= \int_0^1 dx_1\int_{1-x_1}^1 dx_2\,\frac{x_1^2+x_2^2-\varepsilon(2-x_1-x_2)^2}{(1-x_1)^{1+\varepsilon}(1-x_2)^{1+\varepsilon}(x_1+x_2-1)^\varepsilon}$$

eg Mathematica

$$= \frac{2(2-6\varepsilon+5\varepsilon^2-2\varepsilon^3)\Gamma^3(1-\varepsilon)}{\varepsilon^2\,\Gamma(3-3\varepsilon)} \approx \frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\left(\frac{19}{2}-\pi^2\right)+O(\varepsilon)$$

→ compare result with $\sigma_{e^+e^-\to q\bar{q}}$ in same regularization:

repeat calculation of $\left|\right.$  $\left.\right|^2$ (see § 4.1) in d-dimensions,

$$\sigma_{e^+e^-\to q\bar{q}} \overset{d=4-2\varepsilon}{=} \frac{4\pi\alpha^2}{3s}\left(\sum_q Q_q^2\right) N_c \left(\frac{4\pi}{s}\right)^\varepsilon \frac{3(1-\varepsilon)\,\Gamma(2-\varepsilon)}{(3-2\varepsilon)\,\Gamma(2-2\varepsilon)}$$

$$\Rightarrow \sigma_{e^+e^-\to q\bar{q}g} = \sigma_{e^+e^-\to q\bar{q}} \cdot \frac{\alpha_s C_F}{2\pi}\left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)}\, I$$

- note that I is divergent for $\varepsilon\to 0$.

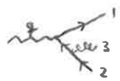
  not a desaster (see pg 48): $\lim\limits_{\varepsilon\to 0}$ (real + virtual corr.) should exist.

  ↳ compute next.

- physical origin of divergences:

  in a 'naive' calculation ($d=4$), $\quad I = \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$

  ⇝ divergences come from $x_1=1$ and $x_2=1$



  but $1-x_1 = 1-\frac{2p_1\cdot q}{q^2} = \frac{(q-p_1)^2-p_1^2}{q^2} = \frac{(p_2+p_3)^2-p_1^2}{q^2} = \frac{2p_2\cdot p_3+p_2^2+p_3^2-p_1^2}{q^2}$

  onshell: $p_i^2=0 \Rightarrow p_i = E_i(1,\vec{e}_i)$

  $$= \frac{2E_2E_3(1-\vec{e}_2\vec{e}_3)}{q^2}$$

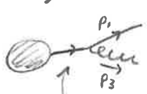
  ⇝ divergences originate from $\begin{cases}\vec{e}_1\cdot\vec{e}_3\to 1 & : q,g\\ \vec{e}_2\cdot\vec{e}_3\to 1 & : \bar{q},g\end{cases}$ collinear ($\theta_{13}\to 0$)

  and from $E_3\to 0$: g soft

  } both are referred to as infrared singularities.

another view of the same fact:

divergences originate from diverging propagators

 $\sim \frac{1}{(p_1+p_3)^2} = \frac{1}{2p_1\cdot p_3} = \frac{1}{2E_1E_3(1-\vec{e}_1\vec{e}_3)} = \frac{1}{2E_1E_3(1-\cos\theta_{13})}$

collinear limit, $\theta\to 0$: $\quad \approx \frac{1}{2E_1E_3\theta^2} \quad \Rightarrow |M|^2 \sim \frac{\theta^2}{\theta^4} \leftarrow$ from numerators

soft limit, $E_3\to 0$: interference term in $\left|\right.$  $\left.\right|^2$

gives $|M|^2 \sim \frac{p_1\cdot p_2}{p_1\cdot p_3\; p_2\cdot p_3} \approx \frac{1}{E_3^2}$

in phase space integral, $\frac{d^3p_3}{2E_3} = \frac{1}{2E_3}E_3^2 dE_3 \sin\theta\, d\theta\, d\varphi \sim E_3\, dE_3\, \theta d\theta$

⇒ logarithmic singularities in both limits.

4.3.2 virtual corrections: $\sigma_{e^+e^-\to q\bar{q}}$ at $\mathcal{O}(\alpha_s)$

structure of cross section computation:

$$|\mathcal{M}_0 + \alpha_s\mathcal{M}_1 + \mathcal{O}(\alpha_s^2)|^2 = |\mathcal{M}_0|^2 + \alpha_s(\mathcal{M}_1\mathcal{M}_0^* + \mathcal{M}_0\mathcal{M}_1^*) + \mathcal{O}(\alpha_s^2)$$

→ need to compute interference term $\mathcal{M}_1\mathcal{M}_0$ only.

recall $\mathcal{M}_0 =$ 

$\alpha_s\mathcal{M}_1 =$ 

- in dimensional regularization,  $\sim \int d^dk \frac{1}{k^2}\frac{\not k+\not p}{(k+p)^2} = \not p\, I(p^2)$

  with $\dim[I(p^2)] = m^{d-4}$

  but $p^2=0$ (due to onshell condition), so no scale ⇒ zero

  $\Rightarrow \mathcal{M}_{1(a)} = 0 = \mathcal{M}_{1(b)}$ in dim. reg.

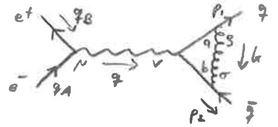
- for diagram (c), need to do some computation

  $$\sigma_{e^+e^-\to q\bar{q}} \overset{\text{pg. 49}}{=} \frac{1}{2s}\left(\prod_{i=1}^{2}\int\frac{d^{d-1}p_i}{(2\pi)^{d-1}2E_{p_i}}\right)(2\pi)^d\,\delta^{(d)}(p_1+p_2-q_A-q_B)\,\langle|\mathcal{M}|^2\rangle$$

  structure of $\alpha\mathcal{M}_{1(c)}$ is similar to $\mathcal{M}_0$!

  recall (pg 44) $\mathcal{M}_0 =$  $= \frac{e^2Q_q}{q^2}\bar{v}_B\gamma^\mu u_A\,\bar{u}_1\gamma_\mu v_2$

  $$\langle|\mathcal{M}_0|^2\rangle \overset{\text{do spin sums}}{=} \frac{e^4Q_q^2}{4q^4}\,\mathrm{tr}(\not q_B\gamma^\mu\not q_A\gamma^{\nu(4)})\,\mathrm{tr}(\not p_1\gamma_\mu\not p_2\gamma_\nu^{(4)})$$

  now, $\alpha_s\mathcal{M}_1 =$ 

  use Feynman gauge $\xi=1$, massless quarks $m_q=0$

  $$= \bar{v}_B(-ie\gamma^\mu)u_A\left(-\frac{ig_{\mu\nu}}{q^2}\right)\bar{u}_1(ig_s\gamma^\rho T^a)\int\frac{d^dk}{(2\pi)^d}\frac{i(\not k+\not p_1)}{(k+p_1)^2}(-ieQ_q\gamma^\nu)\frac{i(\not k-\not p_2)}{(k-p_2)^2} *$$

  $$*\,(ig_s\gamma^\sigma T^b)v_2\left(\frac{-i\delta^{ab}g_{\rho\sigma}}{k^2}\right)$$

use $T^a T^a = T^a_{ij} T^a_{jk} \overset{\text{Fierz, pg. 9}}{=} \frac{1}{2}\left(\delta_{ik}\delta_{jj} - \frac{1}{N}\delta_{ij}\delta_{jk}\right) = \frac{\delta_{ik}}{2}\left(N - \frac{1}{N}\right) = \mathbb{1}\, C_F$

$$= \frac{e^2 Q_q}{q^2} \bar{v}_B \gamma^\mu u_A \; \bar{u}_1 \underbrace{g_s^2 C_F \mathbb{1} \int \frac{d^dk}{(2\pi)^d} \gamma^s \frac{k\!\!/ + p\!\!/_1}{(k+p_1)^2} \gamma_\mu \frac{k\!\!/ - p\!\!/_2}{(k-p_2)^2} \gamma_s \frac{1}{k^2}}_{\equiv \Lambda_\mu} v_2$$

$$\Rightarrow \langle \alpha_s (\mathcal{M}_1 \mathcal{M}_0^+ + \mathcal{M}_0 \mathcal{M}_1^+) \rangle = \frac{e^4 Q_q^2}{4 q^4} \operatorname{tr}(p\!\!/_B \gamma^\mu p\!\!/_A \gamma^\nu) \operatorname{tr}(p\!\!/_1 \Lambda_\mu p\!\!/_2 \gamma_\nu) + \text{c.c.}$$

- note that IF the contribution of Λ_μ inside the 2^{nd} trace can be reduced to $\Lambda_\mu = \gamma_\mu \cdot \alpha_s C_F \frac{\Lambda(q^2)}{2}$

  THEN $\langle \alpha_s (\mathcal{M}_1 \mathcal{M}_0^+ + \mathcal{M}_0 \mathcal{M}_1^+) \rangle = \langle |\mathcal{M}_0|^2 \rangle \cdot \alpha_s C_F \operatorname{Re}[\Lambda(q^2)]$

  and σ follows without additional work!

  → the idea is to profit from the onshell conditions, such that if there is a term, say, $p\!\!/_1 \gamma_\mu p\!\!/_1$ in Λ_μ it gets killed by $p\!\!/_1 \Lambda_\mu = p\!\!/_1 p\!\!/_1 \gamma_\mu p\!\!/_1 = \frac{1}{2}\{\gamma_\rho, \gamma_\nu\} p_1^\rho p_1^\nu \gamma_\mu p\!\!/_1 = p_1^2 \gamma_\mu p\!\!/_1 = 0$

- $$\Lambda_\mu = 4\pi\mu^{2\varepsilon} \alpha_s C_F \gamma^s \sigma_{\mu\nu s} \int \frac{d^4k}{(2\pi)^d} \frac{(k+p_1)^\sigma (k-p_2)^\nu}{(k+p_1)^2 (k-p_2)^2 k^2} \equiv I^{\sigma\nu}$$

  pg. 35:

  $= -2\gamma_{\nu\mu\sigma} + (4-d)\gamma_{\sigma\mu\nu}$ ①

  or $= -4(\gamma_\nu g_{\mu\sigma} - \gamma_\mu g_{\nu\sigma} + \gamma_\sigma g_{\nu\mu}) + (6-d)\gamma_{\sigma\mu\nu}$ ②

  or $= 2(4-d)(\gamma_\sigma g_{\mu\nu} - \gamma_\mu g_{\nu\sigma} + \gamma_\nu g_{\sigma\mu}) - (6-d)\gamma_{\nu\mu\sigma}$ ③

- solve the loop integral $I^{\sigma\nu}$ via Feynman parameters (see pg. 30)

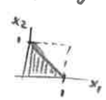
  ($q^2 = (p_1+p_2)^2 = 2p_1p_2$)

$$I^{\sigma\nu} = \Gamma(3) \int_0^1 dx_1 \ldots dx_3\, \delta(1-x_1-x_2-x_3) \int \frac{d^dk}{(2\pi)^d} \frac{(k+p_1)^\sigma (k-p_2)^\nu}{[x_1(k^2+2kp_1+p_1^2) + x_2(k^2-2kp_2+p_2^2) + x_3 k^2]^3}$$

  denominator $[\ldots] = [(k + x_1p_1 - x_2p_2)^2 + x_1x_2 q^2]$ (used δ fct for k^2-prefactor)

  shift $k \to k - x_1p_1 + x_2p_2$, linear terms in k integrate to zero

$$= \Gamma(3) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^dk}{(2\pi)^d} \frac{k^\sigma k^\nu - [(1-x_1)p_1 + x_2p_2]^\sigma [x_1p_1 + (1-x_2)p_2]^\nu}{(k^2 + x_1x_2 q^2)^3}$$



momentum integrals are standard (see pg 31), after $k^\sigma k^\nu \to g^{\sigma\nu}\frac{k^2}{d}$ and Wick rotation: $I_3^1(-x_1x_2q^2)$, $I_3^0(-x_1x_2q^2)$

$$= \frac{1}{(4\pi)^{d/2}} \frac{1}{(-q^2)^{2-d/2}} \left\{ \frac{g^{\sigma\nu}}{d} \frac{d}{2} \Gamma(2-\tfrac{d}{2}) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{(x_1x_2)^{2-d/2}} + \Gamma(3-\tfrac{d}{2}) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{[\,]^\sigma [\,]^\nu}{(x_1x_2)^{3-d/2}} \frac{1}{(-q^2)} \right\}$$

perform the integrals over Fy parameters, e.g. with Mathematica, for $\operatorname{Re}(d) > 4$

$$= \frac{1}{(4\pi)^{d/2}} \frac{1}{(-q^2)^{2-d/2}} \left\{ g^{\sigma\nu} \frac{1}{2} \Gamma(2-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-1)\Gamma(d-1)} + \Gamma(3-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-2)\Gamma(d-1)} \left[p_1^\sigma p_1^\nu + p_2^\sigma p_2^\nu + \frac{8-4d+d^2}{(d-4)(d-2)} p_1^\sigma p_2^\nu + \frac{d-4}{d-2} p_2^\sigma p_1^\nu \right] \frac{1}{(-q^2)} \right\}$$

$I^{\sigma\nu}$; coeffs I_i from above

- $$\Lambda_\mu = 4\pi\mu^{2\varepsilon} \alpha_s C_F \gamma^s \sigma_{\mu\nu s} \left\{ I_1 g^{\sigma\nu} + I_2 p_1^\sigma p_1^\nu + I_3 p_2^\sigma p_2^\nu + I_4 p_1^\sigma p_2^\nu + I_5 p_2^\sigma p_1^\nu \right\}$$

  ① $\to (2-d)\gamma^\nu{}_{\mu\nu}$ $\overset{\text{pg 34}}{=} (2-d)^2 \gamma_\mu$; ① $\to 0$; ① $\to 0$; ② $\to 4\gamma_\mu p_1 p_2$; ③ $\to -2(4-d)\gamma_\mu p_2 \cdot p_1$

  have now specified terms that survive in product $p\!\!/_1 \Lambda_\mu p\!\!/_2$ (in particular, $0 = \{ p\!\!/_1 \gamma_\mu p\!\!/_1, p\!\!/_2 \gamma_\mu p\!\!/_2, p\!\!/_1 \gamma_\mu p\!\!/_2, p\!\!/_2 p_{1\mu}, p\!\!/_1 p_{2\mu} \}$)

$$= 4\pi\mu^{2\varepsilon} \alpha_s C_F \frac{1}{(4\pi)^{d/2}} \frac{1}{(-q^2)^{2-d/2}} \left\{ \frac{1}{2}\Gamma(2-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-1)\Gamma(d-1)} (2-d)^2 \gamma_\mu + \right.$$
$$\left. + \Gamma(3-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-2)\Gamma(d-1)} \left[0 + 0 + \frac{8-4d+d^2}{(d-4)(d-2)} 2\gamma_\mu \left[\frac{2p_1p_2}{-q^2}\right]^{-1} + \frac{d-4}{d-2}(d-4)\gamma_\mu \left[\frac{2p_1p_2}{-q^2}\right]^{-1} \right] \right\}$$

$$\overset{d=4-2\varepsilon}{=} \frac{1}{2}\gamma_\mu \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{-q^2}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left\{ \left(\frac{1}{\varepsilon} + 1 + O(\varepsilon)\right) + 0 + 0 + \left(-\frac{2}{\varepsilon^2} - \frac{4}{\varepsilon} - 9 + O(\varepsilon)\right) + O(\varepsilon) \right\}$$

$$= \frac{1}{2}\gamma_\mu \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + O(\varepsilon) + i\pi\left(-\frac{2}{\varepsilon} - 3 + O(\varepsilon)\right) \right)$$

↑ from $(-1)^\varepsilon$; but irrelevant

($(-1)^\varepsilon = e^{i\pi\varepsilon}$, $\operatorname{Re}[(-1)^\varepsilon] = \cos(\pi\varepsilon)$)

- collecting, we finally have

$$\sigma_{e^+e^- \to q\bar{q}} = \sigma^{(\text{tree})}_{e^+e^- \to q\bar{q}} \cdot \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + O(\varepsilon) \right) + O(\alpha_s^2) \right\}$$

4.3.3 Result

let $\sigma_0 \equiv \sigma_{e^+e^-\to q\bar q}$ (tree level) denote the leading order (LO) result

(see pg. 45; d-dim result on pg 51)

then, in dimensional regularization, real and virtual QCD-corrections are

LO (pg. 51) $\sigma_{e^+e^-\to q\bar q g} = \sigma_0 \frac{\alpha_s C_F}{2\pi}\left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)}\left(\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\frac{19}{2}-\pi^2+O(\varepsilon)\right)+O(\alpha_s^2)$

NLO (pg. 54) $\sigma_{e^+e^-\to q\bar q} = \sigma_0\left\{1+\frac{\alpha_s C_F}{2\pi}\left(\frac{4\pi\mu^2}{s}\right)^\varepsilon\frac{1}{\Gamma(1-\varepsilon)}\left(-\frac{2}{\varepsilon^2}-\frac{3}{\varepsilon}-8+\pi^2+O(\varepsilon)\right)+O(\alpha_s^2)\right\}$

the sum, needed for the hadronic cross section, is finite;

can take $\varepsilon\to 0$

NLO $\Rightarrow \sigma_{e^+e^-\to \text{hadrons}} = \sigma_0\left\{1+\frac{\alpha_s C_F}{2\pi}\frac{3}{2}+O(\alpha_s^2)\right\}$ in QCD, $N_c=3$, $C_F=\frac{N_c^2-1}{2N}=\frac{4}{3}$

$\overset{N_c=3}{=} \sigma_0\left\{1+\frac{\alpha_s}{\pi}+O(\alpha_s^2)\right\}$

Before using this results, a couple of remarks:

- cancellation of soft and collinear divergences between the real and virtual gluon diagrams is not accidental. They are in fact guaranteed by theorems (Bloch/Nordsieck, Kinoshita/Lee/Nauenberg (KLN)): suitably defined inclusive quantities will be IR safe in the massless limit.
  (($\sigma_{e^+e^-\to\text{hadrons}}$ is such a quantity; $\sigma_{e^+e^-\to q\bar q}$ is not))
  → proof in QCD: see e.g. [Collins/Soper, Ann Rev Nucl Sci 37 (1987) 383]
- our result would be worthless if it depended on our choice of regularization procedure, dim. reg.
  Proof of independence is beyond this lecture; but demonstrate it by comparing with gluon mass regularization scheme (m_g = gluon mass)

$\sigma_{e^+e^-\to q\bar q g} = \sigma_0\frac{\alpha_s C_F}{2\pi}\left(\ln^2\frac{s}{m_g^2}-3\ln\frac{s}{m_g^2}+7-\frac{\pi^2}{3}+O(\varepsilon)\right)$

$\sigma_{e^+e^-\to q\bar q} = \sigma_0\left\{1+\frac{\alpha_s C_F}{2\pi}\left(-\ln^2\frac{s}{m_g^2}+3\ln\frac{s}{m_g^2}-\frac{11}{2}+\frac{\pi^2}{3}+O(\varepsilon)\right)\right.$

$\Rightarrow \sigma_{e^+e^-\to\text{hadrons}} = \sigma_0\left\{1+\frac{\alpha_s C_F}{2\pi}\frac{3}{2}+O(\alpha_s^2)\right\}$

→ individual cross sections completely different; sum scheme independent!

- at this order (NLO), and for $m_q=0$, our computation of the QCD correction is independent of the nature of the exchanged weak boson (we took the photon only, cf pg. 48).

→ generalization: the $(1+\frac{\alpha_s}{\pi})$ result is valid also for $R_{Z\,peak}$, see pg. 47

ratios: $R(s) = \frac{\sigma(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)} = N_c\left(\sum_f Q_f^2\right)\left(1+\frac{\alpha_s C_F}{2\pi}\frac{3}{2}+O(\alpha_s^2)\right)$

$s\ll m_Z^2$; otherwise see § 4.2

$R_{Z\,peak} = \left.\frac{\sigma(e^+e^-\to Z\to\text{hadrons})}{\sigma(e^+e^-\to Z\to\mu^+\mu^-)}\right|_{s=m_Z^2} = 19.984\left(1+\frac{\alpha_s}{\pi}+O(\alpha_s^2)\right)$

note that NLO correction is positive.

α_s:

- comparing with the (correctly scaled) experimental result

$R_{Z\,peak}(\text{LEP}) = 20.767 \pm 0.025$ (see pg. 47)

⇒ our first measurement of α_s: $\alpha_s(m_Z) = 0.123 \pm 0.004$

((compare with "world average" from PDG (2012): 0.1184 ± 0.0007))

- as another determination of α_s, let us compare to data taken by PETRA (DESY), at $\sqrt{s}\approx 34$ GeV

$R(s\approx(34\,\text{GeV})^2, \text{PETRA}) = 3.88 \pm 0.03$

we would predict ($udscb$ — t too heavy) (b: 4.5 GeV, t: 172 GeV)

$R((34\,\text{GeV})^2) = \underbrace{3\left(2\left(\tfrac{2}{3}\right)^2+3\left(-\tfrac{1}{3}\right)^2\right)}_{=\frac{11}{3}\approx 3.667}\left(1+\frac{\alpha_s}{\pi}+O(\alpha_s^2)\right)$

or, including Z, $R((34\,\text{GeV})^2) = 3.69\left(1+\frac{\alpha_s}{\pi}+O(\alpha_s^2)\right)$

⇒ our second measurement of α_s: $\alpha_s(34\,\text{GeV}) = 0.162 \pm 0.026$

- recall (§ 3.4, pg. 42) that α_s is running!

$\alpha_s(\mu) = \frac{4\pi}{\beta_0\ln(\mu^2/\Lambda^2)}$, with $\beta_0 = \frac{11}{3}N_c-\frac{2}{3}N_f \overset{N_c=3}{=} 11-\frac{2}{3}N_f$

$\Leftrightarrow \frac{1}{\alpha_s(\mu_1)}-\frac{1}{\alpha_s(\mu_2)} = \frac{\beta_0}{4\pi}\ln\left(\frac{\mu_1^2}{\mu_2^2}\right)$

⇒ 2nd measurement translates into $\alpha_s(m_Z=91.2\,\text{GeV}) = 0.135 \pm 0.018$

→ so far, our NLO correction to R shows correct qualitative features.

→ but what about higher orders ?

4.3.4 Higher-order QCD corrections to $R(s)$

write $R(s) = 3\left(\sum_f Q_f^2\right) \cdot K_{QCD}$

where $K_{QCD} = 1 + 1 \cdot \frac{\alpha_s(\mu)}{\pi} + \sum_{n \geq 2} C_n\left(\frac{s}{\mu^2}\right) \cdot \left(\frac{\alpha_s(\mu)}{\pi}\right)^n$

↑ result of our computation; from { tree-level $q\bar{q}g$, one-loop $q\bar{q}$ } final state

the functions $C_n\left(\frac{s}{\mu^2}\right)$ follow from higher-order computations:

e.g. C_2 from { tree-level $q\bar{q}gg$, $q\bar{q}q\bar{q}$; one-loop $q\bar{q}g$; two-loop $q\bar{q}$ } final states

etc ...

→ note that in our computation, there were no UV divergences
((in fact, those in  cancel exactly)),
so we did not need to renormalize, hence our coefficient did not depend on the renormalization scale μ: $C_1\left(\frac{s}{\mu^2}\right) = 1$.

→ in higher orders, we will encounter UV divs, hence $C_{n \geq 2}$ are renormalization scheme dependent.
If we could sum the whole series, it would be μ-indep.
In a truncated series, μ-dependence is of higher order.

⇒ μ-dependence of $C_n\left(\frac{s}{\mu^2}\right)$ is fixed by knowing μ-dependence of $\alpha_s(\mu)$ (pg 38: $\mu^2 \partial_{\mu^2} \alpha_s = -\frac{\beta_0}{4\pi}\alpha_s^2 - \frac{\beta_1}{(4\pi)^2}\alpha_s^3 - \dots$)

⇒ $C_2\left(\frac{s}{\mu^2}\right) = C_2(1) + C_1(1) \boxed{\frac{\beta_0}{4} \ln\left(\frac{\mu^2}{s}\right)} \equiv L$

$C_3\left(\frac{s}{\mu^2}\right) = C_3(1) + C_1(1) L^2 + \left[C_1(1)\frac{\beta_1}{4\beta_0} + 2C_2(1)\right] L$

etc. (check ?!)

(($\beta_1 = \frac{2}{3}\left(17 N_c^2 - 5 N_c N_f - 3 C_F N_f\right) \overset{N_c=3}{=} 102 - \frac{38}{3} N_f$))

• C_2 and C_3 have been computed [Samuel/Surguladze, PRL 66 (1991) 560] [Gorishny/Kataev/Larin, PL B 259 (1991) 144]
(here $N_c = 3$, ren. scale set to $\mu = \sqrt{s}$, $\overline{MS}$ scheme)

$C_2(1) = \left(\frac{365}{24} - 11\zeta(3)\right) + \left(-\frac{11}{12} + \frac{2}{3}\zeta(3)\right) N_f$

$\approx 1.986 - 0.115\, N_f$

$C_3(1) = \left(\frac{87029}{288} - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5)\right) + \left(-\frac{7847}{216} + \frac{262}{9}\zeta(3) - \frac{25}{9}\zeta(5)\right) N_f$
$+ \left(\frac{151}{162} - \frac{19}{27}\zeta(3)\right) N_f^2 - \frac{\pi^2}{432}(33 - 2N_f)^2 + \eta\left(\frac{55}{72} - \frac{5}{3}\zeta(3)\right)$

$\approx -6.637 - 1.200\, N_f - 0.005\, N_f^2 - 1.240\, \eta$

where $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ is the Riemann Zeta function, $\zeta(3) \approx 1.202057$, $\zeta(5) \approx 1.036928$

and $\eta = \frac{\left(\sum_q Q_q\right)^2}{3\left(\sum_q Q_q^2\right)}$, $\sum_q$ over all quarks with $m_q \ll \sqrt{s}$ (effectively massless)

((for $R_{Z\text{peak}}$, QCD corrections are again the same, except that $\eta \to \frac{\left(\sum_q V_q\right)^2}{3 \sum_q (V_q^2 + A_q^2)}$))

• having a few orders of the perturbative series, can now discuss convergence

→ coefficients are scheme-dependent, so can try to find an "optimal" scheme (from the point of view of convergence), examples are:

FAC (fastest apparent convergence)
choose scale $\mu = \mu_{FAC}$ such that $R^{(1)}(\mu_{FAC}) = R^{(n)}(\mu_{FAC})$

PMS (principle of minimal sensitivity)
choose $\mu = \mu_{PMS}$ such that $\mu \partial_\mu R^{(n)}(\mu)\big|_{\mu_{PMS}} = 0$
[Stevenson, PLB 100 (1981) 61]

BLM ([Brodsky/Lepage/Mackenzie, PR D28 (1983) 228])
absorb all N_f-terms into α_s via β fct

etc ...

for $R^{(3)}(s)$, these are $\{\mu_{FAC}, \mu_{PMS}, \mu_{BLM}\} \overset{(N_f = 5 \text{ massless flavors})}{\approx} \{0.692, 0.587, 0.708\}\sqrt{s}$

→ μ-variation does get weaker, comparing $R^{(1)}(s)$, $R^{(2)}(s)$, $R^{(3)}(s)$

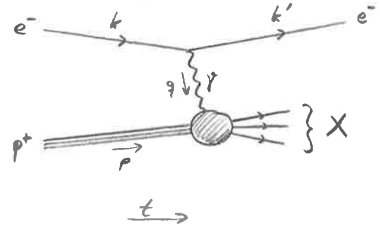
5. Deep inelastic scattering (DIS)

We now (temporarily) go back to tree-level phenomenology.

Q.: are quarks real physical constituents of hadrons,
or just a mathematical convenience for describing the hadron's wavefct?

⇝ DIS gives information on internal proton structure

⇝ structure functions, Bjorken scaling, parton distribution fcts



photon virtuality $Q^2 = -q^2$ $\left(\left(Q^2 \sim \frac{1}{\lambda^2}\right)\right)$
controls resolving power of photon

$Q^2 \ll \frac{1}{R^2}$ → elastic e-p scattering
(R ← proton radius)

$Q^2 \gg \frac{1}{R^2}$ → resolve p constituents, elastic e-q scattering

we are interested in <u>deep</u> $(Q^2 \gg M_p^2)$ <u>inelastic</u> $((p+q)^2 \gg M_p^2)$ scattering

5.1 Structure functions

- want to describe process with Lorentz-invariant variables

  center-of-mass energy $s \equiv (k+p)^2$

  at fixed s, scattered e^- has 2 non-trivial variables (E, θ);

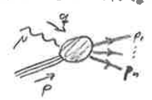
  use $Q^2 \equiv -q^2$, $x \equiv \frac{Q^2}{2p\cdot q}$

  ((other choices: $W^2 \equiv (p+q)^2 = Q^2\,\frac{1-x}{x}$ (invariant mass of γp-system)
  $y \equiv \frac{p\cdot q}{p\cdot k} = \frac{Q^2}{xs}$))

  kinematic limits: $Q^2 < s$, $x > Q^2/s$

  ((of course $k^2 = m_e^2 \approx 0$, $p^2 = M_p^2 \approx 0$; for $Q^2 \gtrsim M_Z^2$, also Z^0 exchange))

- know nothing about detailed structure of proton

  → parametrize $\mathcal{M} =$  $= e\,T_\mu(p,q;\{p_X\})$

  $\Rightarrow \frac{1}{4}\left|\right.$  $\left.\right|^2 = \frac{1}{4}\frac{e^4}{Q^4}\,\underbrace{tr\left(\not{k}'\gamma^\mu\not{k}\gamma^\nu\right)}_{\equiv L^{\mu\nu} = 4(k^\mu k'^\nu + k^\nu k'^\mu - k\cdot k' g^{\mu\nu})}\, T_\mu(p,q,\{p_X\})\, T_\nu^*(p,q,\{p_X\})$ (see p. 49)

  in complete analogy to $e^+e^- \to \mu^+\mu^-, q\bar{q}$ etc.

- for total cross section, need to integrate over phase space

  $$\int d\Phi_{X+1} = \underbrace{\int dQ^2\,dx\,\frac{Q^2}{16\pi^2 s x^2}}_{\text{electron kinematics}}\ \underbrace{\int d\Phi_X}_{n\text{-body phase space for } X}$$

  for an inclusive process (don't measure X → sum over them),

  $$\frac{d^2\sigma}{dQ^2 dx} = \frac{1}{2s}\,\frac{Q^2}{16\pi^2 s x^2}\,\underbrace{\sum_X \int d\Phi_X\,\frac{1}{4}|\mathcal{M}|^2}$$
  $$= \frac{1}{4}\frac{e^4}{Q^4}\,L^{\mu\nu}\,\underbrace{\sum_X \int d\Phi_X\, T_\mu(p,q,\{p_X\})\,T_\nu^*(p,q,\{p_X\})}_{\equiv H_{\mu\nu}}$$

- consider $H_{\mu\nu}$: have summed and integrated all X dependence,

  so $H_{\mu\nu}(p,q)$, must be symm. in μ,ν ((parity cons. in QED, QCD))

  $\Rightarrow H_{\mu\nu} = -H_1 g_{\mu\nu} + H_2\frac{p_\mu p_\nu}{Q^2} + H_3\frac{q_\mu q_\nu}{Q^2} + H_4\frac{p_\mu q_\nu + q_\mu p_\nu}{Q^2}$

  where H_i are scalar fcts, $H_i(q^2 = -Q^2,\ p\cdot q = \frac{Q^2}{2x},\ ~~p^2 = M_p^2~~)$ (neglect in DIS)

  ((including Z^0 exchange, ... $+ H_5\,\varepsilon_{\mu\nu\rho\sigma}\frac{p^\rho q^\sigma}{Q^2}$))

  $\Rightarrow L^{\mu\nu}H_{\mu\nu} = 8(kk')H_1 + 8\frac{(pk)(pk')}{Q^2}H_2 + 0\cdot H_3 + 0\cdot H_4$ (check?!)

  used $d=4$, neglected M_p^2

  now $Q^2 = -q^2 = -(k-k')^2 = 2kk' - ~~2m_e^2~~$

  $s = (p+k)^2 = 2pk + ~~M_p^2 + m_e^2~~$

  $pk' = p(k-q) = pk\left(1-\frac{pq}{pk}\right) = pk(1-y)$

  $= 4Q^2 H_1 + 2\frac{s^2}{Q^2}(1-y)H_2$

- $$\frac{d^2\sigma}{dQ^2\,dx} = \frac{1}{2s}\,\frac{Q^2}{16\pi^2 s x^2}\,\frac{1}{4}\,\frac{\alpha^2 16\pi^2}{Q^4}\left(4Q^2 H_1 + 2\frac{s^2}{Q^2}(1-y)H_2\right) \; ; \text{ def } \begin{matrix} H_1 = 8\pi F_1 \\ H_2 = 16\pi x F_2 \end{matrix}$$

$$\equiv \frac{4\pi\alpha^2}{xQ^4}\left\{ xy^2 F_1(x,Q^2) + (1-y)F_2(x,Q^2)\right\}$$

→ amazing: w/o knowing ep interaction, derived s-dependence of σ ! ((recall $y = Q^2/xs$))

→ the F's are called "structure functions" of the proton

Sometimes, see $F_T(x,Q^2) = 2x F_1(x,Q^2)$ "transverse"

$F_L(x,Q^2) = F_2(x,Q^2) - 2xF_1(x,Q^2)$ "longitudinal"

so $$\frac{d^2\sigma}{d^2Q\,dx} = \frac{2\pi\alpha^2}{xQ^4}\left\{\left(1+(1-y)^2\right)F_T(x,Q^2) + 2(1-y)F_L(x,Q^2)\right\}$$

$$= \frac{2\pi\alpha^2}{xQ^4}\left\{\left(1+(1-y)^2\right)F_2(x,Q^2) - y^2 F_L(x,Q^2)\right\}$$

useful since (for most current data) $y^2 \ll 1$

- have isolated all non-trivial x, Q^2 dependence into F_2, F_L ;
but still don't know anything about these fcts.

Assumption: interaction of γ with innards of proton does not involve any dimensionful scale

⇒ dim-less F's can not depend on dimensionful parameter Q^2

⇒ $$\frac{d^2\sigma}{dQ^2\,dx} = \frac{2\pi\alpha^2}{xQ^4}\left\{\left(1+(1-y)^2\right)F_2(x) - y^2 F_L(x)\right\}$$ Bjorken scaling

→ experimentally, this is true (to a pretty good approximation);
but proton consists of quarks, bound at distance scale ~ $1/m_p$,
so how can the interaction possibly be M_p-indep. ?!

→ answer via parton model

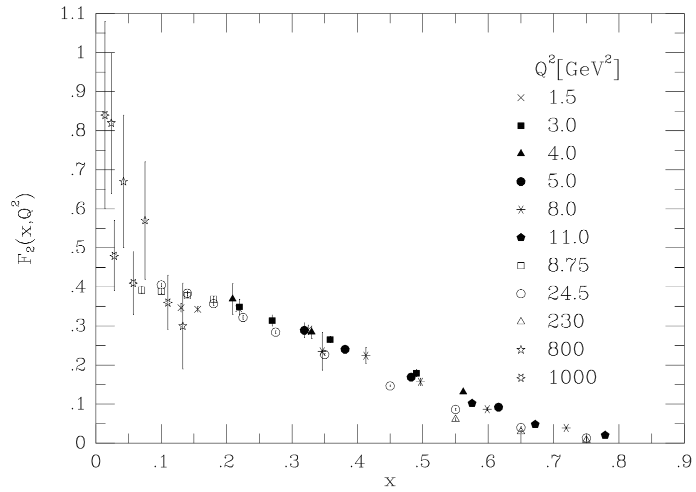


Figure 1: The structure function F_2 as a function of x for various Q^2 values, exhibiting Bjorken scaling, taken from [Ellis/Stirling/Webber]

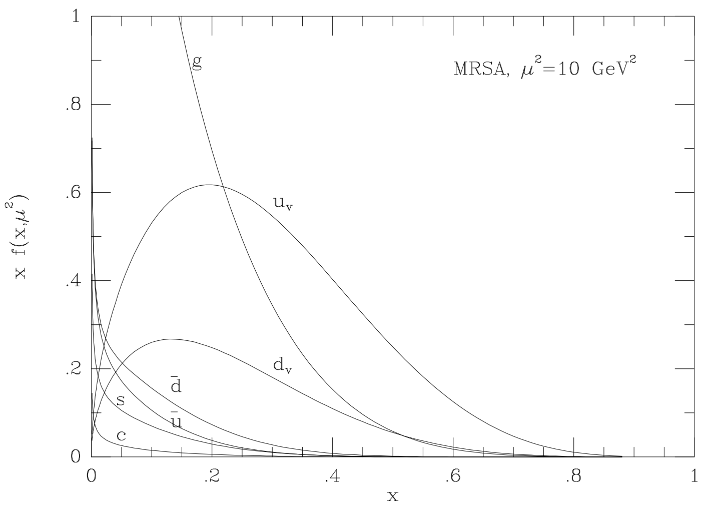


Figure 2: Parton distribution function set A from the Martin-Roberts-Stirling group, taken from [Ellis/Stirling/Webber]. Note that this uses the common notation of defining valence quark distributions, $f_{u_v} \equiv f_u - f_{\bar{u}}$, $f_{d_v} \equiv f_d - f_{\bar{d}}$.

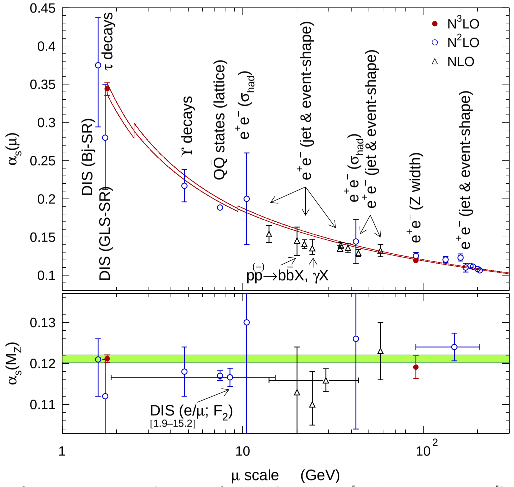


Figure 3: Results of a recent compilation of α_S values, see [arXiv:0803.0979 [hep-ph], arXiv:hep-ex/0606035]. The scale dependence shows excellent agreement with the predictions of perturbative QCD over a wide energy range. When translated into measurements of $\alpha_S(M_z)$, the separate measurements cluster strongly around the average value, $\alpha_S(M_z) = 0.1204 \pm 0.0009$

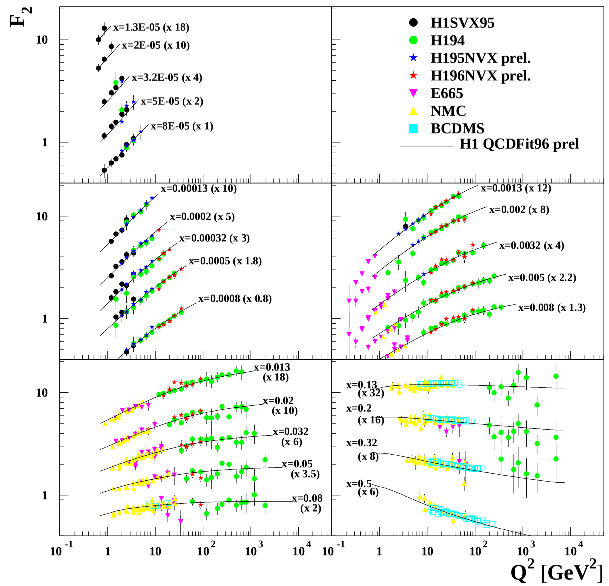


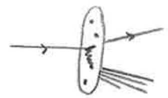
Figure 4: Fit to the F_2 data over a wide range of Q^2 values, exhibiting violation of Bjorken scaling

5.2 Parton distribution functions

- description of process in Lorentz-invariant;
  parton model most easily formulated in "Breit-frame": $E_\gamma = 0$, $E_p = \frac{Q}{2x}$

  proton in its rest frame: ○ ↕2R → Breit frame: ‖ ↕2R L-contraction
  $\frac{4R x M_p}{Q} \ll 2R$

  DIS in Breit frame:



  transverse size of photon $\sim \frac{1}{Q} \ll 2R$
  ⇒ photon interacts with tiny fraction of disk
  → if quarks sufficiently dilute in p, photon does not resolve q interactions; incoherent γq collisions!

  → since quarks act as if they don't interact, their interaction does not introduce a dimensionful scale ⇒ Bjorken scaling

- more precisely:
  proton ≡ bundle of comoving partons, carrying the proton's momentum p.
  parton of type q carries fraction between $(\eta, \eta + d\eta)$ of p during a fraction $d\eta \cdot f_q(\eta)$ of the time
  ↳ pdf, probability / parton distrib. fct.

  assume partons pointlike ($r^2 \ll 1/Q^2$) and dilute ($f_q(\eta) \ll Q^2 R^2$)
  → incoherent γ-parton-scattering
  with
  $$\frac{d^2\sigma(e + p(p))}{dQ^2\, dx} = \sum_q \int_0^1 d\eta\, f_q(\eta)\, \frac{d^2\sigma(e + q(\eta p))}{dQ^2\, dx}$$
  ↳ partonic cross section

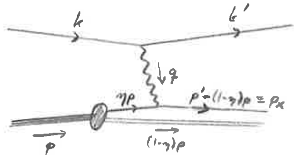
  note: In partonic cross section, elastic scattering
  → outgoing parton is on mass-shell.
  2→2 scattering → only 1 non-trivial kinematical variable (see pg. 45),
  so $\frac{d^2\sigma}{dQ^2 dx} \sim \delta$, where δ fct fixes one of the variables x, Q^2.
  For massless partons, $(q + \eta p)^2 = 2\eta(pq) - Q^2 \overset{!}{=} 0 \Leftrightarrow \eta = x$

  note: partons = quarks = fermions ⇒ (helicity cons.) $F_L = 0$ Callan-Gross relation
  (partons = scalars ⇒ $F_T = 0$)

- to obtain the parton model prediction for structure fcts, need to calculate partonic cross section.

→ need matrix el. for $eq \to eq$

get by "crossing symmetry" from $e^+e^- \to q\bar{q}$



$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4}\frac{1}{N_c} N_c \sum_s |\times|^2$$

(↑ from sum over outgoing color; from average over incoming color)

(see pg. 44, $\mu \to q$, $m=0$)

$$= \frac{8 e^2 (Q_q e)^2}{(q^2)^2} \left[(\eta p\cdot k)^2 + ((p' - (1-\eta)p)\cdot k)^2 \right]$$

convert to our kinematic invariants

$Q^2 = -q^2$, $s = (p+k)^2 = 2p\cdot k + \cancel{p^2} + \cancel{m_e^2}$, $p'\cdot k = (p+q)\cdot k = \frac{s}{2} + q\cdot k$

$q\cdot k = (k-k')\cdot k = \frac{1}{2}(k-k')^2 + \frac{1}{2}\cancel{m_e^2} - \frac{1}{2}\cancel{m_e^2} = -\frac{Q^2}{2}$

$$\Rightarrow [\ldots] = \left[\left(\frac{\eta s}{2}\right)^2 + \left(\frac{s}{2} - \frac{Q^2}{2} - (1-\eta)\frac{s}{2}\right)^2 \right] = \left(\frac{\eta s}{2}\right)^2 \left[1 + \left(1 - \frac{Q^2}{\eta s}\right)^2 \right]$$

$$= 8 (4\pi\alpha)^2 Q_q^2 \, \frac{1 + \left(1 - \frac{Q^2}{\eta s}\right)^2}{4 \left(\frac{Q^2}{\eta s}\right)^2}$$

→ phase space integration, see pg. 60

$$\int d\Phi_{X+1} = \underbrace{\int dQ^2 dx \frac{Q^2}{16\pi^2 s x^2}}_{\text{electron kinematics}} \int d\Phi_X$$

here: X consists of <u>one</u> massless parton

$$\int d\Phi_X = \int \frac{d^4 p_X}{(2\pi)^3} \delta(p_X^2) (2\pi)^4 \delta^4(\eta p + q - p_X)$$

$$= (2\pi)\, \delta\left((\eta p + q)^2\right) = (2\pi)\, \delta\left(\eta^2 \cancel{p^2} + 2\eta p q + q^2\right)$$

$$= (2\pi) \frac{1}{|2pq|} \delta\left(\eta - \frac{Q^2}{2pq}\right) = (2\pi) \frac{x}{Q^2} \delta(\eta - x)$$

→ partonic cross section

$$\sigma(e + q(\eta p)) = \frac{1}{2\eta s} \int d\Phi_{1+1} \langle |\mathcal{M}|^2 \rangle$$

$$\frac{d^2\sigma(e+q(\eta p))}{dQ^2 dx} = \frac{1}{2\eta s} \frac{Q^2}{16\pi^2 s x^2} \frac{2\pi x}{Q^2} \delta(\eta - x) \langle |\mathcal{M}|^2 \rangle \overset{y = \frac{Q^2}{xs}}{=} \frac{y^2}{16\pi Q^4} \delta(\eta - x)\, 2 (4\pi\alpha)^2 Q_q^2 \frac{1 + (1-y)^2}{y^2}$$

$$= \frac{2\pi\alpha^2 Q_q^2}{Q^4} \delta(\eta - x) \left[1 + (1-y)^2 \right]$$

- finally get cross section for $e + p$ (see pg. 62)

$$\frac{d\sigma(e + p(p))}{dQ^2 dx} = \sum_q \int_0^1 d\eta\, f_q(\eta) \cdot \frac{2\pi\alpha^2 Q_q^2}{Q^4} \delta(\eta - x) \left[1 + (1-y)^2 \right]$$

$$= \frac{2\pi\alpha^2}{x Q^4} \left[1 + (1-y)^2 \right] \sum_q Q_q^2\, x\, f_q(x)$$

→ comparing with §5.1 (result in terms of structure fcts F_2, F_L (pg. 61))

$$\Rightarrow F_2(x, Q^2) = \sum_q Q_q^2\, x\, f_q(x) \quad , \quad F_L(x, Q^2) = 0$$

note: F_2 is Q^2-independent: Bjorken scaling!

$F_L = 0$ was the Callan-Gross relation.

→ we will see that QCD corrections do violate Bjorken scaling; in experimental data, however, it is satisfied pretty well → Figure 1

- in practice, measure F_2 from different data sets and extract the f_q's.

$$F_2^{ep} = x \left[\frac{1}{9}(f_d + f_{\bar{d}} + f_s + f_{\bar{s}}) + \frac{4}{9}(f_u + f_{\bar{u}} + f_c + f_{\bar{c}}) \right]$$

since the pdf's contribute differently in different experiments (e.g. F_2^{en}, $F_2^{\nu p}$, $F_2^{\bar{\nu} p}$, ...) can do global fits to extract them. Typical results → Figure 2

→ useful checks via <u>sum rules</u>: $\int_0^1 dx\, f_{u_v}(x) = 2$, $\int_0^1 dx\, f_{d_v}(x) = 1$, etc.

5.3 QCD corrections in DIS

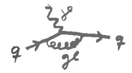
→ α_s is not small, so our above LO treatment of DIS might get important corrections.

→ how does the parton model emerge from QCD?

→ structure fcts will (slowly: logarithmically) depend on Q^2, leading to violation of Bjorken scaling

→ have to compute NLO corrections to DIS; divergences, splitting fcts, factorization, (DGLAP) evolution eqs, data

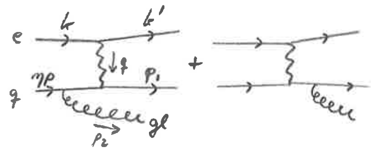
- three sources of NLO corrections

(1) $eq \to eq$ at 1-loop ("virtual corr.") 

(2) $eq \to eqg$ at tree-level 

(3) $eg \to eq\bar{q}$ at tree-level  ← completely new structure, not present in parton model

5.3.1 DIS @ NLO : $eq \to eqg$

- 2 diagrams, get $\langle |\mathcal{M}|^2 \rangle$ by "crossing" from $e^+e^- \to q\bar{q}g$ (see §4.3.1) 

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4}\frac{1}{N_c} N_c \frac{8 C_F e^4 Q_q^2 g_s^2}{(kk')(p_1 p_2)(\eta p \cdot p_2)} \left[(p_1 \cdot k)^2 + (\eta p \cdot k)^2 + (p_1 \cdot k')^2 + (\eta p \cdot k')^2 \right]$$

→ phase space $\int d\Phi_3 = \int dQ^2 dx \frac{Q^2}{16\pi^2 s x^2} \int d\Phi_X$ (electron) here: X consists of 2 partons → non-triv.

$$= \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta) \frac{1}{32\pi^2} = \int_0^{2\pi} d\varphi \int_0^1 dz \frac{1}{16\pi^2}$$

where (φ, θ) refer to direction of p_1 in CMS system of $\eta p + q$

alternatively, use Lorentz-invar. variable $z \equiv \frac{p_1 \cdot p}{q \cdot p} = \frac{1}{2}(1 - \cos\theta)$

→ partonic cross section

$$\frac{d^2\sigma(e+q)}{dQ^2 dx} = \frac{1}{2\eta s} \frac{Q^2}{16\pi^2 s x^2} \int_0^{2\pi} d\varphi \int_0^1 dz \frac{1}{16\pi^2} \frac{2 C_F e^4 Q_q^2 g_s^2}{(kk')(p_1 p_2)(\eta p \cdot p_2)} \left[\text{see} \uparrow \right]$$

$$= \frac{Q^2 C_F \alpha^2 Q_q^2 \alpha_s}{2\eta x^2 s^2} \int_0^1 dz \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{[\ldots]}{(kk')(p_1 p_2)(\eta p \cdot p_2)}$$

rewrite scalar products in terms of kinematic variables, perform φ-integration, use $\bar{x} \equiv x/\eta$

$$= \frac{Q^2 C_F \alpha^2 Q_q^2 \alpha_s}{2\eta x^2 s^2} \int_0^1 dz \frac{1}{y^2 Q^2} \left\{ \left[1 + (1-y)^2 \right] \left(\frac{1+\bar{x}^2}{1-\bar{x}} \frac{1+z^2}{1-z} + 3 - z - \bar{x} + 11 \bar{x} z \right) - y^2 (8 \bar{x} z) \right\}$$

$\left(\ldots\right) \equiv \hat{f}_2(\bar{x}, z)$ — will give divergence in F_2 due to $\int_0^1 dz$

$-y^2(8\bar{x}z)$ — will give a non-zero F_L

- from cross section

$$\frac{d^2\sigma(e+p)}{dQ^2 dx} \overset{(\text{pg. }62)}{=} \sum_q \int_0^1 d\eta\, f_q(\eta) \frac{d^2\sigma(e+q(\eta p))}{dQ^2 dx}$$

$$\overset{(\text{pg. }61)}{=} \frac{2\pi\alpha^2}{x Q^4} \left\{ \left[1 + (1-y)^2 \right] F_2(x, Q^2) - y^2 F_L(x, Q^2) \right\}$$

read off

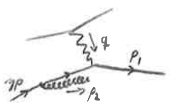
$$F_2(x, Q^2) = \frac{C_F \alpha_s}{2\pi} \sum_q \int_0^1 d\eta\, f_q(\eta)\, Q_q^2 \frac{x Q^4}{2\eta x^2 s^2 y^2} \int_0^1 dz\, \hat{f}_2\left(\frac{x}{\eta}, z\right)$$

with $\frac{xQ^4}{2\eta x^2 s^2 y^2} = \frac{x}{2\eta}$; ($\bar{x} \equiv \frac{x}{\eta}$), $\frac{d\eta}{\eta} = -\frac{d\bar{x}}{\bar{x}}$

$$= \frac{C_F \alpha_s}{2\pi} \sum_q \frac{1}{2} \int_x^\infty d\bar{x}\, f_q\left(\frac{x}{\bar{x}}\right) \frac{x}{\bar{x}} Q_q^2 \int_0^1 dz \left(\frac{1+\bar{x}^2}{1-\bar{x}} \frac{1+z^2}{1-z} + 3 - z - \bar{x} + 11 \bar{x} z \right)$$

- consider the divergence in F_2

comes from $z \to 1$, where $z = \frac{p_1 \cdot p}{q \cdot p}$

→ outgoing gluon collinear with incoming quark: 

$$\eta p \cdot p_2 = \eta p \cdot (\eta p + q - p_1) = \eta^2 p^2 + \eta p \cdot q - \eta p \cdot p_1 = \eta p \cdot q (1 - z)$$

→ internal line becomes on-shell, causing the divergence:

$$(\eta p - p_2)^2 = \eta^2 p^2 - 2\eta p \cdot p_2 + p_2^2 = -2\eta p \cdot q (1-z)$$, quark propagator $\sim \frac{1}{1-z}$

Note also: coefficient of divergence $\sim \frac{1}{1-\bar{x}}$, diverges at $\bar{x} = 1$, when gluon is infinitely soft

- regulate divergence

consider transverse momentum $k_\perp$ of outgoing quark in CMS system of $(\eta p + q)$;

((it turns out that $k_\perp^2 = Q^2 \left(\frac{\eta}{x} - 1 \right) z (1 - z)$))

$z \to 1$ means $k_\perp^2 \to 0$, so restricting $k_\perp^2 > \mu^2$ (with $\mu^2 \ll Q^2$)

regulates the divergence at $z \to 1$ ($\mu \to 0$ gives full result)

$$\rightsquigarrow \int_0^1 dz \to \int_{z_-}^{z_+} dz, \quad \text{where } z_\pm = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\frac{\mu^2}{Q^2(\frac{\eta}{x} - 1)}} \right); \quad \int_0^1 dz \approx \int_0^{1 - \frac{\mu^2}{Q^2}} dz$$

$$\Rightarrow F_2(x, Q^2) = \frac{\alpha_s}{2\pi} \sum_q \frac{1}{2} \int_x^\infty d\bar{x}\, f_q\left(\frac{x}{\bar{x}}\right) \frac{x}{\bar{x}} Q_q^2 \left(2 \hat{P}(\bar{x}) \ln \frac{Q^2}{\mu^2} + \hat{R}(\bar{x}) \right)$$

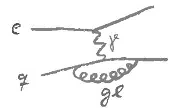
↑ divergence ↑ finite; $\mu \to 0$ done

with "(unregularized) splitting function" $\hat{P}(x) \equiv C_F \frac{1+x^2}{1-x}$

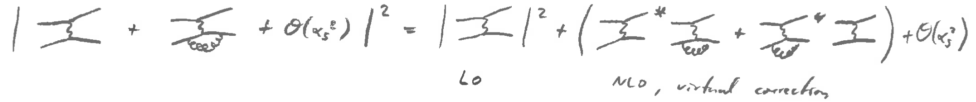
describes probability distribution of outgoing quark in 

note: have not (yet) removed the divergence, only regulated it.

5.3.2 DIS @ NLO: 1-loop $eq \to eq$

diagram:  $\sim \alpha_s$

in $|\mathcal{M}|^2$, only need interference term with tree level diagram



can again take everything from $e^+e^- \to q\bar{q}$ (§ 4.3.2) via crossing

→ here, recall only some of the features of that calculation, to illustrate the physics; interested in divergences.

- external particles same as at LO ⇒ kinematics is the same
  → $\int d\Phi \sim \delta(y-x)$ (see § 5.2, pg. 63)
- as in $e^+e^- \to q\bar{q}$, interference term is divergent (and negative); divergence comes from same kinematic region as in § 5.3.1: if gluon is soft, or collinear with either of the quarks.
- result: same form as $F_2(x, Q^2)$ above, with (unregularized) splitting fct $\hat{P}(\bar{z})$ replaced by $\tilde{P}(\bar{z})$ ((and different $\tilde{R}(\bar{z})$))
  ($\tilde{P}(\bar{z})$ ∼ $\delta(1-\bar{z})$ from $\delta(y-x)$)

⇒ real + virtual ∼ $P(x) \equiv \hat{P}(x) + \tilde{P}(x)$

mathematical trick: plus-distribution

given $f(x)$, well-defined for $0 \le x < 1$

define distribution $f(x)_+$ on $0 \le x \le 1$ as:

$$f(x)_+ \equiv f(x) - \delta(1-x) \int_0^1 dx' f(x')$$

((most useful for $f(x)$ which diverge at $x \to 1$))

⇒ for any test function $g(x)$ which is smooth at $x=1$,

$$\int_0^1 dx\, f(x)_+ \, g(x) = \int_0^1 dx\, f(x) \left[g(x) - g(1) \right]$$

((the latter integral being finite if $g(x) \to g(1)$ sufficiently quickly))

- splitting fct $P(x) = C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right]$

note: - inserting $P(\bar{z})$ into $F_2(x, Q^2)$, the divergence at $\bar{x} \to 1$ cancels; but the divergence at $z \to 1$, parametrized by $\ln \frac{Q^2}{\mu^2}$, remains.

- in fact, $P(x)$ is the first correction to a fct that describes the momentum distribution of quarks within quarks,

$$\mathcal{P}(x) \equiv \delta(1-x) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{Q_0^2} P(x) + \mathcal{O}\left(\alpha_s^2 \ln^2\left(\frac{Q^2}{Q_0^2}\right)\right)$$

((see pg. 66: $p \to xp$ etc; "pure quark" at scale Q_0^2))

5.3.3 Factorization, evolution

⇝ understand why results are still divergent!

compare to $e^+e^- \to q\bar{q}$ case: there, real + virtual = finite

here, main difference is: introduced pdf's f_q

- $z \to 1$ divergence comes from internal quark propagator $\sim \frac{1}{1-z}$ (see pg. 66);
  roughly, vanishingly small virtuality ≙ arbitrarily long time scales (uncertainty princ.)
  ⇝ contradiction to assumptions of parton model: rapid snapshot of proton
- the actual problem is overcounting:
  pdf's ≙ internal proton structure ≙ result of QCD interactions
  our calculation: QCD corrections to q scattering integrated over all final states (hence all E scales)
  ⇝ but are these QCD corr. already somehow in the pdf's?

⇒ to resolve overcounting: separate ("factorize") different types of physics at different energy scales!

introduce factorization scale μ.

physics at scales $< \mu$ ≡ part of hadron wave fct ≡ pdf's

$> \mu$ ≡ part of partonic scattering cross section ≡ coefficient fcts

note: the cutoff introduced in § 5.3.2, $eq \to eqg$, was hence correct.

((• kinematic limits: note that $\int_x^\infty d\bar{x} \to \int_x^1 d\bar{x}$ (cf pg. 66);

outgoing quark (p_1) and gluon (p_2) are real particles

$0 \le (p_1+p_2)^2 = (\eta p+q)^2 = \eta^2 p^2 + 2\eta p\cdot q + q^2 = 2p\cdot q\left(\eta - \frac{Q^2}{2p\cdot q}\right) = 2pq(\eta - x) = 2pq\eta(1-\bar{x})$))

• since pdf's contain physics below μ only, they depend on μ

$$F_2(x,Q^2) = \sum_q Q_q^2 \int_x^1 d\bar{x}\, f_q\left(\frac{x}{\bar{x}},\mu^2\right)\frac{x}{\bar{x}}\left\{\delta(1-\bar{x}) + \frac{\alpha_s}{2\pi}\left(P(\bar{x})\ln\frac{Q^2}{\mu^2} + R(\bar{x})\right) + O(\alpha_s^2)\right\}$$

⇒ structure fct depends on Q^2 now, violates Bjorken scaling!

→ μ^2-dependence ?!? ad-hoc theoretical construct

• physical cross sections cannot depend on μ^2

⇒ $\partial_{\mu^2} F_2(x,Q^2) \stackrel{!}{=} 0$

or, at least, in our calculation $\mu^2\partial_{\mu^2} F_2(x,Q^2) = O(\alpha_s^2)$

$$\Leftrightarrow \quad \mu^2\partial_{\mu^2} f_q(x,\mu^2) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\bar{x}}{\bar{x}} f_q\left(\frac{x}{\bar{x}},\mu^2\right)P(\bar{x}) + O(\alpha_s^2)$$

Dokshitzer-Gribov-Lipatov-Altarelli-Parisi eqn (DGLAP, GLAP, AP)

• try to understand physical content of DGLAP eqn

$$\left(\frac{1+x^2}{1-x}\right)_+ \stackrel{(pg\,67)}{=} \frac{1+x^2}{1-x} - \delta(1-x)\int_0^1 dx' \frac{1+x'^2}{1-x'}$$

($1+x'^2 = 2-(1-x')(1+x')$; $= 1+x^2$ due to δ fct)

$$= (1+x^2)\left\{\frac{1}{1-x} - \delta(1-x)\int_0^1 dx' \frac{1}{1-x'}\right\} + \delta(1-x)\underbrace{\int_0^1 dx'(1+x')}_{=3/2}$$

$$= \frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)$$

$$\Rightarrow P(x) = C_F\left(\frac{1+x^2}{1-x}\right)_+ \quad \text{, (cf. pg 68)}$$

now,
$$\mu^2\partial_{\mu^2} f_q(x,\mu^2) = \underbrace{\frac{\alpha_s}{2\pi}\int_x^1 \frac{d\bar{x}}{\bar{x}} f_q\left(\frac{x}{\bar{x}},\mu^2\right)C_F\frac{1+\bar{x}^2}{1-\bar{x}}}_{\text{pdf increase}} \underbrace{- \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\bar{x}}{\bar{x}} f_q\left(\frac{x}{\bar{x}},\mu^2\right)C_F\delta(1-\bar{x})\int_0^1 dx'\frac{1+x'^2}{1-x'}}_{= -\frac{\alpha_s}{2\pi} f_q(x,\mu^2)\, C_F\int_0^1 d\bar{x}\frac{1+\bar{x}^2}{1-\bar{x}} \text{ — pdf decrease}}$$

so at given x, pdf {increase / decrease} from {higher-x quarks / quarks at x} reducing their momentum fraction by radiating off gluons.

⇝ both pieces are divergent at $\bar{x}\to 1$ (due to infinitesimally soft gluon radiation); div's cancel because # gained quarks = # lost quarks

• solve DGLAP eqn ?

in practice: done numerically

• scheme/scale dependence

the above factorization (of structure fct F in $f_{\text{non-pert}} * \text{coeff.fct}_{\text{pert.}}$) may look pretty arbitrary; can be proven to all orders in pert theory;

→ instead of transverse momentum cutoff μ^2 above, use dim. reg.

⇒ NLO cross section in d dimensions: divergence is pole $\frac{1}{\varepsilon}$ now

$$F_2(x,Q^2) = \sum_q Q_q^2 \int_x^1 d\bar{x}\, \bar{f}_q\left(\frac{x}{\bar{x}}\right)\frac{x}{\bar{x}}\left\{\delta(1-\bar{x}) + \frac{\alpha_s}{2\pi}\left(P(\bar{x})\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\left(-\frac{1}{\varepsilon}\right) + \bar{R}(\bar{x}) + O(\varepsilon)\right) + O(\alpha_s^2)\right\}$$

(from $\frac{g^2_{\text{bare}}}{4\pi} = Z_g^2\alpha_s\mu^{4-d}$)

↑ "bare" pdf ↑ same P as above ↑ different from R

now (cf §3.3), since pdf's are not physical observables, define modified set of pdf's as

$$x f_q(x) = \int_x^1 d\bar{x}\, f_q\left(\frac{x}{\bar{x}},\mu_F^2\right)\frac{x}{\bar{x}}\left\{\delta(1-\bar{x}) - \frac{\alpha_s}{2\pi}\left(P(\bar{x})\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\varepsilon}\left(-\frac{1}{\varepsilon}\right) + K(\bar{x})\right)\right\}$$

↑ again, arbitrary "factorization" scale ↑ arbitrary finite fct

(check?!)

$$\Rightarrow F_2(x,Q^2) = \sum_q Q_q^2\int_x^1 d\bar{x}\, f_q\left(\frac{x}{\bar{x}},\mu_F^2\right)\frac{x}{\bar{x}}\left\{\delta(1-\bar{x}) + \frac{\alpha_s}{2\pi}\left(P(\bar{x})\ln\frac{Q^2}{\mu_F^2} + \bar{R}(\bar{x}) - K(\bar{x})\right) + O(\alpha_s^2)\right\}$$

note:
- same form as pg 69 (top), except for finite piece
- μ-dependence has cancelled
- for $\mu_{(F)}$-evolution of pdf, DGLAP eqn is valid
- f_q also depends on choice of K, "scheme dependence";

factorization theorem proves that
- for any physical quantity, all K and μ_F-dependence cancels
- scheme- and scale-dependent pdf's $f_q(x,\mu^2)$ are universal (i.e. process-independent)

• common choices: $\overline{MS}$ scheme ($K(\bar{x}) \equiv 0$)
DIS scheme ($K(\bar{x}) \equiv \bar{R}(\bar{x})$)

note: • dependence on scheme and scale cancels in physical quantities after calculating infinitely many orders in pert. theory ...

→ at finite order: residual dependence

→ need a procedure to choose value for μ_F

n^{th} order of pert. theory contains terms $\sim \alpha_s^n \ln^m \frac{Q^2}{\mu_F^2}$ ($m \leq n$)

so for reasons of convergence, take μ_F "not too far from" Q^2.

5.3.4 DIS @ NLO : $eg \to eq\bar{q}$

→ have so far not looked at process (3), see pg. 65

• most of § 5.3.1 – 3 carries over

again have a collinear singularity, coming from internal quark going "onshell".

• singularity can again be absorbed into factorized universal gluon-pdf f_g

$$\Rightarrow F_2(x,Q^2) \ni \sum_q Q_q^2 \int_x^1 d\bar{x}\, f_g\left(\frac{x}{\bar{x}},\mu_F^2\right) \frac{x}{\bar{x}} \left\{ \frac{\alpha_s}{2\pi}\left(P_{qg}(\bar{x}) \ln\frac{Q^2}{\mu_F^2} + R_g(\bar{x}) - K_{qg}(\bar{x})\right) + O(\alpha_s^2)\right\}$$

with splitting fct $P_{qg}(x) = \frac{1}{2}\left[x^2 + (1-x)^2\right]$ ((so far, had $P = P_{qq}$))

• in general, DGLAP eqn is a set of coupled eqs

$$\mu^2 \partial_{\mu^2} f_a(x,\mu^2) = \sum_b \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\bar{x}}{\bar{x}}\, f_b\left(\frac{x}{\bar{x}},\mu^2\right) P_{ab}(\bar{x}) + O(\alpha_s^2)$$

(at higher orders, also [gluon line splitting into two gluons] $\to P_{gg}(x)$ contributes)

((systematics of labelling the splitting fcts: $\overset{a}{\longrightarrow}\overset{b}{\diagup}{}^{z}$ ← internal line $\sim \frac{1}{1-z}$))

• Q^2-dependence of $F_2(x,Q^2)$ is entirely driven by μ_F^2-dependence of pdf's, which is predicted by DGLAP evolution eqs;

⇒ structure fct data over a wide range of Q^2 provide a stringent test of perturbative QCD → Figure 4

6. "Anomalies"

[Peskin/Schroeder §19 ; Zee § IV.7]

~ Q.: symmetry of classical physics ≙ symmetry of quantum physics ?

(action $S[\phi]$ invariant under $\phi \to \phi + \delta\phi$) (path integral $\int \mathcal{D}\varphi\, e^{iS[\phi]}$ invariant)

→ A.: not always; measure $\mathcal{D}\varphi$ may or may not be invariant.

example: rotational invariance of QM
would be strange if quantum fluctuations preferred a specific direction!

but: quantum fluctuation can break (some) classical symmetries;
this phenomenon is called "anomaly";
conceptually clear: change of integration variables → don't forget Jacobian

important subject in QFT → many ways of looking at it

here: see a class. symmetry vanishing, by explicit Feynman diag calc.

• theory of one massless fermion: $\boxed{\mathcal{L}_1 \equiv \bar{\psi}\, i\gamma^\mu \partial_\mu \psi}$

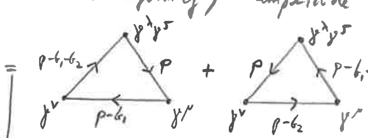
invariant under $\psi \to e^{i\theta}\psi$ and $\psi \to e^{i\theta\gamma^5}\psi$

conserved current $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$ (vector) and $j_5^\mu \equiv \bar{\psi}\gamma^\mu\gamma^5\psi$ (axial)

((check: $\partial_\mu j^\mu = 0 = \partial_\mu j_5^\mu$ via class. eqn. of motion $i\gamma^\mu\partial_\mu\psi = 0$))

• calculate (Fourier transform of) amplitude $\langle 0|T\, j_5^\lambda(0)\, j^\mu(x_1)\, j^\nu(x_2)|0\rangle$

$\Delta^{\lambda\mu\nu}(k_1,k_2) =$



, $q \equiv k_1 + k_2$

$$= (-1)\int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left(\gamma^\lambda\gamma^5 \frac{i}{\not{p}-\not{q}}\gamma^\nu \frac{i}{\not{p}-\not{k}_1}\gamma^\mu \frac{i}{\not{p}} + \gamma^\lambda\gamma^5 \frac{i}{\not{p}-\not{q}}\gamma^\mu \frac{i}{\not{p}-\not{k}_2}\gamma^\nu \frac{i}{\not{p}}\right)$$

if (in QFT) $\partial_\mu j^\mu = 0$ holds, then $k_{1\mu}\Delta^{\lambda\mu\nu} = 0$ and $k_{2\nu}\Delta^{\lambda\mu\nu} = 0$

〃 $\partial_\mu j_5^\mu = 0$ holds, then $q_\lambda \Delta^{\lambda\mu\nu} = 0$

→ can check this by explicit computation.

6.1 Vector current conservation

- would non-conservation of either current be a disaster?

  - j^μ: charge $Q = \int d^3x\, j^0$ counts # of fermions
    → would be difficult to interpret if not conserved!
    couple photon to ψ, photon line coming into vertex γ^μ would have propag $\frac{-i}{k_1^2}\left(g_{\mu\rho} - (1-\xi)\frac{k_{1\mu}k_{1\rho}}{k_1^2}\right)$
    → gauge dependence falls out if $k_{1\mu}\Delta^{\lambda\mu\nu} = 0$

  - j_5^μ: who care if axial charge $Q^5 = \int d^3x\, j_5^0$ changes in time?

- naive calculation

$$k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2) = i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^\lambda\gamma^5\frac{1}{\not p-\not q}\gamma^\nu\frac{1}{\not p-\not k_1}\not k_1\frac{1}{\not p} + \gamma^\lambda\gamma^5\frac{1}{\not p-\not q}\not k_1\frac{1}{\not p-\not k_2}\gamma^\nu\frac{1}{\not p}\right)$$

(with $\not k_1 = \not p - (\not p - \not k_1)$ and $\not k_1 = (\not p - \not k_2) - (\not p - \not q)$)

$$= i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^\lambda\gamma^5\frac{1}{\not p-\not q}\gamma^\nu\frac{1}{\not p-\not k_1} - \gamma^\lambda\gamma^5\frac{1}{\not p-\not k_2}\gamma^\nu\frac{1}{\not p}\right)$$

(shift $p \to p - k_1$)

$$= 0$$

- more careful calculation
  when is it OK to shift integration variables?

  1 dim: $\int_{-\infty}^{\infty} dp\,[f(p+a)-f(p)] = \int_{-\infty}^{\infty} dp\,[a\,\partial_p f(p) + \mathcal{O}(a^2)] = a\,(f(\infty)-f(-\infty)) + \mathcal{O}(a^2)$

  d dim: $\int d^d p_E\,[f(p+a)-f(p)] = \int d^d p_E\,[a^\mu\partial_{p_\mu} f(p) + \dots] \overset{\text{Gauss}}{=} \lim_{P\to\infty}\left\langle a^\mu\left(\frac{P_\mu}{P}\right) f(P)\, S_{d-1}(P)\right\rangle$

  $\left(\!\left(\frac{2\pi^{d/2}}{\Gamma(d/2)}P^{d-1}\right)\!\right)$ "surface" of d dim sphere, radius P

  for our 4 dim Minkowskian integral, (from Wick rot.)

$$\int d^4p\,[f(p+a)-f(p)] = \lim_{P\to\infty}\left\langle i\,a^\mu\left(\frac{P_\mu}{P}\right)f(P)\,(2\pi^2P^3)\right\rangle \quad \text{(angular average)}$$

  use this for $a = -k_1$

  and $f(p) = \mathrm{tr}\left(\gamma^\lambda\gamma^5\frac{1}{\not p-\not k_2}\gamma^\nu\frac{1}{\not p}\right) = \frac{\mathrm{tr}(\gamma^5(\not p-\not k_2)\gamma^\nu\not p\gamma^\lambda)}{(p-k_2)^2p^2} \overset{\text{(check?!)}}{=} \frac{4i\,\varepsilon^{\tau\nu\sigma\lambda}k_{2\tau}p_\sigma}{(p-k_2)^2p^2}$

  $\left(\!\left(\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\,\varepsilon^{\mu\nu\rho\sigma}\right)\!\right)$

$$\Rightarrow k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2) = \frac{i}{(2\pi)^4}\lim_{P\to\infty}\left\langle i(-k_1)^\mu\frac{P_\mu}{P}\,\frac{4i\,\varepsilon^{\tau\nu\sigma\lambda}k_{2\tau}P_\sigma}{(P-k_2)^2P^2}\,2\pi^2P^3\right\rangle$$

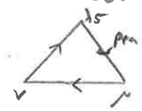
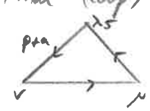
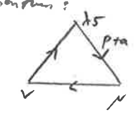
$$= \frac{i}{8\pi^2}k_1^\mu\,\varepsilon^{\tau\nu\sigma\lambda}g_{\mu\sigma}k_{2\tau} = \frac{i}{8\pi^2}\varepsilon^{\lambda\nu\tau\sigma}k_{1\tau}k_{2\sigma}$$

  $\left(\langle P_\mu P_\sigma\rangle = \frac{g_{\mu\sigma}}{4}P^2\right)$

  ⇝ $k_{1\mu}\Delta^{\lambda\mu\nu} \neq 0$?!↯ fermion # not conserved, we disintegrate

- reason for above result: Δ is linearly divergent!
  ⇒ have to make sure (even before calculating $k\Delta$) that integral is well-defined ((i.e. its value does not depend on the physicist doing the calculation))

  freedom of choice to label internal (loop) momentum:

$$\Delta^{\lambda\mu\nu}(a,k_1,k_2) \equiv$$  +  =  $+ \begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix}$

  but which a to choose?
  only sensible answer: choose a such that $k_{1\mu}\Delta^{\lambda\mu\nu}(a,k_1,k_2) = 0 = k_{2\nu}\Delta^{\lambda\mu\nu}$

  compute $\Delta^{\lambda\mu\nu}(a,k_1,k_2) - \Delta^{\lambda\mu\nu}(k_1,k_2)$ with above "careful" way:

  use $f(p) = \mathrm{tr}\left(\gamma^\lambda\gamma^5\frac{1}{\not p-\not q}\gamma^\nu\frac{1}{\not p-\not k_1}\gamma^\mu\frac{1}{\not p}\right)$

  note $\lim_{P\to\infty} f(P) = \lim_{P\to\infty}\frac{\mathrm{tr}(\gamma^\lambda\gamma^5\not P\gamma^\nu\not P\gamma^\mu\not P)}{P^6} \overset{\text{(check?!)}}{=} \frac{-4i\,P^2P_\sigma\,\varepsilon^{\sigma\nu\mu\lambda}}{P^6}$

$$\Rightarrow \Delta^{\lambda\mu\nu}(a,k_1,k_2) - \Delta^{\lambda\mu\nu}(k_1,k_2) = \frac{4i}{8\pi^2}\lim_{P\to\infty}\left\langle a^\omega\frac{P_\omega P_\sigma}{P^2}\varepsilon^{\sigma\nu\mu\lambda} + \begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix}\right\rangle$$

$$= \frac{i}{8\pi^2}\varepsilon^{\sigma\nu\mu\lambda}a_\sigma + \begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix}$$

  in general, $a = \alpha(k_1+k_2) + \beta(k_1-k_2)$ are all possible shifts

$$\Rightarrow \Delta^{\lambda\mu\nu}(a,k_1,k_2) = \Delta^{\lambda\mu\nu}(k_1,k_2) + \frac{i\beta}{4\pi^2}\varepsilon^{\lambda\mu\nu\sigma}(k_1-k_2)_\sigma$$

  ((α dropped out due to antisy of ε))

$$\Rightarrow k_{1\mu}\Delta^{\lambda\mu\nu}(a,k_1,k_2) = \underbrace{k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2)}_{=\frac{i}{8\pi^2}\varepsilon^{\lambda\nu\tau\sigma}k_{1\tau}k_{2\sigma}} + \frac{i\beta}{4\pi^2}\varepsilon^{\lambda\mu\nu\sigma}(k_{1\mu}k_{1\sigma} - k_{1\mu}k_{2\sigma})$$

  (see p. 73)

$$\overset{!}{=} 0 \quad\Rightarrow\quad \beta = -\tfrac{1}{2}$$

- note: Feynman rules are <u>not</u> sufficient to compute $\langle 0|T\,j_5^\lambda(0)j^\mu(x_1)j^\nu(x_2)|0\rangle$
  have to <u>supplement them</u> by vector current conservation!
  ⇝ amplitude $\langle 0|T\,j_5 j j|0\rangle$ is <u>defined</u> by $\Delta^{\lambda\mu\nu}(\alpha(k_1+k_2) - \tfrac{1}{2}(k_1-k_2), k_1, k_2)$

6.2 Axial current non-conservation

$\rightsquigarrow$ in §6.1, learned how to properly define $\Delta^{\lambda\mu\nu}$;

now, check axial current conservation by computing $q_\lambda \Delta^{\lambda\mu\nu}(a, k_1, k_2)$ (fixed by above)

- $q_\lambda \Delta^{\lambda\mu\nu}(a, k_1, k_2)$ $\left(\left(= -\tfrac{1}{2}(k_1-k_2) \right)\right)$ $= q_\lambda \Delta^{\lambda\mu\nu}(k_1,k_2) - \frac{i}{8\pi^2}\varepsilon^{\lambda\mu\nu\sigma}(k_1+k_2)_\lambda (k_1-k_2)_\sigma$

$$\frac{i}{8\pi^2}\varepsilon^{\lambda\mu\nu\sigma}(k_1+k_2)_\lambda (k_1-k_2)_\sigma = +\frac{i}{4\pi^2}\varepsilon^{\mu\nu\lambda\sigma} k_{1\lambda} k_{2\sigma}$$

$$q_\lambda \Delta^{\lambda\mu\nu}(k_1,k_2) = i\int\frac{d^4 l}{(2\pi)^4}\, \mathrm{tr}\left(\not q \gamma^5 \frac{1}{\not l - \not q}\gamma^\nu \frac{1}{\not l - \not k_1}\gamma^\mu \frac{1}{\not l} + \binom{\mu\leftrightarrow\nu}{k_1\leftrightarrow k_2} \right)$$

$\not q = \not l - (\not l - \not q)$; use cyclicity of trace; $\{\gamma^\mu, \gamma^5\} = 0$

$$= i\int\frac{d^4 l}{(2\pi)^4}\, \mathrm{tr}\left(\gamma^\mu\gamma^5 \frac{1}{\not l - \not q}\gamma^\nu \frac{1}{\not l - \not k_1} - \gamma^\nu\gamma^5 \frac{1}{\not l - \not k_1}\gamma^\mu \frac{1}{\not l} + \binom{\mu\leftrightarrow\nu}{k_1\leftrightarrow k_2} - \binom{\mu\leftrightarrow\nu}{k_1\leftrightarrow k_2} \right)$$

(pg. 73)

$$= k_{1\sigma}\Delta^{\mu\sigma\nu}(k_1,k_2) + \binom{\mu\leftrightarrow\nu}{k_1\leftrightarrow k_2} = \frac{i}{8\pi^2}\varepsilon^{\mu\nu\tau\sigma}k_{1\tau}k_{2\sigma}\cdot 2$$

$$= \frac{i}{2\pi^2}\varepsilon^{\mu\nu\lambda\sigma}k_{1\lambda}k_{2\sigma}$$

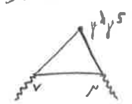
$\Rightarrow$ axial current is <u>not</u> conserved!

this is known as (axial/chiral) anomaly:

quantum fluctuations destroyed the (classical) axial current conservation.

Consequences / remarks (w/o derivations)

- gauge our theory $\mathcal{L}_1$: $\boxed{\mathcal{L}_2 \equiv \bar\psi\, i\gamma^\mu(\partial_\mu - ieA_\mu)\psi}$ ("photon")



$$\rightsquigarrow \partial_\mu j_5^\mu = \begin{cases} 0 & \text{classically} \\ \frac{e^2}{(4\pi)^2}\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} & \text{quantum} \end{cases}$$

((check?! : $k\to\partial$ in $F = \partial A - \partial A$))

historically important! decay $\pi^0 \to \gamma+\gamma$ (massless) forbidden via (wrong) $\partial_\mu j_5^\mu = 0$,

but decay is observed experimentally, as predicted by (correct) $\partial_\mu j_5^\mu \neq 0$.

($\Gamma_i/\Gamma \approx 98.8\,\%$, see PDG

- re-write $\mathcal{L}_2$ with $\psi_{R/L} \equiv \frac{1\pm\gamma^5}{2}\psi$,

introduce left- and right-handed currents $j_{R/L}^\mu \equiv \bar\psi_{R/L}\gamma^\mu\psi_{R/L}$

$$\Rightarrow \partial_\mu j_{R/L}^\mu = \pm\frac{1}{2}\frac{e^2}{(4\pi)^2}\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}$$

((that's why the anomaly is called "chiral"))

- add a fermion mass to $\mathcal{L}_2$: $\boxed{\mathcal{L}_3 \equiv \bar\psi\left[i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right]\psi}$

$\Rightarrow$ invariance under $\psi \to e^{i\theta\gamma^5}\psi$ broken by $m \neq 0$.

classically, $\partial_\mu j_5^\mu = 2m\bar\psi i\gamma^5\psi$, axial current not conserved.

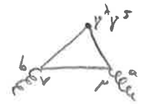
$\rightsquigarrow$ anomaly contributes an additional term (generated by quantum fluct.),

$$\partial_\mu j_5^\mu = 2m\bar\psi i\gamma^5\psi + \frac{e^2}{(4\pi)^2}\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}$$

- generalize to non-abelian case: $\boxed{\mathcal{L}_4 = \bar\psi\, i\gamma^\mu(\partial_\mu - igA_\mu^a T^a)\psi}$

calculation as before; vertex "μ" gets T^a

vertex "ν" gets T^b



summing over fermions in the loop gives $\mathrm{Tr}(T^a T^b)$

$$\Rightarrow \partial_\mu j_5^\mu = \frac{g^2}{(4\pi)^2}\varepsilon^{\mu\nu\lambda\sigma}\,\mathrm{Tr}(F_{\mu\nu}F_{\lambda\sigma})$$

$\uparrow F_{\mu\nu}^a T^a$, see §2.2, pg. 17

$\rightsquigarrow$ since $FF \sim A^2, A^3, A^4$, non-abelian symmetry immediately tells us there are triangle/square/pentagon anomalies

in QCD:

- higher orders? e.g. 3 loop

etc.

expect correction $\sim *\left[1 + \mathrm{fct}(e, g, \dots)\right]$ ← all couplings of theory

anomaly nonrenormalization theorem: $\mathrm{fct}(e,g,\dots) = 0$ (!)

for a proof see [Adler/Bardeen, Phys. Rev. 182 (1969) 1517]

[Collins, Renormalization, pg. 352]

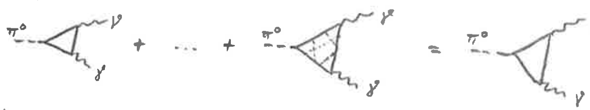
→ we can heuristically <u>understand</u> this:

before integrating over momenta of internal propagators,

the integrand has ≥ 5 fermion propags

⇝ sufficiently <u>convergent</u>, so we can <u>shift</u> momenta naively (cf. p.73)

→ historically, nonrenormalization of the anomaly important for developing concept of <u>color</u>:

$\pi^0 \to \gamma + \gamma$ ~  $+ 0 + \dots + 0$

process could be computed with confidence from <u>one</u> diagram,

decay amplitude does <u>not</u> depend on details of strong interactions;

result was factor of 3 too small ⇒ 3 types of quarks!

- beyond the Standard Model (BSM) – considerations:

are quarks / leptons composed of more fundamental fermions (preons)?

⇝ nonrenormalization theorem severely constrains possible preon models/theories (as long as they are formulated via QFT as we know it):

anomaly at preon level must be the <u>same</u> as at quark/lepton level.

→ <u>anomaly matching conditions</u> ((e.g. el. charges $Q_\nu + Q_e + N_c Q_u + N_c Q_d \stackrel{!}{=} 0$))

see e.g. ['t Hooft, Recent developments in gauge theories, Plenum Press 1980]

[Zee, Phys. Lett. B 95 (1980) 290]

- a last historic note:

after discovery of chiral anomaly, there were claims that path integral is wrong!

→ is $\int \mathcal{D}\bar\psi\, \mathcal{D}\psi\; e^{i\int d^4x\, \bar\psi i\gamma^\mu(\partial_\mu - ieA_\mu)\psi}$ unable to tell us

that it is <u>not</u> invariant under chiral trafo $\psi \to e^{i\theta\gamma^5}\psi$?!

→ it <u>does</u> tell us: action invariant, measure changes ("Jacobian")

see [Fujikawa, Phys. Rev. Lett. 42 (1979) 1195]

7. Outlook

⇝ have discussed aspects of a fascinating theory (QCD), with structures like non-Abelian gauge fields, coupling constant renormalization etc.

⇝ have seen in some examples that these abstract mathematical structures actually correspond to experimental observations

⇝ have so far mostly discussed perturbative QCD;

how about non-perturbative phenomena / techniques?

- can one actually <u>solve QCD</u> analytically?

"holy grail" for field theorists!

prize money: \$1 M ; see www.claymath.org/millennium

as a first step: try to solve pure YM (symm's could help!)

((or, even more symmetric, supersymmetric YM (SYM)))

(one) goal: take "idealized" QCD, u+d+gl, all massless

$$\mathcal{L} = \bar u\, i\not{D} u + \bar d\, i\not{D} d - \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu}$$

calculate e.g. $\frac{m_{\text{g-Meson}}}{m_{\text{Proton}}} \sim \{ \frac{1}{2}, \pi, \ln 2, \zeta(3), \dots \}$

(($\mathcal{L}$ has dimless parameter g^2; but $g^2(\mu)$ due to renormalization; dimensionful scale Λ_{QCD} where $\frac{g^2(\Lambda_{QCD})}{4\pi} \sim 1$; so each $m \sim \Lambda_{QCD}$))

- note that <u>perturbation theory</u> is somewhat <u>unnatural</u> for solving a highly symmetric gauge theory such as YM:

split $F^a_{\mu\nu} F^{a,\mu\nu} \to (\partial A - \partial A)^2 + A^3 + A^4$

= harmonic osc. + rest = $\mathcal{L}_0 + \mathcal{L}_{\text{interaction}}$

(gauge inv.; not g.i.; not g.i.)

⇒ clearly not optimal; solvable/integrable systems <u>need</u> symmetry!

- can one work with QCD in the regime where the strong coupling is actually strong?

  → big open question: confinement

  would like to derive the complete force between quarks from $\mathcal{L}_{QCD}$

  weak coupling limit: Coulomb potential w/ running coupling ✓

  strong coupling limit: linear potential, confines color

  (("string"?)) ↶ derivation needs new tools:

- [Wilson, Phys. Rev. D 10 (1974) 2445]: lattice gauge theory

  violate Lorentz-invariance, not gauge invariance

  formulate theory on 4d Euclidean spacetime lattice  $\}a$,

  perform continuum limit $a\to 0$ in the end, to recover 4d rotational invariance and (after Wick rot.) Lorentz invar.

  fundamental variables: line elements  ; U_{ij} = unitary, $N\times N$ matrix

  gauge invariant quantity: plaquette  $\mathrm{tr}\,(U_{ij}U_{jk}U_{kl}U_{li})$

  invariant under local trafo: $U_{ij}\to V_i^+ U_{ij} V_j$

  def $\quad Z = \int \mathcal{D}U\; e^{\frac{1}{2g^2}\sum_{plaq} \mathrm{Re}\,\mathrm{tr}\,(UUUU)}$ (Wilson)

  → is this equivalent to YM? yes, after $a\to 0$:

  def $U_{ij} = V_i^+\, e^{ia A_\mu(x)} V_j$, where $x \equiv \frac{x_i + x_j}{2}$, $\mu \equiv i\to j$ direction

  ((along $\mu = \frac{x_j - x_i}{a}$))

  then $UUUU = e^{ia^2 F_{\mu\nu} + O(a^3)}$

  and $\mathrm{Re}\,\mathrm{tr}\,(UUUU) \overset{!}{=} \mathrm{Re}\,\mathrm{tr}\,\{1 + ia^2 F_{\mu\nu} - \frac{1}{2}a^4 F_{\mu\nu}F_{\mu\nu} + \dots\}$

  $= \mathrm{tr}\,1 - \frac{a^4}{2}\mathrm{tr}\, F_{\mu\nu}F_{\mu\nu} + \dots$

  ⇝ beautiful formulation: no gauge fixing, no ghosts

  ⇝ as a challenge, try to incorporate fermions!

  highly nontrivial problem, ongoing research, ...

- for practical purposes, lattice gauge theory allows for numerical computations.

  ⇝ lattice Monte Carlo methods;

  huge world-wide efforts, development of algorithms and computers.

  ⇝ principal theoretical tool for quantitative calculations in hadron physics.

  get e.g. mass spectrum of low-lying mesons + baryons to ~10%

  ((as we have seen in §1.2))

- marriage of relativity + QM ⇒ QFT

  + statistical physics ⇒ thermal QFT

  ⇝ hot QCD is very interesting (phase transitions, ...),

  relevant (early universe, ...),

  conceptually clear (hadrons melt → quark-gluon plasma)

  ⇒ can be treated analytically (weak-coupling) ← Bielefeld, E6

  numerically (lattice Monte Carlo) ← Bielefeld, E6

  experimentally (heavy-ion collisions) ← RHIC, LHC

- as in other QFT's, QCD allows for interesting non-perturbative objects (exact sol's of eom; solitons, vortices, monopoles, instantons, ...)

  and non-perturbative methods (large-N expansion, ...)

- what is next?

  → master's thesis, ask everyone, get valuable insight into Bielefeld research

  → lectures in WS 13: lattice gauge theory (Karsch)

  supersymmetry } (Kögerler; maybe)

  electroweak physics }

  symmetries in physics (Akemann)