6. "Anomalies"

[Peskin/Schroeder §19 ; Zee § IV.7]

→ Q.: symmetry of classical physics ≙ symmetry of quantum physics ?
(action $S[\phi]$ invariant under $\phi \to \phi + \delta\phi$) (path integral $\int \mathcal{D}\varphi \, e^{iS[\phi]}$ invariant)

→ A.: not always ; measure $\mathcal{D}\varphi$ may or may not be invariant.

example: rotational invariance of QFT
would be strange if quantum fluctuations preferred a specific direction!

but: quantum fluctuation can break (some) classical symmetries ;
this phenomenon is called "anomaly" ;
conceptually clear: change of integration variables → don't forget Jacobian

important subject in QFT → many ways of looking at it

here: see a class. symmetry vanishing, by explicit Feynman diag calc.

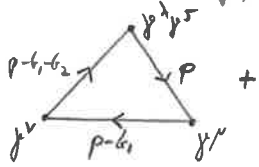
- theory of one massless fermion: $\boxed{\mathcal{L}_1 \equiv \bar{\psi} i \gamma^\mu \partial_\mu \psi}$

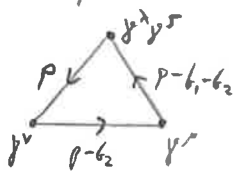
  invariant under $\psi \to e^{i\theta}\psi$ and $\psi \to e^{i\theta\gamma^5}\psi$

  conserved current $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$ (vector) and $j_5^\mu \equiv \bar{\psi}\gamma^\mu\gamma^5\psi$ (axial)

  ((check: $\partial_\mu j^\mu = 0 = \partial_\mu j_5^\mu$ via class. eqn. of motion $i\gamma^\mu\partial_\mu\psi = 0$))

- calculate (Fourier transform of) amplitude $\langle 0 | T \, j_5^\lambda(0) \, j^\mu(x_1) \, j^\nu(x_2) | 0 \rangle$

$\Delta^{\lambda\mu\nu}(k_1, k_2) =$ 

$+$ 

$, \quad q \equiv k_1 + k_2$

$$= (-1) \int \frac{d^4p}{(2\pi)^4} \, \mathrm{tr}\left(\gamma^\lambda\gamma^5 \frac{i}{\not p - \not q} \gamma^\nu \frac{i}{\not p - \not k_1} \gamma^\mu \frac{i}{\not p} + \gamma^\lambda\gamma^5 \frac{i}{\not p - \not q} \gamma^\mu \frac{i}{\not p - \not k_2} \gamma^\nu \frac{i}{\not p} \right)$$

if (in QFT) $\partial_\mu j^\mu = 0$ holds, then $k_{1\mu}\Delta^{\lambda\mu\nu} = 0$ and $k_{2\nu}\Delta^{\lambda\mu\nu} = 0$

〃 $\partial_\mu j_5^\mu = 0$ holds, then $q_\lambda \Delta^{\lambda\mu\nu} = 0$

⇒ can check this by explicit computation.

6.1 Vector current conservation

- would non-conservation of either current be a disaster?

  - J^μ: charge $Q = \int d^3x\, J^0$ counts # of fermions
    → would be difficult to interpret if not conserved!
    couple photon to ψ, photon line coming into vertex γ^μ
    would have propag $\frac{-i}{k_1^2}\left(g_{\mu\rho} - (1-\xi)\frac{k_{1\mu}k_{1\rho}}{k_1^2}\right)$
    → gauge dependence falls out if $k_{1\mu}\Delta^{\lambda\mu\nu} = 0$

  - J_5^μ: who cares if axial charge $Q^5 = \int d^3x\, J_5^0$ changes in time?

- naive calculation

$$k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2) = i\int\frac{d^4p}{(2\pi)^4}\, tr\left(\gamma^\lambda\gamma^5\frac{1}{\slashed{p}-\slashed{q}}\gamma^\nu\frac{1}{\slashed{p}-\slashed{k}_1}\overset{=\slashed{p}-(\slashed{p}-\slashed{k}_1)}{\slashed{k}_1}\frac{1}{\slashed{p}} + \gamma^\lambda\gamma^5\frac{1}{\slashed{p}-\slashed{q}}\overset{=(\slashed{p}-\slashed{k}_2)-(\slashed{p}-\slashed{q})}{\slashed{k}_1}\frac{1}{\slashed{p}-\slashed{k}_2}\gamma^\nu\frac{1}{\slashed{p}}\right)$$

$$= i\int\frac{d^4p}{(2\pi)^4}\, tr\left(\gamma^\lambda\gamma^5\frac{1}{\slashed{p}-\slashed{q}}\gamma^\nu\frac{1}{\slashed{p}-\slashed{k}_1} - \underbrace{\gamma^\lambda\gamma^5\frac{1}{\slashed{p}-\slashed{k}_2}\gamma^\nu\frac{1}{\slashed{p}}}_{\text{shift } p\to p-k_1}\right)$$

$$= 0$$

- more careful calculation
  when is it OK to shift integration variables?

1 dim: $\int_{-\infty}^{\infty} dp\,[f(p+a)-f(p)] = \int_{-\infty}^{\infty} dp\,[a\,\partial_p f(p) + \mathcal{O}(a^2)] = a\,(f(\infty)-f(-\infty)) + \mathcal{O}(a^2)$

d dim: $\int d^dp_E\,[f(p+a)-f(p)] = \int d^dp_E\,[a^\mu\partial_{p_\mu} f(p) + \ldots] \overset{\text{Gauss}}{=} \lim_{P\to\infty}\left\langle a^\mu\left(\frac{P_\mu}{P}\right) f(P)\, S_{d-1}(P)\right\rangle$

$\left(\frac{2\pi^{d/2}}{\Gamma(d/2)}P^{d-1}\right)$ "surface" of d dim sphere, radius P

for our 4 dim Minkowskian integral, (from Wick rot.)

$$\int d^4p\,[f(p+a)-f(p)] = \lim_{P\to\infty}\left\langle i\,a^\mu\left(\frac{P_\mu}{P}\right)f(P)\,(2\pi^2P^3)\right\rangle \quad \text{angular average}$$

$tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda) = -4i\,\varepsilon^{\mu\nu\sigma\lambda}$

use this for $a = -k_1$

and $f(p) = tr\left(\gamma^\lambda\gamma^5\frac{1}{\slashed{p}-\slashed{k}_2}\gamma^\nu\frac{1}{\slashed{p}}\right) = \frac{tr(\gamma^5(\slashed{p}-\slashed{k}_2)\gamma^\nu\slashed{p}\gamma^\lambda)}{(p-k_2)^2p^2} \overset{\text{(check?!)}}{=} \frac{4i\,\varepsilon^{\tau\nu\sigma\lambda}k_{2\tau}p_\sigma}{(p-k_2)^2p^2}$

$$\Rightarrow k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2) = \frac{i}{(2\pi)^4}\lim_{P\to\infty}\left\langle i\,(-k_1)^\mu\frac{P_\mu}{P}\,\frac{4i\,\varepsilon^{\tau\nu\sigma\lambda}k_{2\tau}P_\sigma}{(P-k_2)^2P^2}\,2\pi^2P^3\right\rangle$$

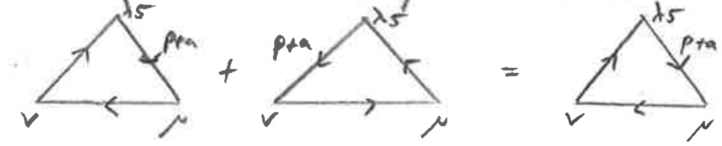
$$\left(\langle P_\mu P_\sigma\rangle = \frac{g_{\mu\sigma}}{4}P^2\right) \quad = \frac{i}{8\pi^2}k_1^\mu\,\varepsilon^{\tau\nu\sigma\lambda}g_{\mu\sigma}k_{2\tau} = \frac{i}{8\pi^2}\varepsilon^{\lambda\nu\tau\sigma}k_{1\tau}k_{2\sigma}$$

⇝ $k_{1\mu}\Delta^{\lambda\mu\nu} \neq 0$?! ↯ fermion # not conserved, we disintegrate

- reason for above result: Δ is linearly divergent!

  $\Rightarrow$ have to make sure (even before calculating $b\Delta$) that integral is well-defined ((i.e. its value does not depend on the physicist doing the calculation))

  freedom of choice to label internal (loop) momentum:

  $\Delta^{\lambda\mu\nu}(a, b_1, b_2) \equiv$  $+ \begin{pmatrix} \mu \leftrightarrow \nu \\ b_1 \leftrightarrow b_2 \end{pmatrix}$

  but which a to choose?

  only sensible answer: choose a such that $b_{1\mu} \Delta^{\lambda\mu\nu}(a, b_1, b_2) = 0 = b_{2\nu} \Delta^{\lambda\mu\nu}$

  compute $\Delta^{\lambda\mu\nu}(a, b_1, b_2) - \Delta^{\lambda\mu\nu}(b_1, b_2)$ with above "careful" way:

  use $f(p) = tr\left(\gamma^\lambda \gamma^5 \frac{1}{\not p - \not b} \gamma^\nu \frac{1}{\not p - \not b_1} \gamma^\mu \frac{1}{\not p}\right)$

  note $\lim\limits_{p\to\infty} f(p) = \lim\limits_{p\to\infty} \frac{tr(\gamma^\lambda \gamma^5 \not p \gamma^\nu \not p \gamma^\mu \not p)}{p^6} \overset{(\text{check?!})}{=} \frac{-4i\, p^2 p_\sigma\, \varepsilon^{\sigma\nu\mu\lambda}}{p^6}$

  $$\Rightarrow \Delta^{\lambda\mu\nu}(a, b_1, b_2) - \Delta^{\lambda\mu\nu}(b_1, b_2) = \frac{4i}{8\pi^2} \lim_{p\to\infty} \left\langle a^\omega \frac{p_\omega p_\sigma}{p^2} \varepsilon^{\sigma\nu\mu\lambda} + \begin{pmatrix} \mu \leftrightarrow \nu \\ b_1 \leftrightarrow b_2 \end{pmatrix} \right\rangle$$

  $$= \frac{i}{8\pi^2} \varepsilon^{\sigma\nu\mu\lambda} a_\sigma + \begin{pmatrix} \mu \leftrightarrow \nu \\ b_1 \leftrightarrow b_2 \end{pmatrix}$$

  in general, $a = \alpha(b_1 + b_2) + \beta(b_1 - b_2)$ are all possible shifts

  $$\Rightarrow \Delta^{\lambda\mu\nu}(a, b_1, b_2) = \Delta^{\lambda\mu\nu}(b_1, b_2) + \frac{i\beta}{4\pi^2} \varepsilon^{\lambda\mu\nu\sigma}(b_1 - b_2)_\sigma$$

  ((α dropped out due to antisym of ε))

  $$\Rightarrow b_{1\mu} \Delta^{\lambda\mu\nu}(a, b_1, b_2) = \underbrace{b_{1\mu} \Delta^{\lambda\mu\nu}(b_1, b_2)}_{= \frac{i}{8\pi^2} \varepsilon^{\lambda\nu\tau\sigma} b_{1\tau} b_{2\sigma} \ (\text{see pg. } 73)} + \frac{i\beta}{4\pi^2} \varepsilon^{\lambda\mu\nu\sigma}(b_{1\mu} b_{1\sigma} - b_{1\mu} b_{2\sigma})$$

  $$\overset{!}{=} 0 \quad \Rightarrow \quad \beta = -\frac{1}{2}$$

- note: Feynman rules are <u>not</u> sufficient to compute $\langle 0|T\, J_5^\lambda(0) J^\mu(x_1) J^\nu(x_2)|0\rangle$

  have to <u>supplement them</u> by vector current conservation!

  $\rightsquigarrow$ amplitude $\langle 0|T J_5 J J|0\rangle$ is <u>defined</u> by $\Delta^{\lambda\mu\nu}(\alpha(b_1+b_2) - \frac{1}{2}(b_1 - b_2), b_1, b_2)$